Scaling limits for randomly trapped random walks

Ben Hambly

with David Croydon and Takashi Kumagai

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Random walk in random environment

Let $G = (V, E)$ be a graph. For each $x \in V$ we assign a random probability $p_\omega(x, y)$ to each edge $e(x, y) \in E$. The collection

$$\{p_\omega(x, .) : x \in V, \sum_{y:e(x,y)\in E} p_\omega(x, y) = 1\}$$

is called the random environment. We write $\mathbf{P}, \mathbf{E}$ for the law and expectation with respect to the environment.

There are two natural measures for RWRE.

- Quenched: we fix the environment and consider the law $P^x_\omega(X \in \cdot)$ and study the random walker for $\mathbf{P}$-a.e. ω.
- Annealed: We consider the law of the random walker averaged over the environments $\mathbf{E}P^x_\omega(X \in \cdot)$

Random Conductance Model

A reversible version of RWRE is the random conductance model.

We define a random resistor network on the graph $G = (V, E)$. For each edge $e \in E$ we have a random variable $\mu_e = \mu_{xy}$, the conductance of the edge e between x, y. We then define

$$\mu_x = \sum_{y:(x,y)\in E} \mu_{xy}, \quad p_\omega(x,y) = \frac{\mu_{xy}}{\mu_x}.$$

By construction these are the transition probabilities for a discrete time reversible random walk with invariant measure μ_x on G.

Continuous time random walks

- The constant speed random walk (CSRW), $X = (X_t, t \geq 0, P_\omega^x, x \in \mathbb{Z}^d)$, with holding time 1 at each vertex; generator

$$\mathcal{L}_C f(x) = \mu_x^{-1} \sum_y \mu_{xy}(f(y) - f(x)).$$

- The variable speed random walk (VSRW), $Y = (Y_t, t \geq 0, P_\omega^x, x \in \mathbb{Z}^d)$, with holding time with mean $1/\mu_x$ at vertex x; generator

$$\mathcal{L}_V f(x) = \sum_y \mu_{xy}(f(y) - f(x)) = \mu_x \mathcal{L}_C f(x).$$

Scaling limits

Let $X^{(\varepsilon)}_t = \varepsilon X_{t/\varepsilon^2}, \quad Y^{(\varepsilon)}_t = \varepsilon Y_{t/\varepsilon^2}, \quad t \geq 0.$

Theorem (Andras-Barlow-Deuschel-H)

Let $d \geq 2$ and suppose that $(\mu_e, e \in E_d)$ are i.i.d., $\mu_e \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(\mu_e > 0) > p_c$.
(a) Let Y be the VSRW with $Y_0 = 0$. Then, $\mathbf{P}$-a.s. $Y^{(\varepsilon)}$ converges (under P^0_ω) in law to a Brownian motion on $\mathbb{R}^d$ with covariance matrix $\sigma_V^2 I$, where $\sigma_V > 0$ is non-random.
(b) Let X be the CSRW with $X_0 = 0$. Then, $\mathbf{P}$-a.s. $X^{(\varepsilon)}$ converges (under P^0_ω) in law to a Brownian motion on $\mathbb{R}^d$ with covariance matrix $\sigma_C^2 I$, where

$$\sigma_C^2 = \begin{cases} \sigma_V^2/(2d\mathbf{E}\mu_e), & \text{if } \mathbf{E}\mu_e < \infty, \\ 0, & \text{if } \mathbf{E}\mu_e = \infty. \end{cases}$$

Fractional Kinetics

In the case of $\mathbb{Z}^d$ for $d \geq 2$ with the CSRW with $\mathbf{E}\mu_e = \infty$ there is a scaling limit - the fractional kinetics process (Barlow-Cerny).

Let B be a Brownian motion and V an independent α-stable subordinator for $0 < \alpha < 1$. Let $\tau_t = \inf\{u > 0 : V_u > t\}$ be the inverse of V. The fractional kinetics process is given by

$$X_t^{FK(\alpha)} = B_{\tau_t}.$$

A continuous non-Markov self-similar process; $X_t = \lambda^{-\alpha/2} X_{\lambda t}$.

Theorem

For $d \geq 3$, if $\mathbf{P}(\mu_e > u) \sim u^{-\alpha}$, then for the CSRW X_t we have that $\mathbf{P}$-a.s. as $n \to \infty$

$$(\frac{1}{n} X_{tn^{2/\alpha}})_t \to (X_t^{FK(\alpha)})_t, \quad \textit{in law}$$

Random walk in a random trapping environment

Let $G = (V, E)$, be a graph. Traps are now just random holding times at the vertices. At each $x \in V$ we have a probability measure π_x on $(0, \infty)$; this is the random trapping environment. The randomly trapped random walk is then the simple random walk on G, which at each visit to vertex x chooses an i.i.d holding time according to π_x.

- The CSRW has $\pi_x \sim \text{Exp}(1)$.
- The VSRW has $\pi_x \sim \text{Exp}(\sum_y \mu_{xy})$.
- The Bouchaud trap model has $\pi_x \sim \text{Exp}(\xi)$ where $1/\xi$ is randomly chosen from a heavy tailed distribution.

Bouchaud trap model on $\mathbb{Z}^d$

- For $\mathbb{Z}^d$ with $\pi_x \sim \mathrm{Exp}(\xi)$ where ξ_x^{-1} for $x \in \mathbb{Z}^d$ chosen according to the distribution with $P(\xi > u) \sim u^{-\alpha}$ for $0 < \alpha < 1$.
- X is simple random walk on $\mathbb{Z}^d$ with holding times given by the ξ_x; That is with generator

$$\mathcal{L}_{BTM} f(x) = \xi_x^{-1} \sum_y (f(y) - f(x)).$$

- In $\mathbb{Z}^d$ for $d \geq 2$ we will see a fractional kinetics process in the scaling limit. For $d \geq 3$ the transience of the random walk means we do not always revisit the same traps.
- In $\mathbb{Z}$ the recurrence means we return to the deep traps.

The FIN diffusion

The Fontes-Isopi-Newman (FIN) diffusion is a singular diffusion on $\mathbb{R}$. It is defined as follows

- Let (x_i, v_i) be an inhomogeneous Poisson process on $\mathbb{R} \times (0, \infty)$ with intensity measure $dx \alpha v^{-\alpha-1} dv$. Set $\rho = \sum_i v_i \delta_{x_i}$.
- Let B be a Brownian motion with local time process $l_t(x)$.
- Given ρ we define the FIN diffusion as a time change of Brownian motion by the additive functional

$$A_t = \int_{\mathbb{R}} l_t(x) \rho(dx).$$

- Set $\tau_t = \inf\{u > 0 : A_u > t\}$. The FIN diffusion is

$$X_t^{FIN} = B_{\tau_t}.$$

Scaling limit

The FIN diffusion arises as the scaling limit of the BTM on $\mathbb{Z}$. Let $X_t^\epsilon = \epsilon X_{t\epsilon^{-1-1/\alpha}}$ and $\eta^\epsilon(dx) = \epsilon^{1/\alpha} \sum_{y \in \mathbb{Z}} \xi_y \delta_{\epsilon y}(dx)$.

Theorem

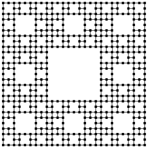
As $\epsilon \to 0$ the distribution of $(X_t^\epsilon, \eta^\epsilon)$ converges weakly to (X_t^{FIN}, ρ) under the annealed law.

In one dimension Ben Arous et al have classified the limits of trapping models, showing that the scaling limits are Brownian motion, Fractional Kinetics, FIN and a so-called spatially subordinated Brownian motion.

Aim

The aim of our project was to

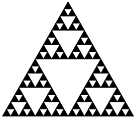
- generalize this type of result to families of recurrent graphs such as discrete trees or fractal graphs.
- Establish FIN limits for these processes by proving a general theorem about scaling limits of processes when we have Gromov-Hausdorff convergence of spaces.
- Consider the heat kernel estimates for these scaling limits.

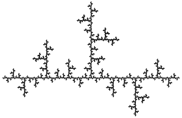


The set up

We work in a metric space F equipped with a resistance metric R.

- One dimension - the resistance is the Euclidean length
- Finitely ramified fractals such as the Sierpinski gasket
- Real trees - resistance is the length along the shortest path
- The two-dimensional Sierpinski carpet





We will have a base measure μ, the 'natural' measure on F

Assumptions

Let $\mathbb{F}$ be a collection of (F, R, μ, ρ) with ρ a distinguished point.

(UVD) There exist constants c_d, c_l, c_u and a non-decreasing function $v : (0, \infty) \to (0, \infty)$ satisfying $v(2r) \leq c_d v(r)$ for all $r \in (0, R_F + 1)$ such that

$$c_l v(r) \leq \mu(B_R(x, r)) \leq c_u v(r), \qquad \forall x \in F, \; r \in (0, R_F + 1).$$

(MC) The function $d : F \times F \to \mathbb{R}$ is a metric on F such that $d \asymp R^\beta$ for some $\beta > 0$.

(GMC) MC holds and also d is a geodesic metric.

Convergence results

Let $(F_n, R_n, \mu_n, \rho_n)_{n\geq 1}$ in $\mathbb{F}$ satisfies UVD, and also

$$(F_n, R_n, \mu_n, \rho_n) \to (F, R, \mu, \rho)\,,$$

in the Gromov-Hausdorff-vague topology, where $(F, R, \mu, \rho) \in \mathbb{F}$.

Theorem

1. (F_n, R_n), $n \geq 1$, and (F, R) can be isometrically embedded into a common metric space (M, d_M) in such a way that

$$(X_t^n)_{t\geq 0} \to (X_t)_{t\geq 0}$$

in distribution in $D(\mathbb{R}_+, M)$.

2. Moreover, the local times of L^n are equicontinuous, and if $d_M(x_i^n, x_i) \to 0$ for $i = 1, .., k$, then it simultaneously holds that

$$(L_t^n(x_i^n))_{i=1,\dots,k,t\geq 0} \to (L_t(x_i))_{i=1,\dots,k,t\geq 0}\,,$$

in distribution in $C(\mathbb{R}_+, \mathbb{R}^k)$.

The Bouchaud Trap Model

For the BTM we have the following set up.

Assumption

Suppose $(G_n)_{n\geq 1}$ *is a sequence of locally finite, connected graphs with vertex sets* V_n*, resistance metrics* R_n *where individual edges have unit resistance, counting measures* μ_n*, and distinguished vertices* ρ_n*.*

Assume that there exist scaling factors $(a_n)_{n\geq 1}$*,* $(b_n)_{n\geq 1}$ *such that* $(V_n, a_n R_n, b_n \mu_n, \rho_n)_{n\geq 1}$ *converges to* $(F, R, \mu, \rho) \in \mathbb{F}$*. Finally, we suppose that each* G_n *is equipped with a trapping landscape* $\xi^n = (\xi^n_x)_{x\in V_n}$ *such that for some fixed* $\alpha \in (0,1)$*,*

$$\mathbf{P}\left(\xi^n_x > u\right) \sim u^{-\alpha}.$$

The FIN diffusion on F

- Let
$$\nu(dx) := \sum_i v_i \delta_{x_i}(dx),$$
where (v_i, x_i) are the points of a Poisson process on $(0,\infty) \times F$ with intensity $\alpha v^{-1-\alpha} dv \mu(dx)$. This is a locally finite, Borel regular measure on (F, R) of full support, **P**-a.s.
- Let $l_t(x)$ denote the local times for the diffusion X on F
$$A_t = \int_{\mathbb{R}} l_t(x) \nu(dx).$$
- Set $\tau_t = \inf\{u > 0 : A_u > t\}$. The α-FIN diffusion on (F, R, μ) is
$$X_t^\nu = X_{\tau_t}.$$

The convergence result

Proposition

Suppose the BTM assumption holds. It is then possible to isometrically embed $(V_n, a_n R_n, b_n \mu_n, \rho_n)$*,* $n \geq 1$*, and* (F, R, μ, ρ) *into a common metric space* (M, d_M) *so that*

$$\mathbb{P}^{\mathrm{BTM}_n}_{\rho_n}\left(\left(X^{n,\xi^n}_{t/a_n b_n^{1/\alpha}}\right)_{t\geq 0} \in \cdot\right) \to \mathbb{P}^{\mathrm{FIN}}_{\rho}\left((X^{\nu}_t)_{t\geq 0} \in \cdot\right)$$

weakly as probability measures on $D(\mathbb{R}_+, M)$*.*

The random conductance model on fractal graphs with heavy tails also converges to the FIN diffusion on the limit fractal.

Heat kernels

- The classical heat kernel for Brownian motion in $\mathbb{R}$.

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

- For uniformly elliptic operators in $\mathbb{R}^d$ (Aronson)

$$p_t(x,y) \sim \frac{1}{(2\pi t)^{d/2}} \exp\left(-c\frac{|x-y|^2}{2t}\right).$$

- On manifolds we have (Li-Yau)

$$p_t(x,y) \sim \frac{c}{Vol_M(B_{\sqrt{t}}(x))} \exp\left(-c\frac{d(x,y)^2}{t}\right).$$

Heat kernels

- On fractals such as the Sierpinski gasket we have sub-diffusivity (Barlow+Perkins)

$$p_t(x,y) \sim ct^{-d_s/2} \exp\left(-c\left(\frac{d(x,y)^{d_w}}{t}\right)^{1/(d_w-1)}\right).$$

  where $1 < d_s = 2\log 3/\log 5 < 2$ and $d_w = \log 5/\log 2 > 2$.
- In the case of random fractals such as the CRT (Croydon): Quenched result is that $\mathbf{P}$-a.s.

$$ct^{-2/3}|\log(t)|^{-\beta} \le \inf_x p_t(x,x) \le \sup_x p_t(x,x) \le ct^{-2/3}|\log(t)|^{\beta'}.$$

- Annealed case

$$\mathbf{E}p_t(\rho,\rho) \sim ct^{-2/3}.$$

- What happens for the FIN diffusion?

Heat kernel estimates for FIN

- (Kigami) For **P**-a.e. realisation of ν, X^ν admits a jointly continuous transition density $(p_t^\nu(x,y))_{x,y\in F, t>0}$; the quenched heat kernel for the FIN diffusion.
- In the averaged case we consider $\mathbf{E}(p_t^\nu(x,y))$. The anomalous behaviour of the tail of the exit time of the FIN diffusion in $\mathbb{R}$ is due to Cerny and Cabezas.
- Let $V(\rho, r) = \nu(B_R(\rho, r))$ be the volume growth function of the FIN measure.

  Key to the study of the heat kernel is a connection with stable subordinators.

Volume growth

Let $\mathcal{L}$ be an α-stable subordinator. We can construct this by setting $\mathcal{L}_t = \sum_i v_i \mathbf{1}_{\{t_i \leq t\}}$, where (v_i, t_i) are the points of a Poisson process on $(0, \infty) \times \mathbb{R}_+$ with intensity $\alpha v^{-\alpha-1} dvdt$.

Recall $v(r) \asymp \mu(B_R(\rho, r))$ - typically $v(r) = r^{d_f}$.

Lemma

It is possible to couple $(\mathcal{L}_t)_{t \geq 0}$ and $(V(\rho, r))_{r \geq 0}$ so that, $\mathbf{P}$-a.s.,

$$\mathcal{L}_{c_l v(r)} \leq V(\rho, r) \leq \mathcal{L}_{c_u v(r)}, \qquad \forall r \in (0, R_F).$$

Stable subordinators

- Upper bounds:
  There is an integral test for the upper bound on the behaviour of $\mathcal{L}$ near 0 in that, **P**-almost surely,

$$\limsup_{t\to 0} \frac{\mathcal{L}_t}{h_t} = \begin{cases} \infty, & \text{if } \int_0^1 h_t^{-\alpha} dt = \infty, \\ 0, & \text{if } \int_0^1 h_t^{-\alpha} dt < \infty. \end{cases}$$

- Lower bounds:
  We have smaller fluctuations in that, **P**-almost surely,

$$\liminf_{t\to 0} \frac{\mathcal{L}_t}{t^{1/\alpha}(\log|\log t|)^{1-1/\alpha}} = C_\alpha (= \alpha(1-\alpha)^{(1-\alpha)/\alpha}).$$

Volume growth locally

These estimates give local volume growth estimates:

Lemma

(1) For any $\varepsilon > 0$ there exists a $c > 0$ such that

$$V(\rho, r) \leq cv(r)^{1/\alpha}|\log v(r)|^{(1+\varepsilon)/\alpha}, \qquad \forall r < R_F, \qquad \mathbf{P}\text{-a.s.}$$

(2) There is a $c > 0$ such that

$$V(\rho, r) \geq cv(r)^{1/\alpha}(\log|\log v(r)|)^{1-1/\alpha}, \qquad \forall r < R_F, \qquad \mathbf{P}\text{-a.s.}$$

Volume growth globally

For the global bounds we see the atoms in the upper bound:

Lemma

There exist random constants $0 < c_1, c_2$ *such that*

$$c_1 \leq \sup_{x \in F} V(x,r) \leq c_2, \qquad \forall r < R_F, \qquad \mathbf{P}\text{-}a.s.$$

For the lower:

Lemma

There exist positive constants c_1, c_2 *such that,* $\mathbf{P}$*-a.s.,*

$$c_1 \leq \liminf_{r \to 0} \frac{\inf_{x \in F} V(x,r)}{v(r)^{1/\alpha} |\log v(r)|^{1-1/\alpha}} \leq \limsup_{r \to 0} \frac{\inf_{x \in F} V(x,r)}{v(r)^{1/\alpha} |\log v(r)|^{1-1/\alpha}} \leq c_2$$

Quenched results - local

We can observe local and global on-diagonal results:

Suppose $v(r) = r^{d_f}$. Then the following hold.
(1) We have

$$0 < \limsup_{t\to 0} \frac{p_t^{\nu}(\rho,\rho)}{t^{-d_f/(d_f+\alpha)}\left(\log|\log t|\right)^{(1-\alpha)/(d_f+\alpha)}} < \infty, \qquad \mathbf{P}\text{-a.s.}$$

(2) For any $\varepsilon > 0$, there exists a constant c_3 such that

$$\liminf_{t\to 0} \frac{p_t^{\nu}(\rho,\rho)}{t^{-d_f/(d_f+\alpha)}|\log t|^{-3(1+\varepsilon)/\alpha}} \geq c_3 \qquad \mathbf{P}\text{-a.s.}$$

(3) Also there is a constant c_4 such that

$$\liminf_{t\to 0} \frac{p_t^{\nu}(\rho,\rho)}{t^{-d_f/(d_f+\alpha)}|\log t|^{-1/(d_f+\alpha)}} \leq c_4 \qquad \mathbf{P}\text{-a.s.}$$

Quenched results - global

(1) There exist random constants c_1, c_2 and a deterministic constant t_F such that

$$0 < c_1 \leq \inf_{x \in F} p_t^{\nu}(x,x) \leq c_2, \qquad \forall t < t_F, \qquad \mathbf{P}\text{-a.s.}$$

(2) Suppose $v(r) = r^{d_f}$. Then we have

$$0 < \limsup_{t \to 0} \frac{\sup_{x \in F} p_t^{\nu}(x,x)}{t^{-d_f/(d_f+\alpha)} \left|\log t\right|^{(1-\alpha)/(d_f+\alpha)}} < \infty, \qquad \mathbf{P}\text{-a.s.}$$

Annealed case

Let $T_D := T_{B_d(\rho,D)}$ be the exit time of the ball $B_d(\rho, D)$ by X^ν.

Theorem

Under UVD and MC there exist constants a, c_1, c_2 *such that*

$$\mathbf{E}P_\rho\left(T_D \le T\right) \le c_1 e^{-c_2 N(a)}, \qquad \forall D \in (0, D_F/2),\ t \in (0, h(R_F)),$$

where $N(a)$ *is defined as*

$$N(a) := \inf\left\{ n : \frac{at}{n} \le h\left(\left(\frac{D}{n}\right)^{1/\beta}\right)\right\}, \qquad (1)$$

and $h(r) = rv(r)^{1/\alpha}$

There is a corresponding lower bound under GMC as well.

The quenched case in 1-d

Theorem

For a fixed x, $\mathbf{P}$-a.s there exist constants such that for all $t < t_0$,

$$c_1 \exp(-c_2 N(D,t)) \le P_0^{\nu}(T_D \le t) \le c_3 \exp(-c_4 N(D,t)),$$

where $N(D,t) = \inf\{n : (D/n)^{1+1/\alpha} \ge c_5 t/n\}$. *Hence*

$$P_0^{\nu}(T_D \le t) \sim c \exp\left(-c\left(\frac{D^{1+1/\alpha}}{t}\right)^{\alpha}\right),$$

Proof idea: We decompose the path into visits to intermediate points on scale N and use the independence of the environment within these segments. The strong law of large numbers then gives the result.

Annealed heat kernel examples

For α large enough we have an annealed heat kernel estimate.

We assume that $v(r) = r^{d_f}$ and $D = d(x,y) \sim R(x,y)^{\beta}$.

- For the classical one-dimensional FIN diffusion we have $d_f = 1, \beta = 1$ and our estimate becomes, provided $\alpha > 0.618$;

$$\mathbf{E}\left(p_t^{\nu}(x,y)\right) \sim c_1 t^{-1/(\alpha+1)} e^{-c_2\left(\frac{D^{1+1/\alpha}}{t}\right)^{\alpha}}.$$

- For the FIN on the SG we have $d_f = \log 3/\log 2$, $\beta = \log 2/\log(5/3)$ and provided $\alpha > 0.743$;

$$\mathbf{E}\left(p_t^{\nu}(x,y)\right) \sim c_1 t^{-d_f/(\alpha\beta+d_f)} e^{-c_2\left(\frac{D^{1+d_f/\alpha}}{t^{\beta}}\right)^{1/(1+d_f/\alpha-\beta)}}.$$