Scaling limits of planar random growth models

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Work in progress with
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DLA aggregate formed on electrode in copper sulphate solution

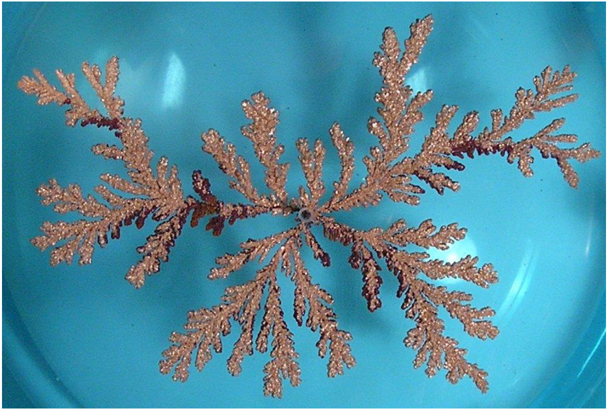


Photo by Kevin R Johnson

Eden cluster formed by lichen growth

Photo by James Wearn

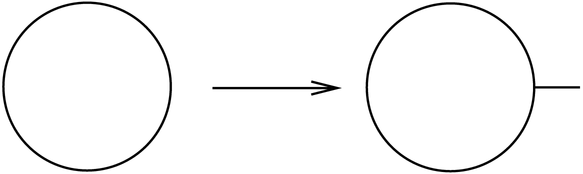
Electrical “tattoo” on survivor of lightning strike

From “Lichtenberg Figures Due to a Lightning Strike” by Yves Domart, MD, and Emmanuel Garet, MD

Conformal mapping representation of single particle

Let D_0 denote the exterior unit disk in the complex plane $\mathbb{C}$ and P denote a particle of size d attached at the point 1.

We typically take P to be the "slit" $(1, d]$ and use the unique conformal mapping $f_P : D_0 \to D_0 \setminus (1, d]$ that fixes ∞ as a mathematical description of the particle.

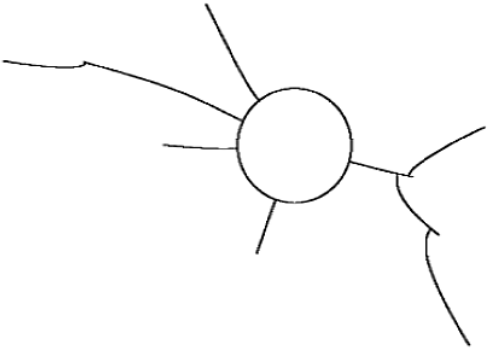


(Usually talk in terms of logarithmic capacity $c = \log f_P'(\infty)$, instead of size d. For slit maps, $e^c = 1 + \frac{d^2}{4(1+d)}$ so $c \asymp d^2/4$.)

Conformal mapping representation of a cluster

- Suppose $P_1, P_2, \dots$ is a sequence of particles, where P_n has capacity c_n and attachment angle θ_n, $n = 1, 2, \dots$.
  - Set $\Phi_0(z) = z$.
  - Recursively define $\Phi_n(z) = \Phi_{n-1} \circ f_{P_n}(z)$, for $n = 1, 2, \dots$.
- This generates a sequence of conformal maps $\Phi_n : D_0 \to K_n^c$, where $K_{n-1} \subset K_n$ are growing compact sets, which we call clusters.
- By varying the sequences $\{\theta_n\}$ and $\{c_n\}$, it is possible to describe a wide class of growth models.

Cluster formed by iteratively composing slit mappings

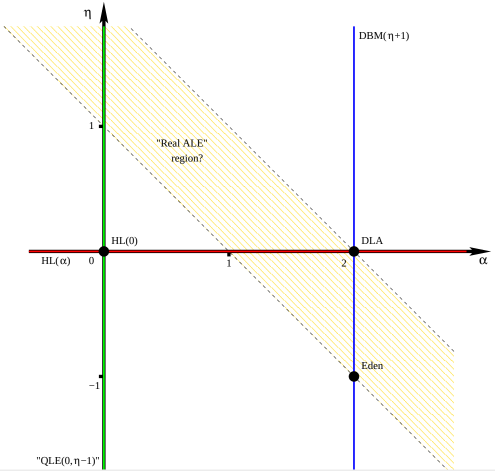


Examples of models within this framework

- Hastings-Levitov family, HL(α) [1998]:
  - θ_n are i.i.d. $U(-\pi,\pi)$ random variables;
  - $c_n = c|\Phi'_{n-1}(e^{i\theta_n})|^{-\alpha}$.
- Dielectric-breakdown models, DBM(η) [due to Mathiesen-Jensen, 2002]:
  - θ_n distributed $\propto |\Phi'_{n-1}(e^{i\theta})|^{1-\eta} d\theta$;
  - $c_n = c|\Phi'_{n-1}(e^{i\theta_n})|^{-2}$.
- Quantum Loewner Evolution, QLE(γ,η) [due to Miller-Sheffield, 2013]:
  - θ_n "distributed" $\propto e^{a(\gamma) h\circ\Phi_{n-1}(e^{i\theta})}|\Phi'_{n-1}(e^{i\theta})|^{b(\gamma)-1-\eta} d\theta$;
  - $c_n = c$ for all n, P_n a SLE_κ conditionally independent of the GFF h, given θ_n (a, b, functions depending on κ).

Aggregate Loewner Evolution, ALE(α, η)

- θ_n distributed $\propto |\Phi'_{n-1}(e^{i\theta})|^{-\eta} d\theta$; $c_n = c|\Phi'_{n-1}(e^{i\theta_n})|^{-\alpha}$.



Previous results

- Primary interest has been in asymptotic behaviour of large clusters.
- Almost all previous work relates to HL(0) as particle maps are i.i.d. so the model is mathematically the most tractable.
  - Norris and T. (2012) showed scaling limit of HL(0) is a growing disk with a branching structure related to the Brownian web.
  - Silvestri (2015) showed fluctuations form a Gaussian field.
- Results for HL(α) with $\alpha \neq 0$ have only been shown for regularized versions of the model.
  - Rohde and Zinsmeister (2005) analysed the dimension of scaling limits for HL(0) and for a regularized version of HL(α) when $\alpha > 0$.
  - Sola, T., Viklund (2015) showed scaling limit of regularized HL(α) is a growing disk for all α provided regularization is strong enough.

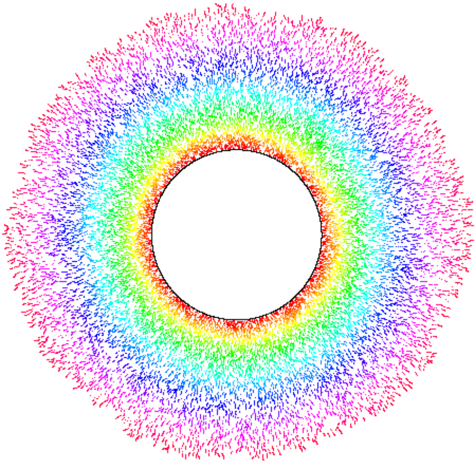
Open problems

- Does ALE(α, η) have phase transitions from disks to non-disks along the line $\alpha + \eta = 1$ (within some compact region)?
  Longstanding conjectures:
  - HL(α) has a phase transition at $\alpha = 1$.
  - DBM(η) has a phase transition at $\eta = 0$.
- Does ALE(α, η) have phase transitions to simple paths when η or α are large?
  Longstanding conjectures:
  - There exists some η_0 such that DBM(η) converges to a simple path for $\eta > \eta_0$.
  - There exists some α_0 such that HL(α) converges to a simple path for $\alpha > \alpha_0$.
- What other limit shapes are possible?

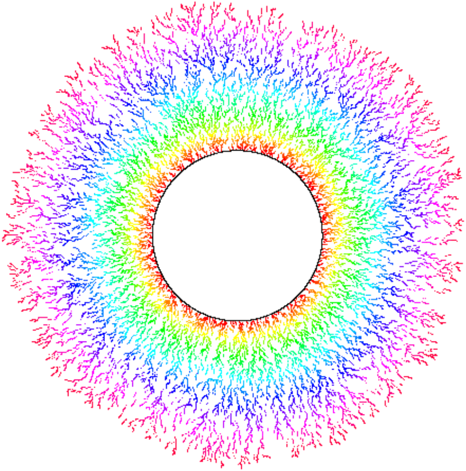
Scaling limits for ALE(0,η)

- Natural to consider particle sizes that are very small compared to the overall size of the cluster and scaling limits where $n \to \infty$ while $c \to 0$.
- Models are difficult to analyse mathematically as all models (except HL(0)) exhibit long-range dependencies.
- Additional difficulty, when $\alpha \neq 0$, is total capacity of cluster is random and cannot, a priori, be bounded above or below, so unclear at what rate to let $n \to \infty$.
- When $\alpha = 0$, K_n has capacity cn, so natural to look for scaling limits when $n = \lfloor T/c \rfloor$.

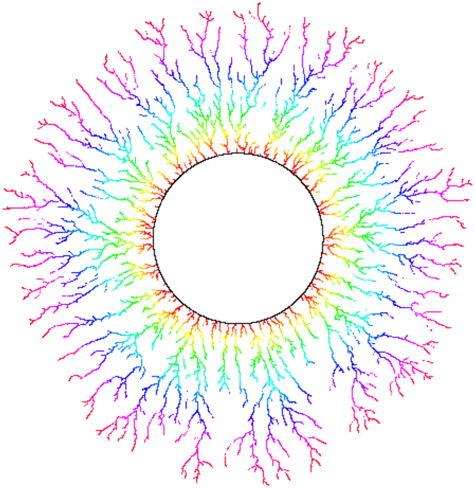
ALE(0,-1) cluster with 10,000 particles for $d = 0.02$



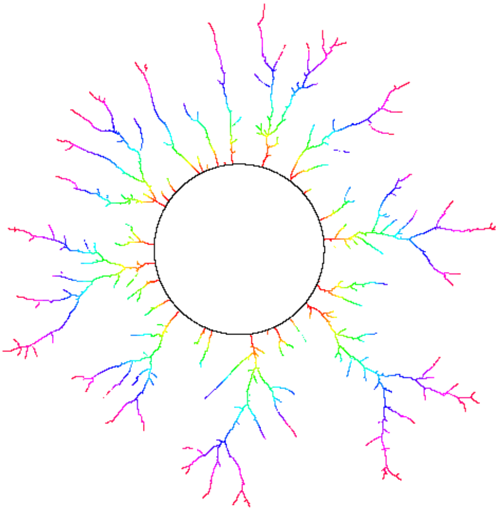
ALE(0,0) cluster with 10,000 particles for $d = 0.02$



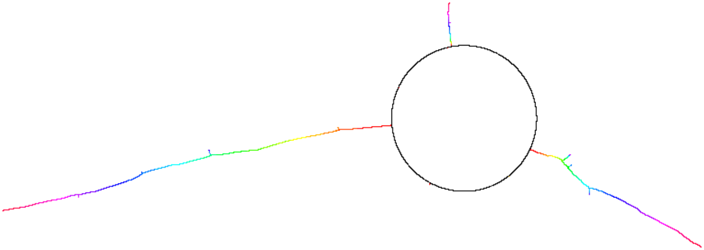
ALE(0,1) cluster with 10,000 particles for $d = 0.02$



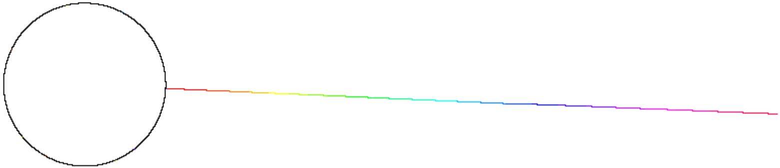
ALE(0,1.5) cluster with 10,000 particles for $d = 0.02$



ALE(0,2) cluster with 10,000 particles for $d = 0.02$



ALE(0,4) cluster with 10,000 particles for $d = 0.02$



Regularization for ALE(0,η)

- Even after the arrival of a single slit particle, the map $\theta \mapsto |\Phi_n'(e^{i\theta})|$ is badly behaved and takes the values 0 and ∞.
- For some values of η,

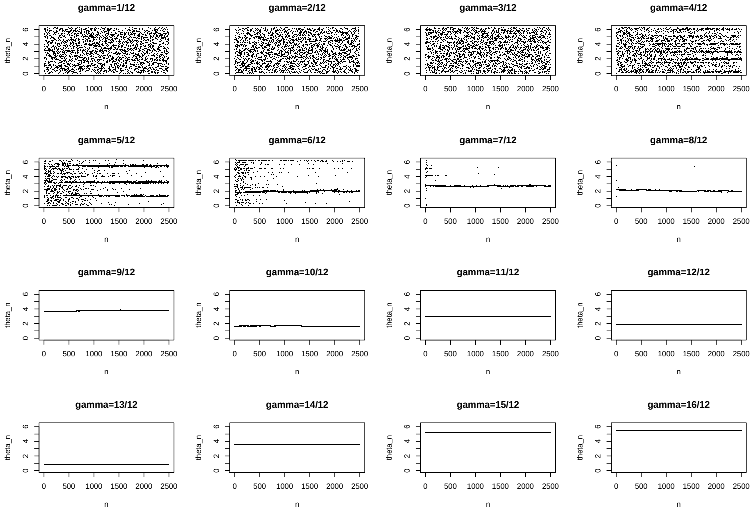
$$\int_{\pi}^{\pi} |\Phi_{n-1}'(e^{i\theta})|^{-\eta} d\theta = \infty,$$

  so regularization is necessary to even define the measure.
- A solution is to let θ_n have distribution

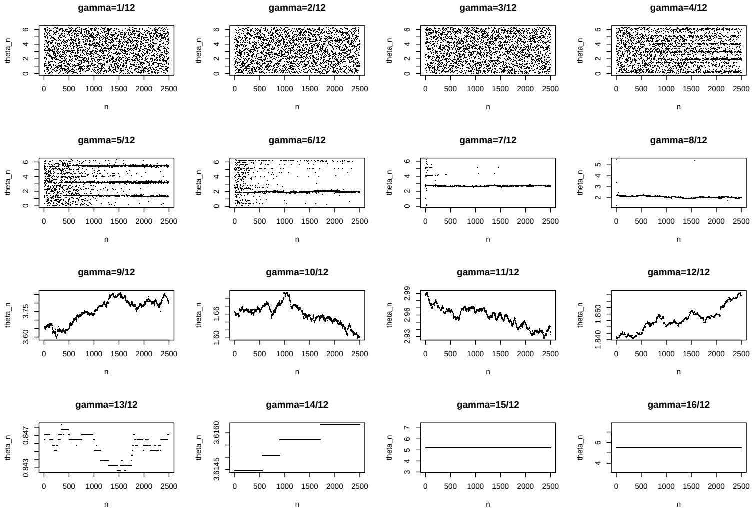
$$\propto |\Phi_{n-1}'(e^{\sigma+i\theta})|^{-\eta} d\theta$$

  for $\sigma > 0$ and take the limit $\sigma \to 0$.

Sequences $\{\theta_n\}$ in ALE(0,4) for varying γ where $\sigma = c^\gamma$



Sequences $\{\theta_n\}$ in ALE(0,4) for varying γ where $\sigma = c^\gamma$



Results for ALE(0,η)

Suppose $n = \lfloor T/c \rfloor$, and ALE(0,η) is regularized by σ.

- **Stick Theorem:**
  There exist η_0 and γ_0 such that for all $\eta > \eta_0$ and all $\sigma \ll c^{\gamma_0}$,

  $$\Phi_n(z) \to e^{i\theta_1} f_T(e^{-i\theta_1} z) \text{ in probability as } c \to 0,$$

  where f_t is the map corresponding to a slit of capacity t at 1.
- **Ball Theorem:**
  For every $\eta \in \mathbb{R}$, there exists a γ_1 such that for all $\sigma \gg c^{\gamma_1}$

  $$\Phi_n(z) \to e^T z \text{ in probability as } c \to 0.$$

(Refining the values of η_0 and γ_i is work in progress.)

Fluctuations about disk ($\eta \leq 1$)

Set

$$\mathcal{F}_n(z) = c^{-1/2}(\Phi_n(z) - e^{cn}z).$$

Then $\mathcal{F}_n(z) \to \mathcal{W}_t(z)$ where

$$\dot{\mathcal{W}_t}(z) = (1-\eta)z\mathcal{W}_t'(z) + \sqrt{2}\dot{\xi}_t(z)$$

where $\xi_t(z)$ is complex space-time white noise on the circle, analytically continued to the exterior unit disk.

(Note that if $\eta > 1$ would need $|z| > e^{(\eta-1)t}$ for this SPDE to make sense – beginnings of a phase transition at $|\eta| = 1$?)

Implication of results

- Have family of random growth process for which we are able to prove that, by varying a single parameter, scaling limits transition from being:
  - Deterministic to random;
  - Absolutely continuous to singular.
- Specifically, have shown:
  - Existence of transition from disks to simple paths in ALE(0,η) for fixed η as σ varies.
  - Existence of transition from disks to simple paths in ALE(0,η) as η varies?
  - Existence of phase transition in fluctuations in ALE(0,η) at $\eta = 1$?

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