Optimal Stopping, Smooth Pasting and the Dual Problem

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The general optimal stopping problem:

Given a filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$ and an adapted gains process G find

$$S_t \stackrel{def}{=} \text{ess sup}_{\text{optional } \tau \geq t} \mathbb{E}[G_\tau | \mathcal{F}_t]$$

Recall, under very general conditions

- S is the minimal supermartingale dominating G
- $\tau_t \stackrel{def}{=} \inf\{s \geq t : \ S_s = G_s\}$ is optimal
- for any t, S is a martingale on $[t, \tau_t]$
- when

$$G_t = g(X_t) \tag{1}$$

  for some (continuous-time) Markov Process X, S_t can be written as a function, $v(X_t)$.

Remark

1.1 Condition $G_t = g(X_t)$ is less restrictive than might appear. With θ being the usual shift operator, can expand statespace of X by appending adapted functionals F with the property that

$$F_{t+s} = f(F_s, (\theta_s \circ X_u;\ 0 \leq u \leq t)). \tag{2}$$

The resulting process $Y \stackrel{def}{=} (X, F)$ is still Markovian. If X is strong Markov and F is right-cts then Y is strong Markov.

e.g if X is a BM,

$$Y_t = (X_t, L_t^0, \sup_{0 \leq s \leq t} X_s, \int_0^t \exp(-\int_0^s \alpha(X_u)du)g(X_s)ds$$

is a Feller process on the filtration of X.

1. *When is v is in the domain of the generator, $\mathcal{L}$, of X?* (Surprisingly, unable to find any general results about this.)
2. Recall that the dual problem is to find

$$V = \inf_{M \in \mathcal{M}_0} \mathbb{E}[\sup_t (G_t - M_t)],$$

where $\mathcal{M}_0$ is the collection of uniformly integrable martingales started at 0.
Is the dual of the Markovian problem a controlled Markov Process problem?

3. The smooth pasting principle is used to find explicit solutions to optimal stopping problems essentially by "pasting together a martingale (on the continuation region) and the gains process (on the stopping region)"
Can we say anything about smooth pasting?

Define $\mathbb{G} = \{\text{semimartingales such that } \mathbb{E}\Big[\sup_{0 \le t \le T} |G_t|\Big] < \infty\}$.

Theorem

If $G \in \mathbb{G}$ then

- *the Snell envelope S of G, admits a right-continuous modification and is the minimal supermartingale that dominates G.*
- *both G and S are class (D).*
- *G and S admit unique decompositions*

$$G = N + D, \qquad S = M - A \tag{3}$$

where $N \in \mathcal{M}_{0,loc}$ and D is a predictable finite-variation process, $M \in \mathcal{M}_0$, and A is a predictable, increasing process of integrable *variation (in IV).*

Remark

It is more normal to assume that the process A in the Doob-Meyer decomposition of S is started at zero. The dual problem is one reason why we do not do so here.

Recall that

$$H^1 = \{\text{special semimartingales } N+D \text{ where } \sup_t |N_t| + \int_0^\infty |dD_t| \in L^1\}.$$

The main assumption in this section is the following:

Assumption

3.1 G is in $\mathbb{G}$ and in H^1_{loc}.

Under Assumption 3.1, the previous theorem's conclusions hold and, in the decomposition $G = N + D$, D is a predictable IV_{loc} process.

We finally arrive to the main result:

Theorem

3.2 Suppose Assumption 3.1 holds. Let D^- (D^+) denote the decreasing (increasing) components of D. Then $A << D^-$, and μ, defined by

$$\mu_t := \frac{dA_t}{dD_t^-}, \quad 0 \leq t \leq T,$$

satisfies $0 \leq \mu_t \leq 1$.

Remark

As is usual in semimartingale calculus, we treat a process of bounded variation and its corresponding Lebesgue-Stiltjes signed measure as synonymous.

Proof First localise G and S so they are both in H^1. Recall the characterisation of a predictable IV process V: we have:

$$V_t - V_s = \lim_{\delta \downarrow 0} \sum_{i=0}^{\lfloor (t-s)/\delta \rfloor} \mathbb{E}[V_{s+(i+1)\delta} - V_{s+i\delta} | \mathcal{F}_{s+i\delta}], \qquad (4)$$

with limit being in L^1 (taking a subsequence if necessary).

Now, set

$$\Delta \overset{def}{=} \mathbb{E}[A_v - A_u|\mathcal{F}_u] = \mathbb{E}[S_u - S_v|\mathcal{F}_u]$$
$$= \mathbb{E}\left[\mathbb{E}[G_{\tau_u}|\mathcal{F}_u] - \text{ess sup}_{\sigma \geq v}\mathbb{E}[G_\sigma|\mathcal{F}_v]\middle|\mathcal{F}_u\right] \tag{5}$$

Taking $\sigma = \tau_u \vee v$ in (5), we obtain

$$\Delta \leq \mathbb{E}[G_{\tau_u} - G_{\tau_u \vee v}|\mathcal{F}_u] = \mathbb{E}[D_{\tau_u} - D_{\tau_u \vee v}|\mathcal{F}_u]$$
$$= \mathbb{E}[(D^+_{\tau_u} - D^+_{\tau_u \vee v}) + D^-_{\tau_u \vee v} - D^-_{\tau_u}|\mathcal{F}_u] \leq \mathbb{E}[D^-_{\tau_u \vee v} - D^-_{\tau_u}|\mathcal{F}_u]$$
$$\leq \mathbb{E}[D^-_v - D^-_u|\mathcal{F}_u]. \tag{6}$$

The last inequalities following since: D^+ and D^- are increasing; $\tau_u \geq u$; and, on the event that $\tau_u \geq v$, the term inside the penultimate expectation vanishes. Applying (4) to inequality (6) we get that $0 \leq A_t - A_s \leq D^-_t - D^-_s$ for all $s \leq t$, giving the result $\square$

Assumption

3.4 X is a strong Markov process with quasi continuous filtration.

Remark

Note that if X satisfies the asumption then expanding the state by a right-continuous functional F of the form in Remark 1.1, (X, F) also satisfies Assumption 3.4. If X is Feller then it satisfies the assumption.

Finally,

Assumption

3.6 $\sup_t |g(X_t)| \in L^1$ *and* $g \in \mathbb{D}(\mathcal{L})$*, i.e.*

$$g(X_t) = g(x) + M_t^g + \int_0^t \mathcal{L}g(X_s)ds, \quad 0 \leq t \leq T, x \in E, \quad (7)$$

so that G is a semimartingale and the FV process in the semimartingale decomposition of $G = g(X)$ is absolutely continuous with respect to Lebesgue measure, and therefore predictable. Moreover, we deduce that $g(X)$ satisfies Assumption 3.1.

The result of this section is the following:

Theorem

Suppose X and g satisfy Assumptions 3.4 and 3.6, then $v \in \mathbb{D}(\mathcal{L})$.

Proof Since $D_t := g(X_0) + \int_0^t \mathcal{L}g(X_s)ds, \quad 0 \le t \le T$, (ignoring initial values) D^+ and D^- are explicitly given by

$$D_t^+ := \int_0^t \mathcal{L}g(X_s)^+ ds,$$
$$D_t^- := \int_0^t \mathcal{L}g(X_s)^- ds,$$

so D^- is absolutely continuous with respect to Lebesgue measure.

Applying Theorem 3.2,we conclude that

$$v(X_t) = v(x) + M_t - \int_0^t \mu_s \mathcal{L}g(X_s)^- ds, \quad 0 \le t \le T, \qquad (8)$$

where μ is a non-negative Radon-Nikodym derivative with $0 \le \mu_s \le 1$.

Setting $\lambda_t = \mu_t \mathcal{L}g(X_t)^-$, all that remains is to show that λ_t is $\sigma(X_t)$-measurable (since then there exists $\beta : E \to \mathbb{R}_+$, such that $\lambda = \beta(X)$).

This is fairly elementary (by the Markov property and quasi-continuity of the filtration) and thus $v \in \mathbb{D}(\mathcal{L})$.

$\square$

Recall that the dual problem is to find

$$V = \inf_{M \in \mathcal{M}_0} \mathbb{E}[\sup_t (G_t - M_t)],$$

and we know that the optimal M is the martingale appearing in the decomposition of V. Since $v \in \mathbb{D}(\mathcal{L})$, this is $M_t^v \stackrel{def}{=} v(X_t) - v(X_0) - \int_0^t \mathcal{L}v(X_s)ds$. It follows that the dual problem is

$$V(x) = \inf_{h \in \mathbb{D}(\mathcal{L})} \mathbb{E}_x[\sup_t (g(X_t) - h(X_t) - \int_0^t \mathcal{L}h(X_s)ds)]$$

and a little thought shows that this is a controlled Markov process problem, with controlled MP Y^h given by $Y^h = (X, F^h)$ where

$$F_t^h = (\int_0^t \mathcal{L}h(X_s)ds, \sup_{s \le t}(g(X_s) - h(X_s) - \int_0^s \mathcal{L}h(X_u)du)).$$

We assume that X is a one-dimensional regular diffusion on E, a possibly infinite interval.. Let $s(\cdot)$ denote a scale function of X.

Theorem

Suppose Assumption 3.6 holds, then $v \in \mathbb{D}(\mathcal{L})$*. Let* $Y = s(X)$*. If* $s \in \mathcal{C}^1$ *and* $< Y, Y >_t$ *is absolutely continuous with respect to Lebesgue measure, then* $v(\cdot)$ *is* $\mathcal{C}^1$*.*

Proof Note that $Y = s(X)$ is a Markov process, and let $\mathcal{G}$ denote its martingale generator. Then $v(x) = W(s(x))$, where

$$W(y) = \sup_{\tau} \mathbb{E}_{s^{-1}(y)}[g \circ s^{-1}(Y_\tau)]. \tag{9}$$

Then, since $v \in \mathbb{D}(\mathcal{L})$,

$$v(X_t) = v(x) + M_t^v + \int_0^t \mathcal{L}v(X_s)ds,$$

and thus

$$W(Y_t) = W(y) + M_t^v + \int_0^t (\mathcal{L}v) \circ s^{-1}(Y_s)ds, \quad 0 \le t \le T.$$

Therefore, $W \in \mathbb{D}(\mathcal{G})$, i.e.

$$W(Y_t) = W(y) + M_t^W - \int_0^t \mathcal{G}W(Y_s)ds, \qquad (10)$$

with $\mathcal{G}W(\cdot) \geq 0$.

Y is a local martingale and so it's easy to show that $W(\cdot)$ is a concave function. Using the generalised Ito formula we have

$$W(Y_t) = W(y) + \int_0^t W_-^{'}(Y_s)dY_s + \int L_t^z \nu(dz), \qquad (11)$$

where L_t^z is the local time of Y at z, and ν is a non-negative, σ-finite measure corresponding to the derivative $W^{''}$ in the sense of distributions.

By the Lebesgue decomposition theorem, $\nu = \nu_c + \nu_s$, where ν_c and ν_s are measures, absolutely continuous and singular (with respect to Lebesgue measure), respectively. Denoting the Radon-Nykodym derivative of ν_c by ν_c', the occupation time formula gives

$$W(Y_t) - W(y) = \int_0^t W_-^{'}(Y_s)dY_s + \int L_t^z \nu_c'(z)dz + \int L_t^z \nu_s(dz)$$
$$= \int_0^t W_-^{'}(Y_s)dY_s + \int_0^t \nu_c'(Y_s)d < Y, Y >_s + \int L_t^z \nu_s(dz). \tag{12}$$

By hypothesis, the quadratic variation process $(< Y, Y >_t)_{t \geq 0}$ is absolutely continuous with respect to Lebesgue measure. Then, by comparing (12) with (10), we conclude that

$$\int L_t^z \nu_s(dz) = 0 \text{ for each } t. \tag{13}$$

Since Y is a semimartingale, $L^z_\cdot$ is carried by the set $\{t : Y_t = z\}$. We conclude that ν_s does not charge points, and therefore, left and right derivatives of $W(\cdot)$ must be equal. So $W \in \mathcal{C}^1$, and since $s \in \mathcal{C}^1$ by assumption and is strictly increasing, $v \in \mathcal{C}^1$. □

Example

Suppose X is an Itô diffusion, i.e. X is a diffusion with infinitesimal generator (on $\mathcal{C}^2$)

$$\mathcal{L} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \quad x \in E,$$

where $\sigma(\cdot)$ and $b(\cdot)$ are continuous functions and $\sigma(\cdot)$ does not vanish. Then the scale function $s \in \mathcal{C}^2$, and since $< X, X >$ is absolutely continuous with respect to Lebesgue measure, so is $< s(X), s(X) >$. It follows that if g is C^2 then v is C^1.

Technical results used are all in Kallenberg, Protter and Revuz & Yor.