

Asymptotics of killed one-dimensional diffusions, with applications to biodemography

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Stationary Distributions (finite state space) Finite-state Markov processes converge to

stationary distributions.

Generator Q: Q_{ij} =rate of jumping to j from i. For positive t, $\mathbb{P}^i \{X_t = j\} = (e^{tQ})_{ij}$.

By Perron-Frobenius, top eigenvalue (which is 0) is simple, with a positive left-eigenvector π . Then

$$\mathbb{P}^i \{ X_t = j \} \xrightarrow{t \to \infty} \pi_j$$

Example: $Q_{12}=1$, $Q_{21}=2$.

 $Q = \begin{pmatrix} -1 & 1\\ 2 & -2 \end{pmatrix}$

Converges to 2/3 state 1, 1/3 state 2.

Quasistationary Distributions

Finite-state sub-Markov processes conditioned on long-time survival converge to quasistationary distributions. Generator Q: Q_{ii}=rate of jumping to j from i.

For positive t, $\mathbb{P}^i \{X_t = j\} = (e^{tQ})_{ij}$. By Perron-Frobenius, top eigenvalue $-\lambda$ (which is negative) is simple, with a positive left-eigenvector π , and positive right-eigenvector v. Then

$$e^{\lambda t} P^i \{ X_t = j \} \xrightarrow{t \to \infty} v_i \pi_j.$$

Conditional Convergence

Let τ_{∂} be the time when the process is killed. Then

$$P^i \{ X_t = j \mid \tau_\partial > t \} \xrightarrow{t \to \infty} \pi_j.$$

 $P^{i}\left\{\tau_{\partial} > t + s \mid \tau_{\partial} > t\right\} \xrightarrow{t \to \infty} e^{-\lambda s}.$

Example redux: State 1 healthy, state 2 death rate 1.

$$Q = \begin{pmatrix} -1 & 1\\ 2 & -3 \end{pmatrix}$$

 $e^{tQ} = \begin{pmatrix} 0.789e^{-.268t} + 0.21e^{-3.73t} & 0.289e^{-.268t} - 0.289e^{-3.73t} \\ 0.577e^{-.268t} - 0.577e^{-3.73t} & 0.211e^{-.268t} + 0.789e^{-3.73t} \end{pmatrix}$

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 $\lim_{t \to \infty} e^{.268t} e^{tQ} = \begin{pmatrix} 0.789 & 0.289 \\ 0.577 & 0.211 \end{pmatrix}$ $\lim_{t \to \infty} P\{\text{sick} \mid \text{survive to time } t\}$ $= \frac{.289}{.289 + .789} = \frac{.211}{.211 + .577} = 0.268.$

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Adjoint Generator

$$\mathcal{L}^* f = -\frac{1}{2}f'' + (b \cdot f)' + \kappa \cdot f$$

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Adjoint Generator

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If *f* is the starting density, the density of X_t is $e^{-t\mathcal{L}^*}f$.

Crucial fact: The operator can be made self-adjoint.

Γ is a measure with density

 $|\gamma(x) = e^{2\int_0^x b(s)ds}|$

In the space $L^2((0,\infty),\Gamma)$ the generator is self-adjoint. That is, with

$$\langle f,g \rangle = \int_0^\infty f(x)g(x)\gamma(x)dx$$

 $\langle \mathcal{L}f,g\rangle = \langle f,\mathcal{L}g\rangle$

Goal: General conditions for convergence to a quasistationary distribution with density ϕ , the top eigenfunction of the adjoint?

 $\lim_{t \to \infty} \mathbb{P}\{X_t \in A \mid \tau_\partial > t\} = \int_A \phi(x) \,\mathrm{d}\Gamma(x)$

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"Yaglom convergence" from Yaglom 1947 on limits of branching processes conditioned on survival. Goal: General conditions for convergence to a quasistationary distribution with density ϕ , the top eigenfunction of the adjoint?

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"Yaglom convergence" from Yaglom 1947 on limits of branching processes conditioned on survival. Also "asymptotic killing rate": $\lim_{t\to\infty} P\{\tau_{\partial} > t + s \mid \tau_{\partial} > t\} = e^{-\lambda s},$ where - λ is the top eigenvalue.

Main Results (S.-Evans 2006; Kolb - S. 2012) **Dichotomy:** On $(0,\infty)$, with ∞ inaccessible, if $\lim_{z \to \infty} \kappa(z) \neq \underline{\lambda} \quad \text{then either}$ $(1) \int_{0}^{\infty} \phi_{\underline{\lambda}}(x) \gamma(x) dx < \infty \text{ and}$ $\lim_{t \to \infty} \mathbb{P} \{ X_t \in A \mid \tau_\partial > t \} = \frac{\int_A \phi_{\underline{\lambda}}(x) \gamma(x) dx}{\int_0^\infty \phi_{\lambda}(x) \gamma(x) dx}$ (2) $\int_{0}^{\infty} \phi_{\underline{\lambda}}(x)\gamma(x)dx = \infty \text{ and } \lim_{t \to \infty} \mathbb{P}\{X_t \in A \mid \tau_{\partial} > t\} = 0$ for bounded A.

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Main Results

Yaglom convergence with high killing at infinity: On (0, ∞), with ∞ inaccessible, if $\liminf_{z \to \infty} \kappa(z) > \underline{\lambda}$

 $\int_{0}^{\infty} \phi_{\underline{\lambda}}(x)\gamma(x)dx < \infty \text{ and}$ $\lim_{t \to \infty} \mathbb{P}\{X_t \in A \mid \tau_{\partial} > t\} = \frac{\int_A \phi_{\underline{\lambda}}(x)\gamma(x)dx}{\int_0^{\infty} \phi_{\underline{\lambda}}(x)\gamma(x)dx}$

In this case $\underline{\lambda}$ is an isolated eigenvalue, so there is a spectral gap.

Main Results

Yaglom convergence with low killing at infinity: On $(0,\infty)$, with ∞ inaccessible, assume

 $K := \lim_{z \to \infty} \kappa(z) < \underline{\lambda}$ $\int_{0}^{\infty} \phi_{\underline{\lambda}}(x) \gamma(x) dx < \infty \text{ and}$

 $\lim_{t \to \infty} \mathbb{P} \{ X_t \in A \mid \tau_{\partial} > t \} = \frac{\int_A \phi_{\underline{\lambda}}(x) \gamma(x) dx}{\int_0^\infty \phi_{\underline{\lambda}}(x) \gamma(x) dx}$ if and only if $\int_0^\infty \gamma(x)^{-1} dx = \infty.$

(This is equivalent to saying the unkilled process is recurrent.) In this case $\underline{\lambda}$ is in the essential spectrum, and there is no spectral gap

Other results

For diffusions, only the case κ=0 with killing at 0 had been considered. (Mandl 1961; Collett, Martínez, San Martín 1994; Champagnat, Villemonais 2016)

In this case, escape to infinity can be blocked only by inward drift.

Champagnat, Villemonais 2017 showed Yaglom convergence for a class of multidimensional Lotka—Volterra models.

Other results

 Seneta & Vere-Jones (1966), van Doorn (1991): Partial results for birth-death chains killed only at 0.

Gosselin (AoP 2001): Discrete chains with general killing. Lyapunov-like condition for quasicompactness.



Quasistationary convergence.

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Borel functional calculus $f(\mathcal{L})u = \int_{\Sigma(\mathcal{L})} f(\lambda) dE(\lambda)u$ for continuous *f*.

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 J_0

Weyl's eigenfunction expansion

$$e^{-t\mathcal{L}}u(x) = \int_{\underline{\lambda}}^{\infty} \langle u, \phi_{\lambda} \rangle \phi_{\lambda}(x) \, d\rho(\lambda)$$

 $= \int_{\underline{\lambda}}^{\infty} e^{-t\lambda} \phi_{\lambda}(x) \int_{\underline{\lambda}}^{\infty} u(y) \phi_{\lambda}(y) \gamma(y) \, dy \, d\rho(\lambda)$

 $J\lambda$

Finite state space:

$$Q = \sum_{i=1}^{n} -\lambda_i u_i v$$
$$^{tQ} = \sum_{i=1}^{n} e^{-t\lambda_i} u_i v_i^*$$

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 $f(\mathcal{L})u = \int_{\Sigma(\mathcal{L})} f(\lambda) dE(\lambda)u$ for continuous f.

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Note: The ϕ_{λ} here are solutions to the appropriate ODE, but may not be eigenfunctions; but ρ is supported on the spectrum.

 $=\int_{\lambda}^{\infty} e^{-t\lambda} \phi_{\lambda}(x) \int_{0}^{\infty} u(y) \phi_{\lambda}(y) \gamma(y) \,\mathrm{d}y \,\mathrm{d}\rho(\lambda)$

If u is an L² function and f is an L² density, $\mathbb{E}_f \left[u(X_t) \right] = \langle f, e^{-t\mathcal{L}}u \rangle = \int_{\underline{\lambda}}^{\infty} e^{-t\lambda} \langle f, \phi_{\lambda} \rangle \langle u, \phi_{\lambda} \rangle \, \mathrm{d}\rho(\lambda)$

In particular, we have Yaglom convergence on compacta: If $A \subset B$ are bounded measurable sets,

 $\lim_{t \to \infty} \mathbb{P}\{X_t \in A \,|\, X_t \in B, \, \tau > t\} = \frac{\int_A \phi_{\underline{\lambda}}(x) \gamma(x) dx}{\int_B \phi_{\underline{\lambda}}(x) \gamma(x) dx}$

We also have the strong ratio theorem

 $\lim_{s \to \infty} \frac{p(t+s, x, y)}{p(s, a, a)} = e^{-\underline{\lambda}t} \frac{\phi_{\underline{\lambda}}(x)\phi_{\underline{\lambda}}(y)}{\phi_{\underline{\lambda}}(a)\phi_{\underline{\lambda}}(a)},$

where p is the transition kernel with respect to Γ .

Key analytic fact (Simon 1993): If $\underline{\lambda}$ is an eigenvalue (i.e., if $\Phi_{\underline{\lambda}}$ is L²) then $\lim_{t \to \infty} e^{\underline{\lambda}t} p(t, x, y) = c \phi_{\underline{\lambda}}(x) \phi_{\underline{\lambda}}(y).$

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Easy consequence: If the top eigenfunction is L^2 and L^1 then we have Yaglom convergence. When $\underline{\lambda}$ <K it is an isolated eigenvalue, so L^2 . Probabilistic proof required to show L^1 .

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When K< $\underline{\lambda}$ and $\int_0^{\infty} \gamma(x)^{-1} dx = \infty$ then we show that the asymptotic killing rate is $\underline{\lambda}$, implying the process can't escape to infinity.

Application: Mortality plateaux

Germany Mortality Rates (1990s)

(from Human Mortality Database http://www.mortality.org)



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Gompertz Mortality









Plateaux and beyond...



Figure 2.24 Mortality rates of a population of 1.2 million medflies maintained in cages of 7,200 animals each. Note that the age-specific mortality rates initially rose exponentially with age but then leveled off at about 20 days of age (16 percent survival), slowly increased to a peak at 58 days of age (0.2 percent survival), and declined thereafter. (Redrawn from data in Carey et al. 1992.)

(Carey et al., Science 1992)

Traditional explanations for plateaus:

- population heterogeneity: Population composed of distinct frailties. Flattening from selection.
- temporal heterogeneity: Whatever drives the physical aging process slows down at advanced ages. Example: Cancer growth.

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"Cascading failures" model

 $_{\odot}$ Start at senescence state X₀=1. $_{\odot}$ Rate of jumping to next higher state λX_{+} . \odot Rate of killing μX_{t} . Le Bras (1976) pointed out that when $\lambda >> \mu$, the mortality rate is about $\mu e^{\lambda t}$ for small t. • Not surprising: Acts like $dX_{+}/dt = \lambda X_{+}$.

Weitz-Fraser Diffusion Model

- "Vitality" at age t is a Brownian motion (continuous random walk) with constant downward drift.
- "Death" is the time when vitality reaches 0.
- Time of death has "inverse Gaussian" distribution.

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Example -- Continuous version of the Le Bras model: $\sigma(x)=\sigma x$, b(x)=bx, $\kappa(x)=\kappa x$.

Reflected at 1.

Example -- Weitz-Fraser model:

σ(x)=1, b(x)=b, κ(x)=0. Killed at 0.

Example: Continuous version of the Le Bras model

 $\sigma(x)=\sigma x,$ b(x)=bx, $\kappa(x)=\kappa x.$ Reflected at 1.

 $\kappa(x) \rightarrow \infty$, so convergence to quasistationary distribution is immediate from Result 2.

(In this case, eigenfunctions are computable.)

What does this mean?

Traditional explanations:

population heterogeneity

temporal heterogeneity

New idea "Evolving heterogeneity": Late-life mortality flattens out because of an equilibrium between the drift toward lower vitality, and the pruning of mortality

Explanatory link between abstract models and the real phenomenon. Possibly testable.



Le Bras evolution

Reaches equilibrium between spread and killing. Application: Superprocesses with "damage accumulation"



Diagrams from Stewart, Madden, Paul, Taddei (2005).

Senescence in E. coli



Stewart, Madden, Paul, Taddei (PLoS 2005)





How do we model the aging that comes from accumulated damage in protozoans?

How do we estimate the advantage that comes from segregating damage?

Can the organism gain a selective advantage by accumulating more damage, but segregating more often?

Branching diffusion model

- Protozoans may increase growth rate by segregating damage.
- State of system is a random measure on R+.
- Position=amount of damage.
- Each individual performs independent diffusion. Dies at rate increasing in damage.
 Splits at rate decreasing in damage.
- At split, two daughters jump to ±random variable. That is, parent's damage is divided randomly between the daughters

Superprocess model

Superprocess=measure-valued diffusion.

- State measures quantity of damage. At a split, daughters jump up and down.
- Semigroup of nonlinear differential operators.
- Rescaling limit behavior determined by

spectrum of $\mathcal{L}^*\phi = -\frac{1}{2}\sigma^2\phi'' + (b\phi)' + (\kappa - \beta)\phi.$ Here κ =killing rate, β =splitting rate, b and σ depend on the underlying motion, and on the jump process. In particular, the jump at fission increases σ .

$P_t = e^{-t\mathcal{L}^*}$ determines the mean. More generally,

 $\overline{\mathbf{E}}_{\nu} \left[\langle \varphi, X_t \rangle \right] = \langle P_t \varphi, \nu \rangle,$ $\overline{\mathbf{E}}_{\nu} \left[\langle \varphi, X_t \rangle \langle \varphi', X_{t'} \rangle \right] = \langle P_t \varphi, \nu \rangle \langle P_{t'} \varphi', \nu \rangle$

 $+ \left\langle \int_0^t P_s \left[\rho \cdot P_{t-s} \varphi \cdot P_{t'-s} \varphi' \right] ds, \nu \right\rangle.$

Theorem (Evans & S. 2006) - If $-\underline{\lambda}>0$ — that is, asymptotic growth rate is positive — and $\beta-\kappa$ is nonincreasing, then the rescaled random measure $\widetilde{X}_t := \langle P_t \mathbf{1}, \nu \rangle^{-1} X_t$ converges in L² to a random multiple of the measure with density $\phi_{\underline{\lambda}}$.

The long-term growth rate of the total mass is $-\underline{\lambda}$.

What does this say about the advantage of damage segregation?

Theorem: If drift b is bounded below, away from 0, then the maximum $-\underline{\lambda}$ is attained for a nonzero finite value of σ . In particular, if inherent damage accumulation is deterministic, then unequal segregation of damage increases the long-term growth rate.

MCMC application

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Long answer: Andi Wang's talk Thursday.