Localisation and delocalisation in the parabolic Anderson model

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joint with Stephen Muirhead (Oxford) and Richard Pymar (Birkbeck)

LMS-EPSRC Durham Symposium: Markov Processes, Mixing Times and Cutoff
4 August 2017

Parabolic Anderson model

The Parabolic Anderson model is the heat equation on $\mathbb{Z}^d$

$$\frac{\partial u}{\partial t} = \Delta u + \xi u$$

with independent identically distributed random potential $\{\xi(z) \colon z \in \mathbb{Z}^d\}$ and localised initial condition $u(0, z) = \mathbf{1}_0(z)$.

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How does $u(t, \cdot)$ behave as $t \to \infty$?

Branching CTRW in random environment

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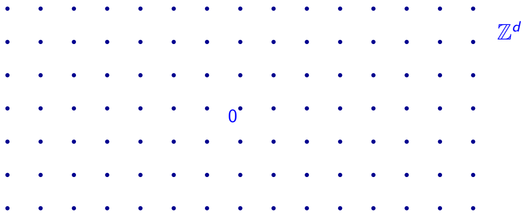

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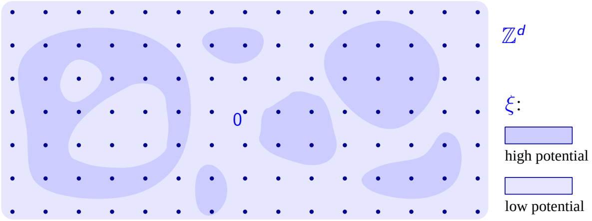

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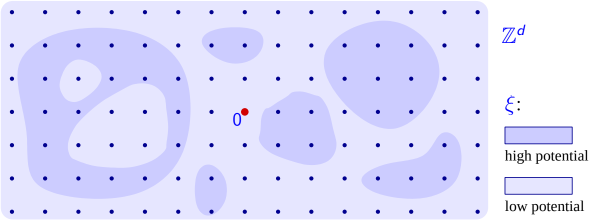

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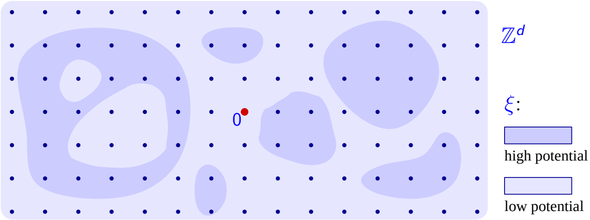

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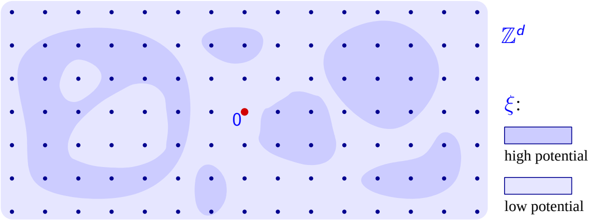

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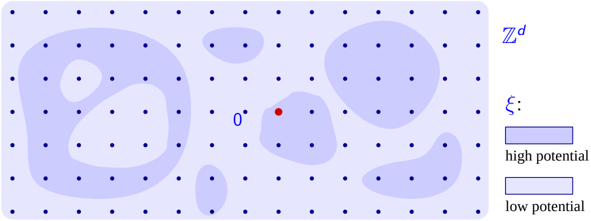

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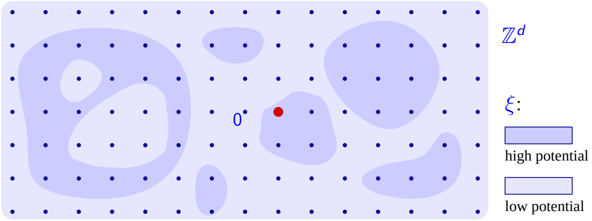

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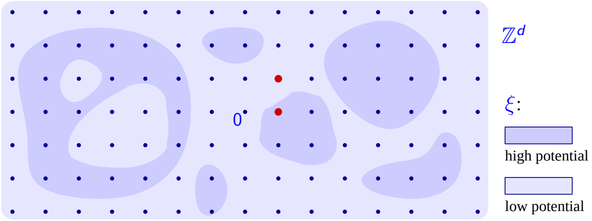

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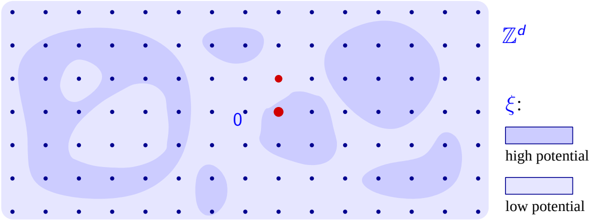

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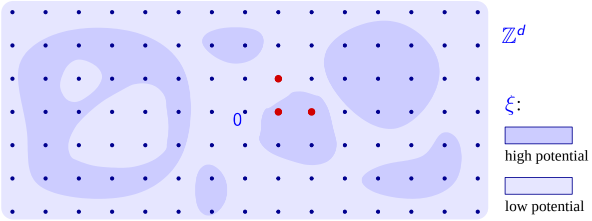

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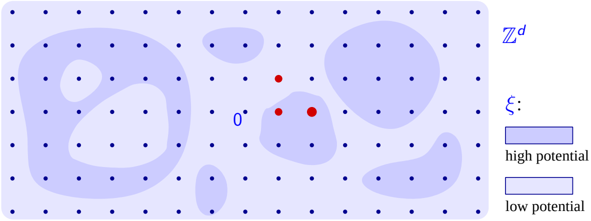

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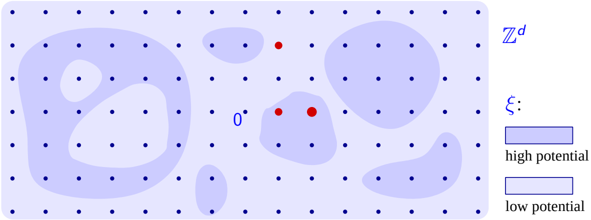

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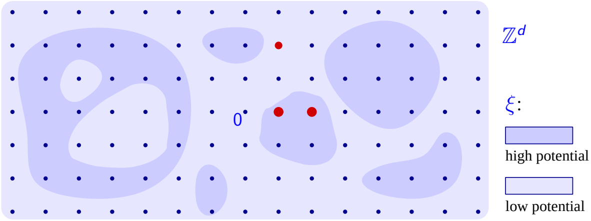

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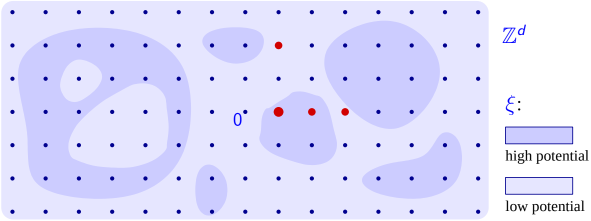

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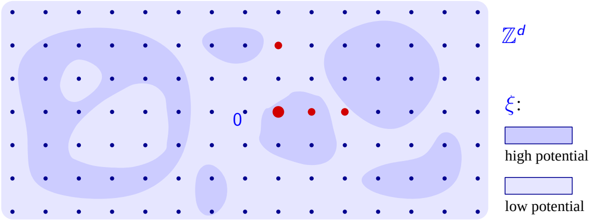

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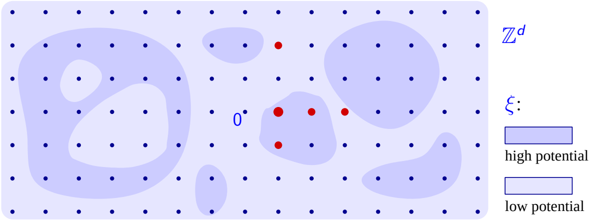

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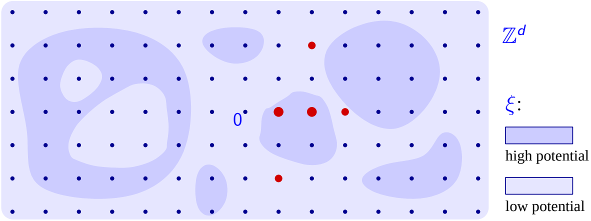

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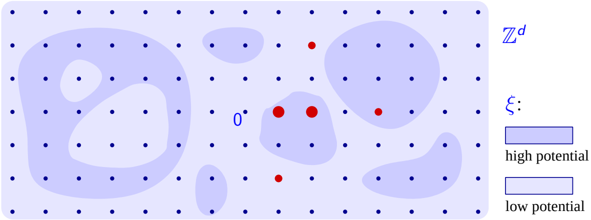

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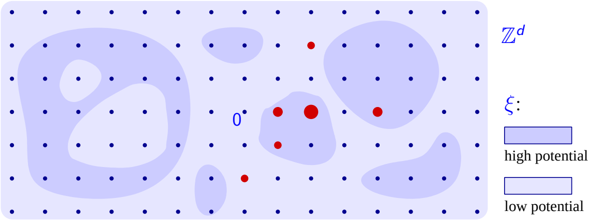

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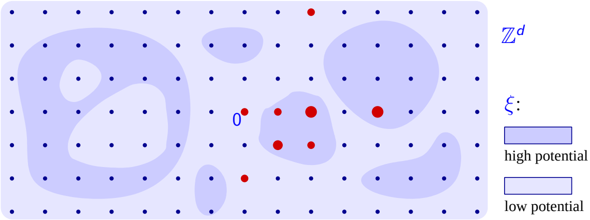

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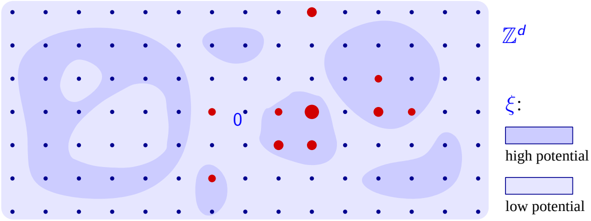

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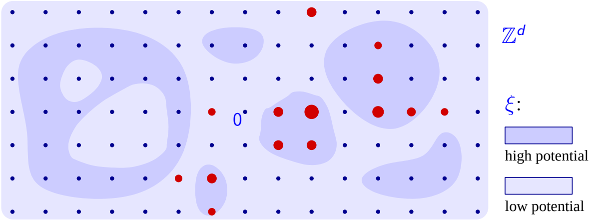

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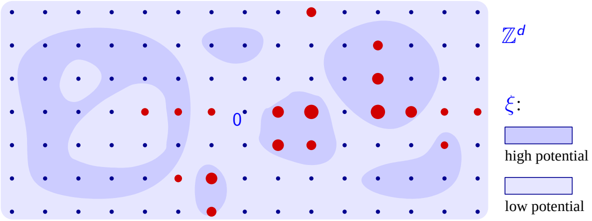

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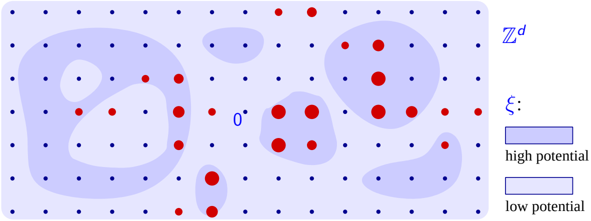

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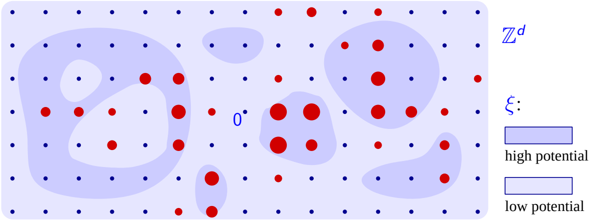

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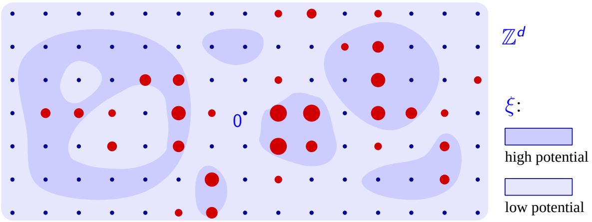

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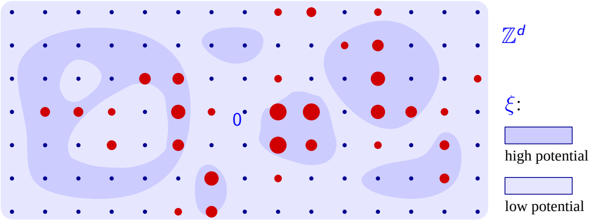

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$u(t, z) = \mathbb{E}N(t, z)$ is the average number of particles at time t at site z, still random.

Two approaches to study $u(t, z)$

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- Probabilistic: use path analysis to analyse the Feynman–Kac Formula

$$u(t, z) = \mathbb{E}\Big\{ e^{\int_0^t \xi(X_s)\, \mathrm{d}s} \mathbf{1}_{\{X_t = z\}} \Big\},$$

where (X_s) is a continuous-time random walk starting at zero.

Heat equation

The propagation of temperature $u(t,x)$ at time t at the point $x \in \mathbb{R}$ is described by

$$\frac{\partial u}{\partial t} = \Delta u.$$

If the initial temperature is δ_0 then

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

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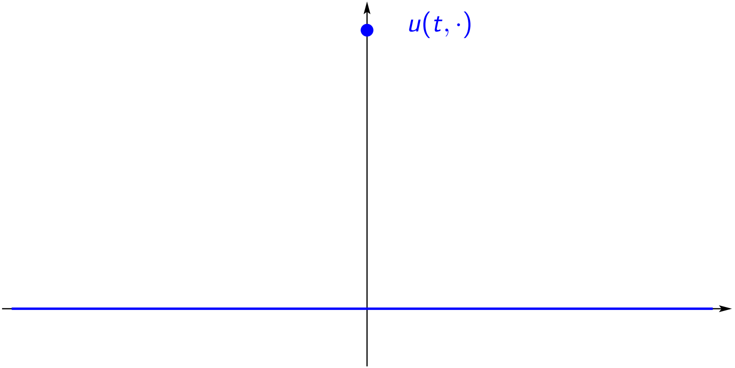

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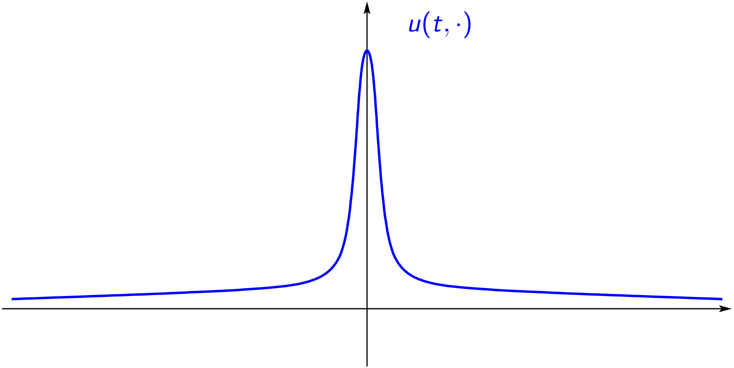

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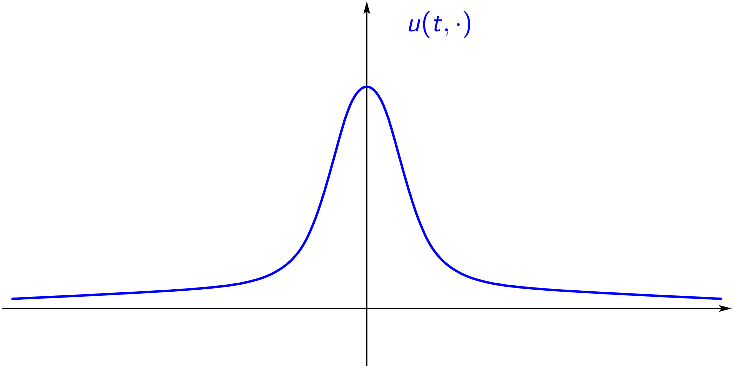

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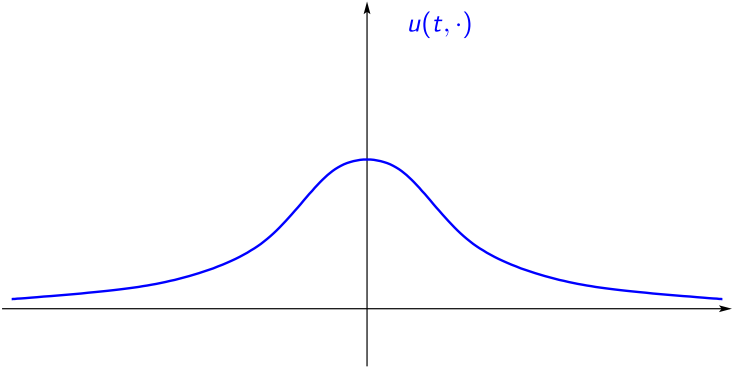

Heat equation with a potential

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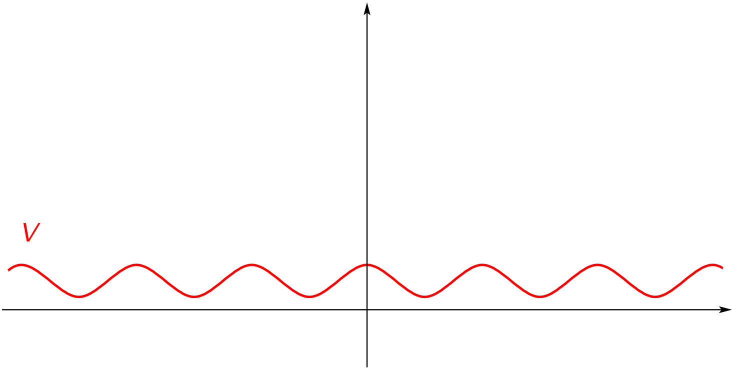
$$\frac{\partial u}{\partial t} = \Delta u + V u,$$

where $V : \mathbb{R} \to \mathbb{R}$ is a reasonably nice potential.



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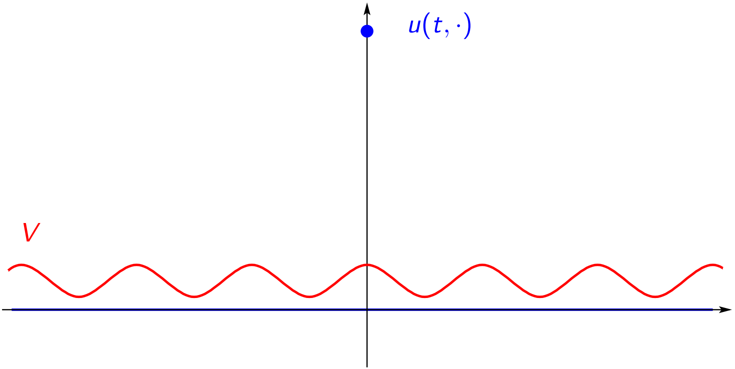

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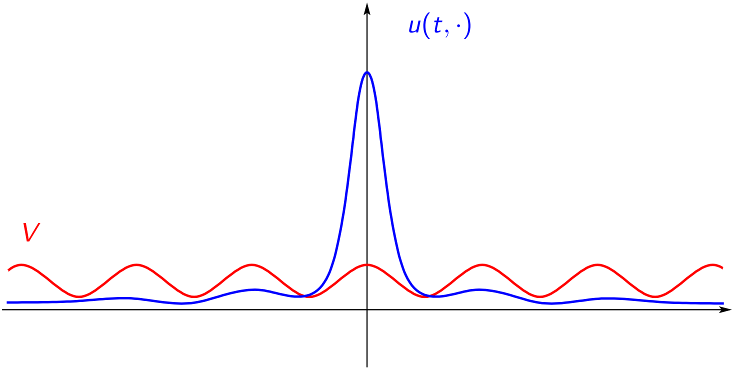

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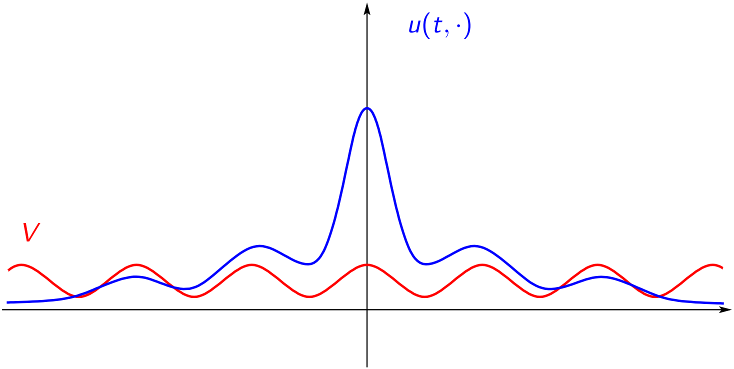

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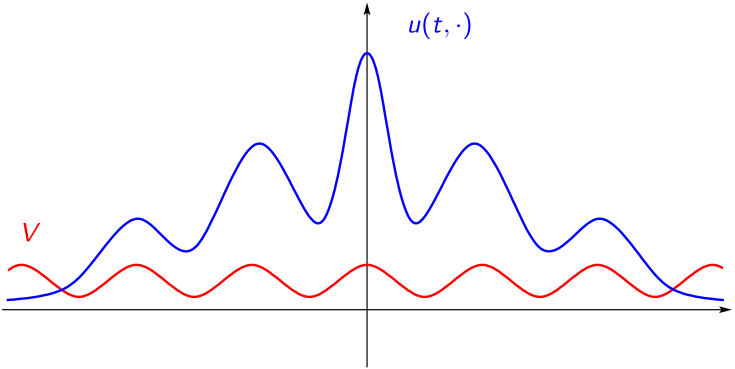

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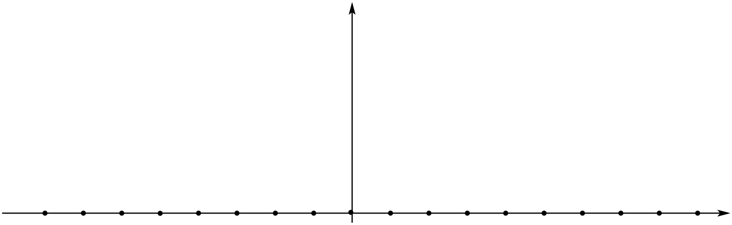

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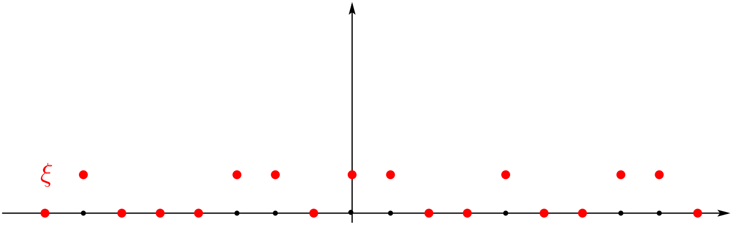

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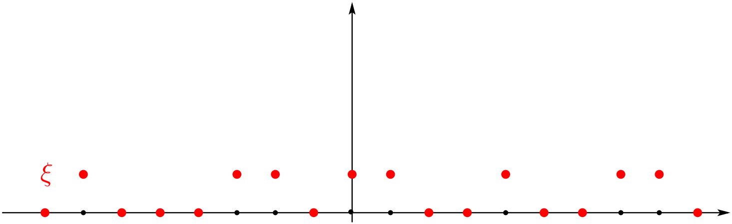
$$\frac{\partial u}{\partial t} = \Delta u + \xi u,$$

where $\xi : \mathbb{Z}^d \to \mathbb{R}$ is a random i.i.d. potential.



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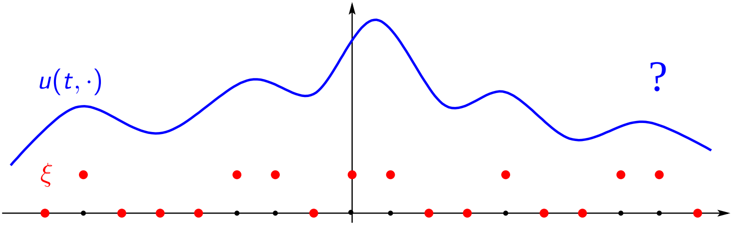

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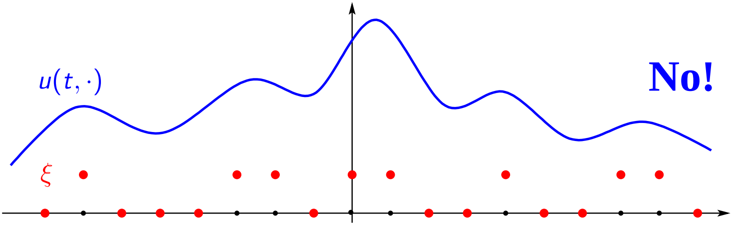

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Why not?

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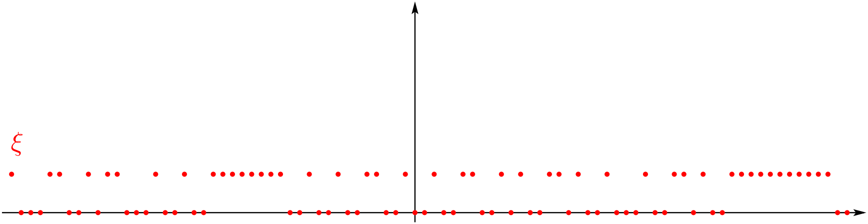

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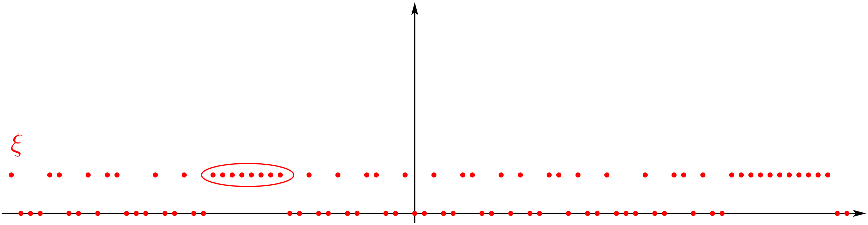

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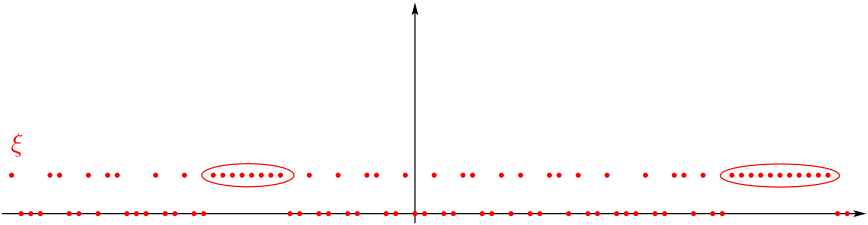

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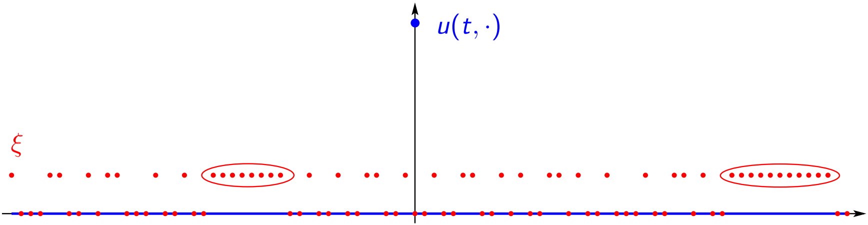

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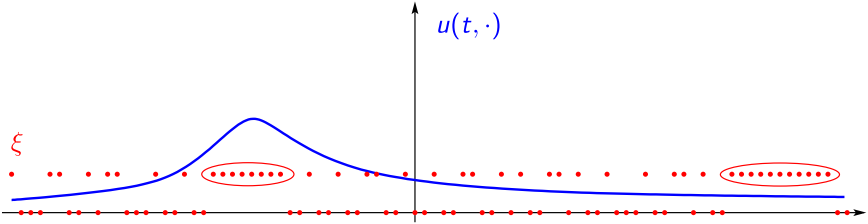

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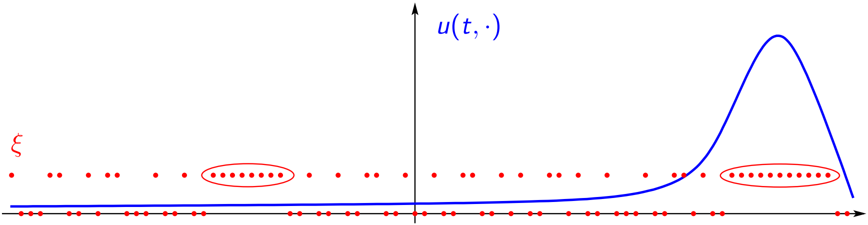

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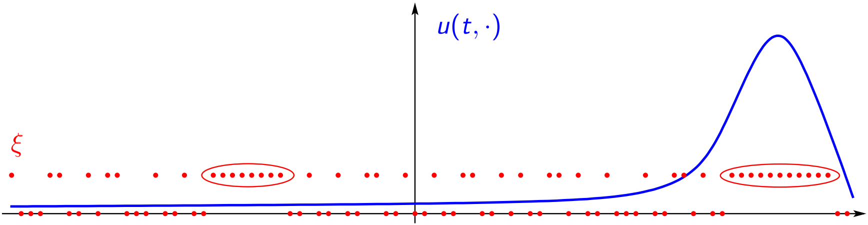

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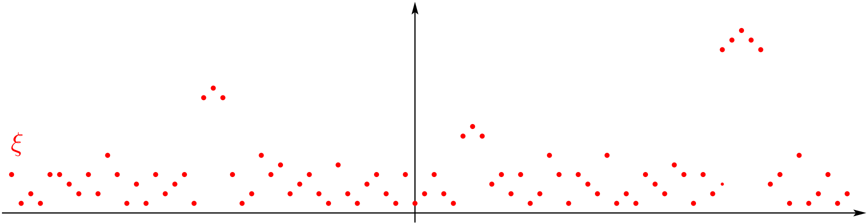

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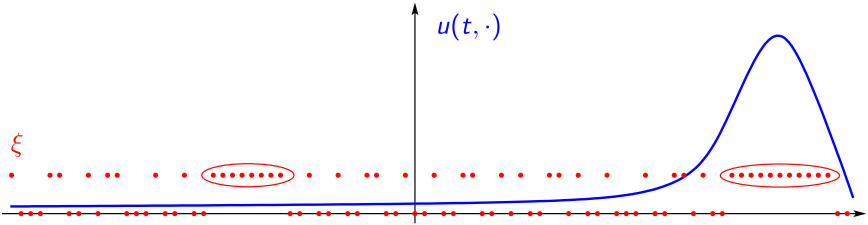

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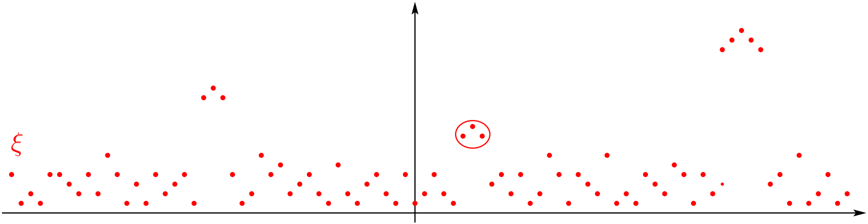

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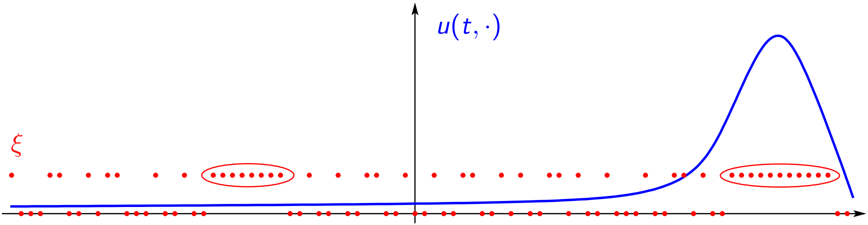

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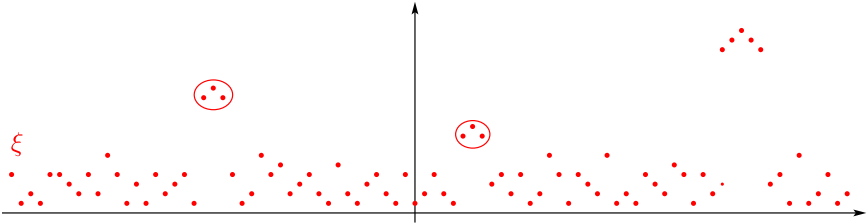

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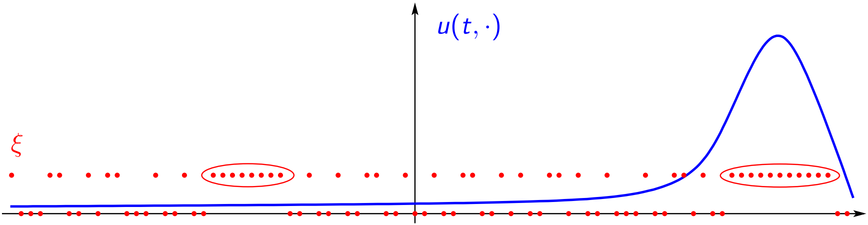

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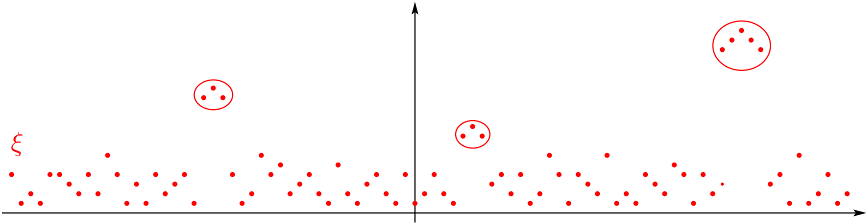

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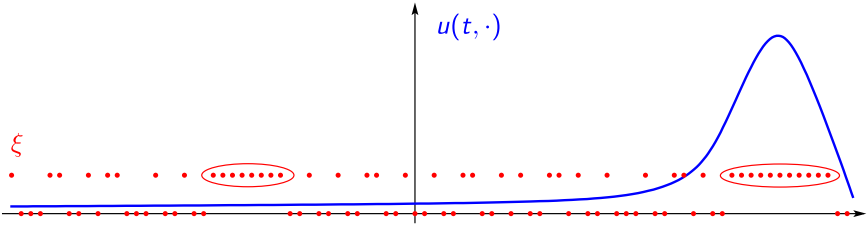

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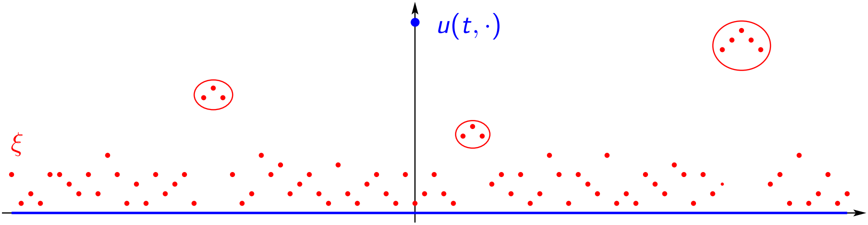

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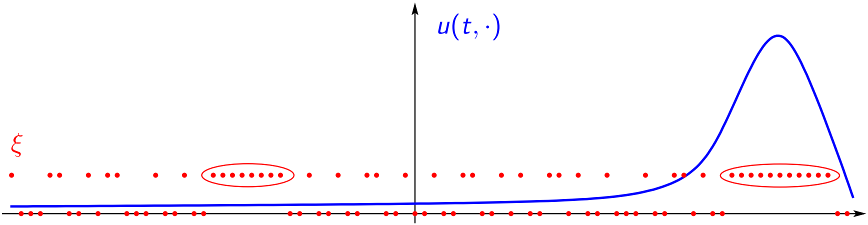

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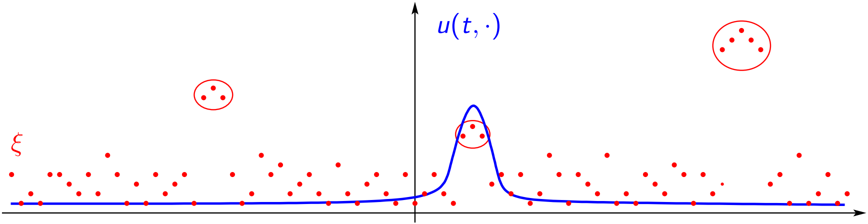

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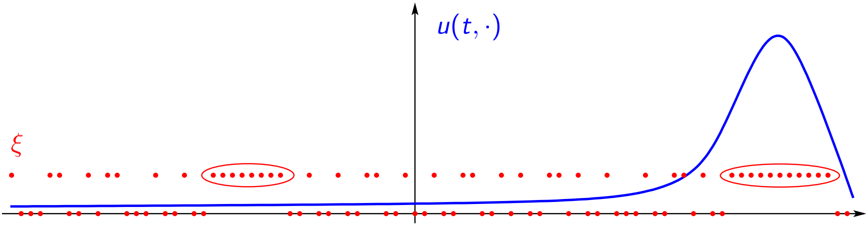

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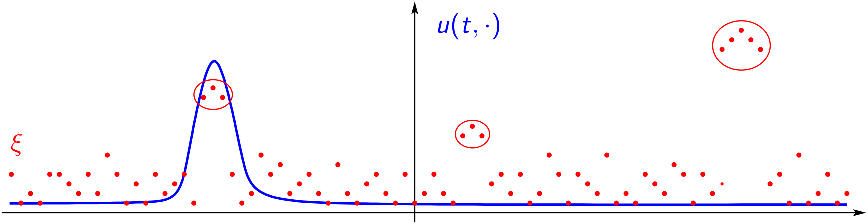

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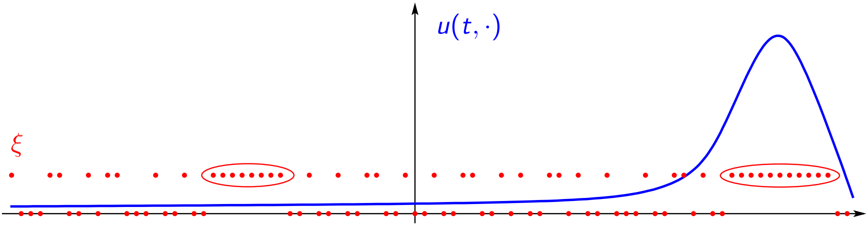

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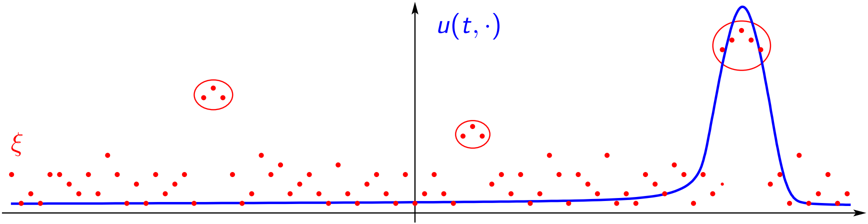

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What is known for unbounded potentials?

- Pareto: $P(\xi(0) > x) = x^{-\alpha}$, $\alpha > d$
- Weibull: $P(\xi(0) > x) = \exp\{-x^{\gamma}\}$, $\gamma > 0$
- Double-exponential: $P(\xi(0) > x) = \exp\{-e^{x/\rho}\}$, $\rho > 0$
- 'Almost bounded' — quite different, not in this talk

What is known

[König, Mörters, S. '06] Pareto
[S., Twarowski '12] Weibull with $\gamma < 2$
[Fiodorov, Muirhead '13] Weibull with any γ

[Biskup, König, dos Santos, '16] Double-exponential potentials

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Main question of this talk: How can we break this and make the solution spread between two (or more) independent locations?

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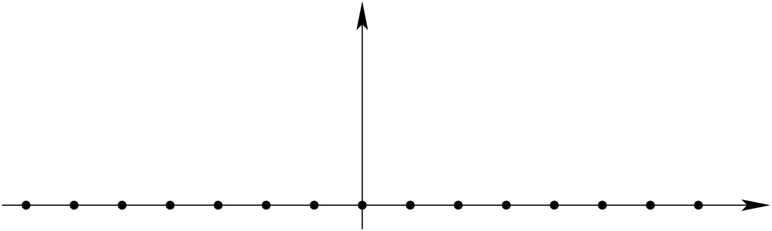

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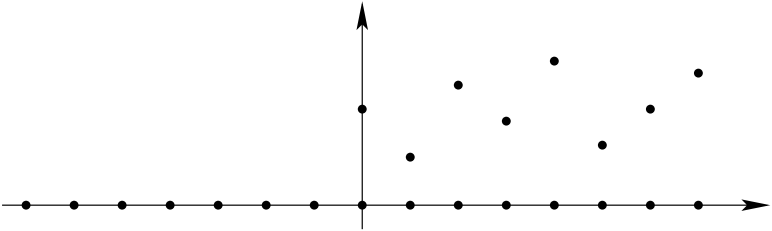

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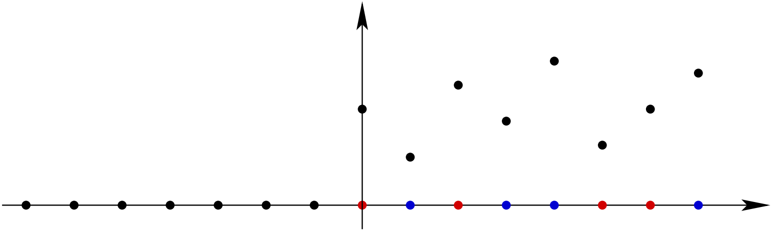

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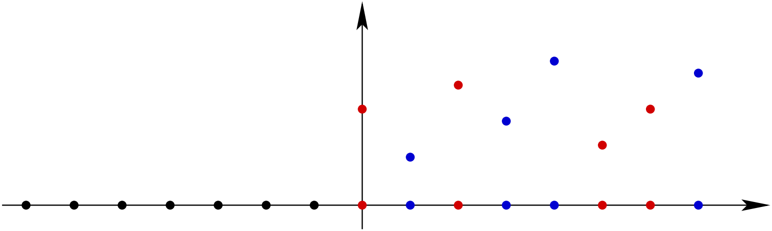

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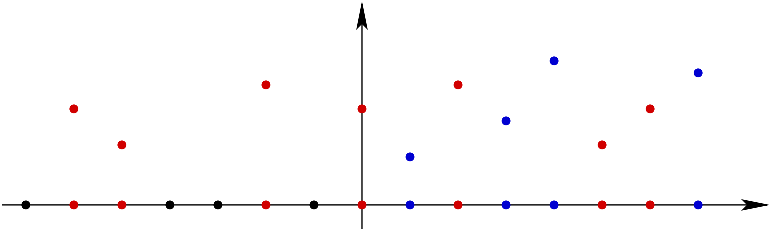

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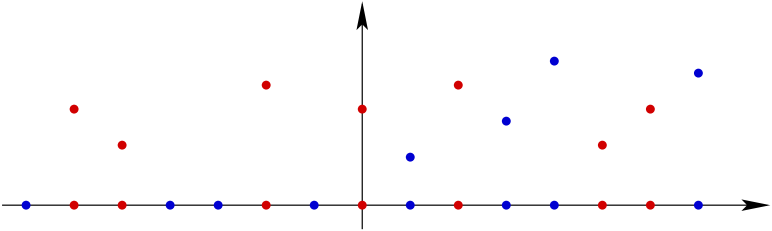

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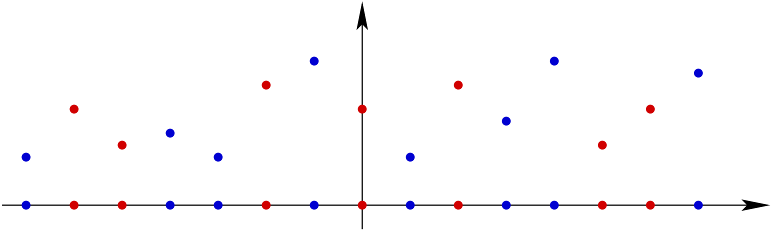

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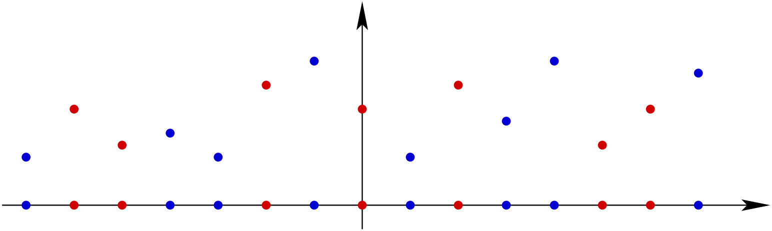

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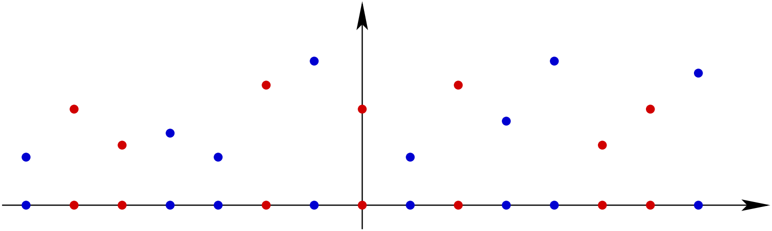

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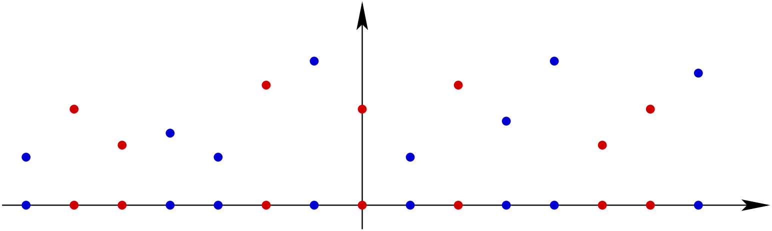

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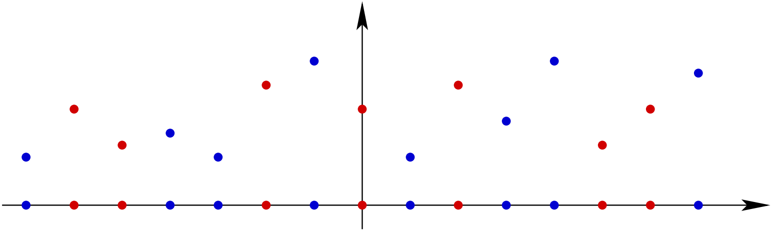

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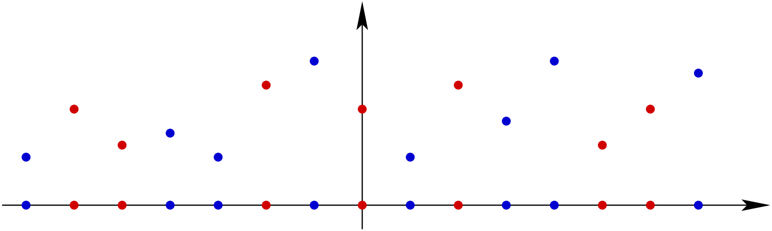

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- If so, what is the proportion of the mass carries by each point?

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For $t > 0$ and $z \in \mathbb{Z}$, let

$$\Psi_t(z) = \xi(z) - \frac{|z|}{t} \log \xi(z).$$

Very roughly,

$$u(t,z) \approx e^{t\Psi_t(z)}.$$

Let Z_t be a maximiser of Ψ_t. Denote

$$\mathcal{D}_t = \{\text{duplication at } Z_t\}.$$

- The standard PAM would localise at Z_t.
- On the event $\mathcal{D}_t$ the points Z_t and $-Z_t$ are equally good in terms of the value of ξ and distance from the origin.
- Will the PAM with duplication localise at both points on the event $\mathcal{D}_t$?
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Denote the total mass of the solution by

$$U(t) = \sum_{z \in \mathbb{Z}} u(t,z).$$

Answers

Theorem 1 (Muirhead, Pymar, S. '16)

Let $1 < \alpha < 2$.

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Theorem 2 (Muirhead, Pymar, S. '16) — still localised

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Increasing duplication for $\alpha \geq 2$

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where $\mathcal{P}_t^{\pm}$ are the sets of paths on $\mathbb{Z}$ starting at 0 and ending at $\pm Z_t$.

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$$u(t, \pm Z_t) = \sum_{y \in \mathcal{P}_t^{\pm}} U(t, y)$$

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$$U(t, y) = \sum_{i=0}^{n} e^{tc_i - 2t} \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{1}{c_i - c_k}$$

where $c_0 \;\; \ldots\ldots \;\; c_n$ are the values of ξ along y

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In particular, for $1 < \alpha < 2$ we have

$$u(t, \pm Z_t) \sim e^{t\xi(Z_t) - 2t} \prod_{k=0}^{Z_t} \frac{1}{\xi(Z_t) - \xi(\pm k)}$$

$1 < \alpha < 2$

$$\frac{u(t, Z_t)}{u(t, -Z_t)} \sim \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t) - \xi(k)} : \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t) - \xi(-k)}$$

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$$\uparrow \text{ normal} \qquad \uparrow\ \infty \text{ for Pareto}(\alpha)$$

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$$= \exp\Big\{ -\sum_{k:\text{non-dupl}} \log\Big(1-\frac{\xi(k)}{\xi(Z_t)}\Big) + \sum_{k:\text{non-dupl}} \log\Big(1-\frac{\xi(-k)}{\xi(Z_t)}\Big)\Big\}$$

Insightful cheating:

$$-\sum_{k:\text{non-dupl}} \log\Big(1-\frac{\xi(\pm k)}{\xi(Z_t)}\Big) \approx \frac{1}{\xi(Z_t)} \sum_{k:\text{non-dupl}} \xi(\pm k) \approx$$

$1 < \alpha < 2$

$$\sum_{i=1}^{n} X_i \approx n\mu + n^{1/2}\mathcal{N}$$

$\uparrow$ normal Pareto(α)

$$\frac{u(t,Z_t)}{u(t,-Z_t)} \sim \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(k)} : \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(-k)}$$

$$= \exp\Big\{ - \sum_{k:\text{non-dupl}} \log\Big(1-\frac{\xi(k)}{\xi(Z_t)}\Big) + \sum_{k:\text{non-dupl}} \log\Big(1-\frac{\xi(-k)}{\xi(Z_t)}\Big)\Big\}$$

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$\uparrow$ stable

Pareto(α)

$$\frac{u(t,Z_t)}{u(t,-Z_t)} \sim \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(k)} : \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(-k)}$$

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same for $\pm Z_t$

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same for $\pm Z_t$

Hence everything is determined by

$$\frac{|Z_t|^{1/\alpha}}{\xi(Z_t)} \asymp$$

$1 < \alpha < 2$

$$\sum_{i=1}^{n} X_i \approx n\mu + n^{1/\alpha}\mathcal{N}$$

↑ stable   Pareto(α)

$$\frac{u(t,Z_t)}{u(t,-Z_t)} \sim \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(k)} : \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(-k)}$$

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Hence everything is determined by

$$\frac{|Z_t|^{1/\alpha}}{\xi(Z_t)} \asymp \begin{cases} \infty & \Rightarrow \text{one point dominates} \\ 1 & \Rightarrow \text{random proportion of mass at each point} \\ 0 & \Rightarrow 1/2 \text{ of the mass at each point} \end{cases}$$

$1 < \alpha < 2$

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$\uparrow$ stable Pareto(α)

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Hence everything is determined by

$$\frac{|Z_t|^{1/\alpha}}{\xi(Z_t)} \asymp \frac{r_t^{1/\alpha}}{a_t} \asymp \begin{cases} \infty & \Rightarrow \text{one point dominates} \\ 1 & \Rightarrow \text{random proportion of mass at each point} \\ 0 & \Rightarrow 1/2 \text{ of the mass at each point} \end{cases}$$

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Miracle! $a_t = r_t^{1/\alpha}$

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LLN: same for $\pm Z_t$

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Miracle! $a_t = r_t^{1/\alpha}$

The scale of fluctuations remains finite for all values of $1 < \alpha < 2$.

$1 < \alpha < 2$

$$\sum_{i=1}^{n} X_i \approx n\mu + n^{1/\alpha}\mathcal{N}$$

$\uparrow$ stable — Pareto(α)

$$\frac{u(t,Z_t)}{u(t,-Z_t)} \sim \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(k)} : \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(-k)}$$

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