Random Cluster Dynamics for the Ising model is Rapidly Mixing

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The Model

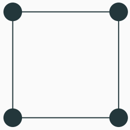
The random cluster model (Fortuin, Kasteleyn 69)

Parameters $0 \leqslant p \leqslant 1$ (edge weight), $q \geqslant 0$ (cluster weight).

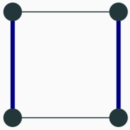
Given graph $G = (V, E)$, the measure on subgraph $r \subseteq E$ is defined as

$$\pi_{RC}(r) \propto p^{|r|}(1-p)^{|E \setminus r|} q^{\kappa(r)},$$

where $\kappa(r)$ is the number of connected components in (V, r).



$(1-p)^4 q^4$



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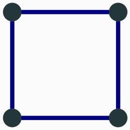

$p^4 q$

The random cluster model (Fortuin, Kasteleyn 69)

The partition function (normalizing factor):

$$Z_{RC}(p,q) = \sum_{r \subseteq E} p^{|r|}(1-p)^{|E \setminus r|} q^{\kappa(r)}.$$

Equivalent to the Tutte polynomial $Z_{Tutte}(x,y)$:

$$q = (x-1)(y-1) \qquad\qquad p = 1 - \frac{1}{y}$$

The random cluster model (Fortuin, Kasteleyn 69)

$$\pi_{RC}(r) \propto p^{|r|}(1-p)^{|E\setminus r|}q^{\kappa(r)}$$

The motivation is to unify:

- Ising model $q = 2$
- Potts model $q > 2$, integer
- Bond percolation $q = 1$ (On K_n, Erdős-Rényi random graph)
- Electrical network $q \to 0$ (Spanning trees if $p \to 0$ and $\frac{q}{p} \to 0$)

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Glauber dynamics

Glauber dynamics (single edge update) P_{RC} (Metropolis):

Current state $x \subseteq E$

1. With prob. 1/2 do nothing. (Lazy)
2. Otherwise, choose an edge e u.a.r.
3. Move to $y = x \oplus \{e\}$ with prob. $\min\left\{1, \frac{\pi_{RC}(y)}{\pi_{RC}(x)}\right\}$.

Detailed balance:

$$\pi(x)P(x,y) = \pi(y)P(y,x) = \min\{\pi(x), \pi(y)\}$$

Glauber dynamics

Glauber dynamics (single edge update) P_{RC} (Metropolis):

$$P_{RC}(x,y) = \begin{cases} \frac{1}{2m} \min\left\{1, \frac{\pi_{RC}(y)}{\pi_{RC}(x)}\right\} & \text{if } |x \oplus y| = 1; \\ 1 - \frac{1}{2m} \sum_{e \in E} \min\left\{1, \frac{\pi_{RC}(x \oplus \{e\})}{\pi_{RC}(x)}\right\} & \text{if } x = y; \\ 0 & \text{otherwise.} \end{cases}$$

We are interested in:

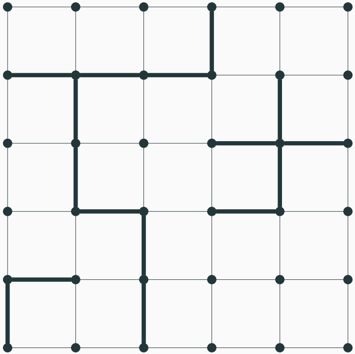
$$T_{mix}(P_{RC}) = \min\left\{t : \|P_{RC}^t(x_0, \cdot) - \pi\|_{TV} \leqslant \epsilon\right\},$$

$$T_{rel}(P_{RC}) = \frac{1}{1 - \lambda_2(P_{RC})}.$$

A simple example

Let $p < 1/2$.

$$\min\left\{1, \frac{\pi_{RC}(x \cup \{e\})}{\pi_{RC}(x)}\right\} = \begin{cases} \frac{p}{1-p} & \text{if } e \text{ is not a cut edge} \\ \frac{p}{q(1-p)} & \text{if } e \text{ is a cut edge} \end{cases}$$



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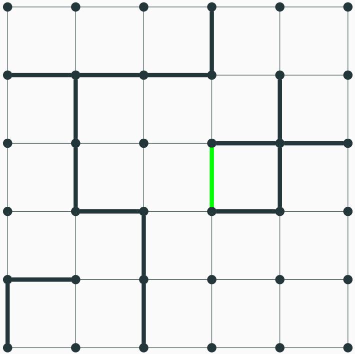

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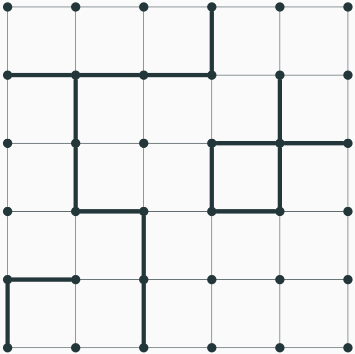

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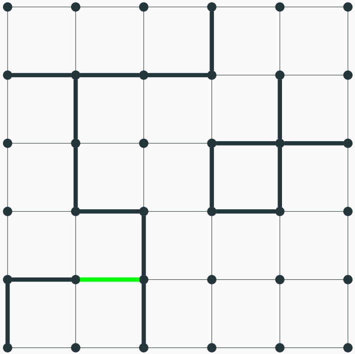

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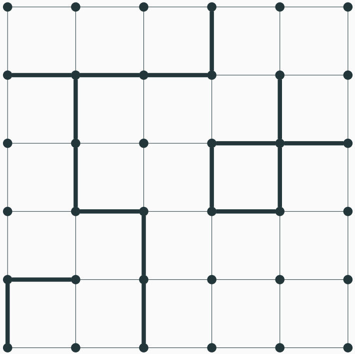

Previous results focus on special graphs.

- On the complete graph (mean-field):
  [Gore, Jerrum 99] [Blanca, Sinclair 15]
- On the 2D lattice $\mathbb{Z}^2$:
  [Borgs et al. 99] [Blanca, Sinclair 16] [Gheissari, Lubetzky 16]

$q > 2$: Slow mixing for the complete graph.
$0 \leqslant q \leqslant 2$: No known fast mixing bound for general graphs.

Main theorem

Theorem

For the random cluster model with parameters $0 < p < 1$ and $q = 2$,

$$T_{rel}(P_{RC}) \leqslant 8n^4m^2,$$
$$T_{mix}(P_{RC}) \leqslant 8n^4m^2(\ln \pi_{RC}(x_0)^{-1} + \ln \epsilon^{-1}).$$

($n = $ #vertices, $m = $ #edges.)

- For $q > 2$, there exists p such that $T_{mix}(P_{RC})$ is exponential on complete graphs. [Gore, Jerrum 99] [Blanca, Sinclair 15] [Gheissari, Lubetzky, Peres 17]
- For $q > 2$ and $0 < p < 1$, it is #BIS-hard to approximate $Z_{RC}(p, q)$. [Goldberg, Jerrum 12]
- For $0 \leqslant q < 2$, there is no known obstacle.

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Swandsen-Wang algorithm

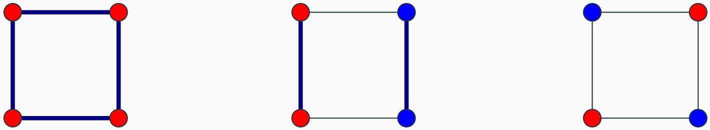
Ferromagnetic Ising model (Ising, Lenz 25)

A configuration $\sigma : V \to \{$ ●, ● $\}$. Parameter $\beta > 1$.

$$w(\sigma) = \beta^{|mono(\sigma)|}$$

Gibbs distribution: $\pi(\sigma) \sim w(\sigma)$.

Partition function: $Z_{Ising}(\beta) = \sum_{\sigma} w(\sigma)$.



β^4 β^2 β^0

Exact evaluation of Z_{Ising} is #P-hard even for $\beta \in \mathbb{C}$ unless $\beta = 0, \pm 1, \pm i$.

FPRAS for Z_{Ising} for $\beta > 1$ [Jerrum, Sinclair 93]
Efficient sampling [Randall, Wilson 99]

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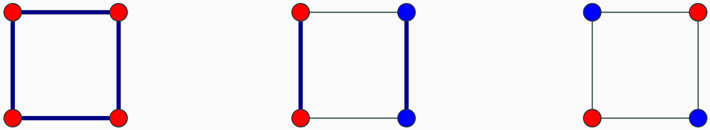

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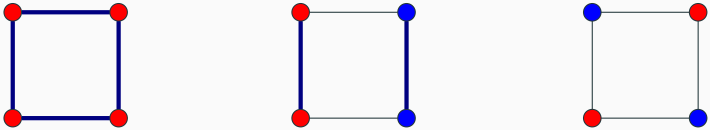

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Equivalence at $q = 2$

Let $\beta = \frac{1}{1-p}$.

$$Z_{Ising}(\beta) = \beta^{|E|} Z_{RC}(p, 2)$$

Joint distribution on vertices and edges [Edwards, Sokal 88]:

vertex colors assigned uniformly, edges chosen with prob. p, conditioned on no chosen edge is bichromatic.

Marginal on vertices ⇒ Ising model.

Marginal on edges ⇒ random cluster model.

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Swendsen-Wang algorithm [Swendsen, Wang 87]

→ 1. Select monochromatic edges.
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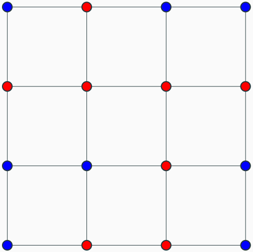

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"This algorithm appears to work extremely well but there are no quantitative theoretical results to support this experimental finding." (Saloff-Coste 97)

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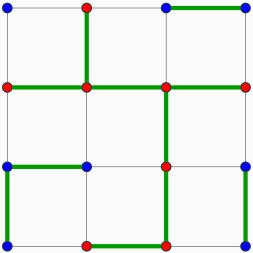

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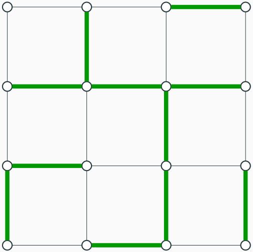

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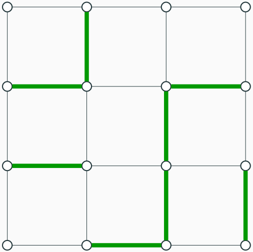

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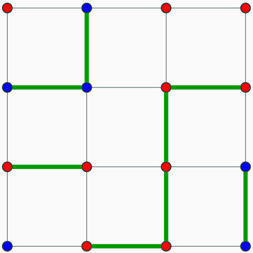

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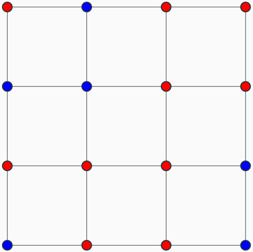

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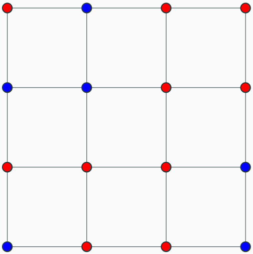

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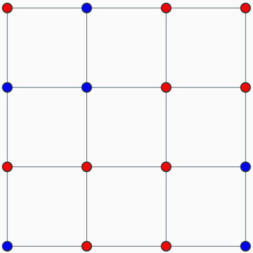

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Previous Results

Again, most previous results focus on special graph families.

- On the complete graph:
  [Gore, Jerrum 99] [Cooper, Dyer, Frieze, Rue 00]
  [Long, Nachimus, Ning, Peres 11] [Borgs, Chayes, Tetali 11]
  [Galanis, Štefankovič, Vigoda 15] [Gheissari, Lubetzky, Peres 17]
- On trees (or bounded tree-width):
  [Cooper, Frieze 99] [Ge, Štefankovič 10]

Concequence — Swendsen-Wang algorithm is rapidly mixing

Theorem (Ullrich 14)

$$T_{rel}(P_{SW}) \leqslant T_{rel}(P_{RC})$$

Combine with our theorem:
Swendsen-Wang is rapidly mixing at $q = 2$,
namely, for the ferromagnetic Ising model at any temperature.

However, our mixing time bound is $O(n^4 m^3)$.

Conjecture (Peres)

The mixing time of Swendsen-Wang at $q = 2$ is $O(n^{1/4})$.

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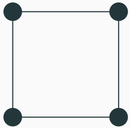
Even subgraphs

Another equivalent formulations at $q = 2$

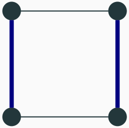
Subgraph $r \subseteq E$ is even if every vertex in (V, r) has an even degree.

$$\pi_{even}(r) \propto p^{|r|}(1-p)^{|E \setminus r|}$$

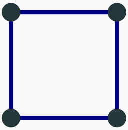
Partition function $Z_{even}(p)$



$(1-p)^4$



NOT EVEN



p^4

Equivalence at $q = 2$

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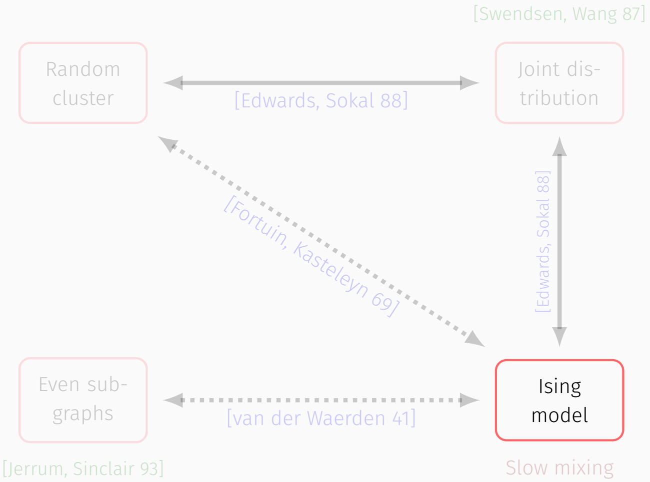
$$Z_{Ising}(\beta) \;=\; \beta^{|E|} Z_{RC}(p, 2) \;=\; 2^{|V|}\beta^{|E|} Z_{even}\left(\frac{p}{2}\right)$$

Overview


[Swendsen, Wang 87]
Random cluster
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Joint distribution
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[Edwards, Sokal 88]
Even subgraphs
[van der Waerden 41]
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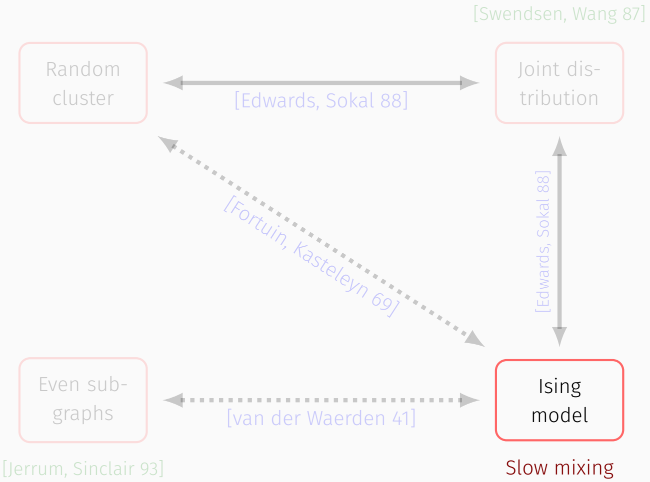
Slow mixing

Overview



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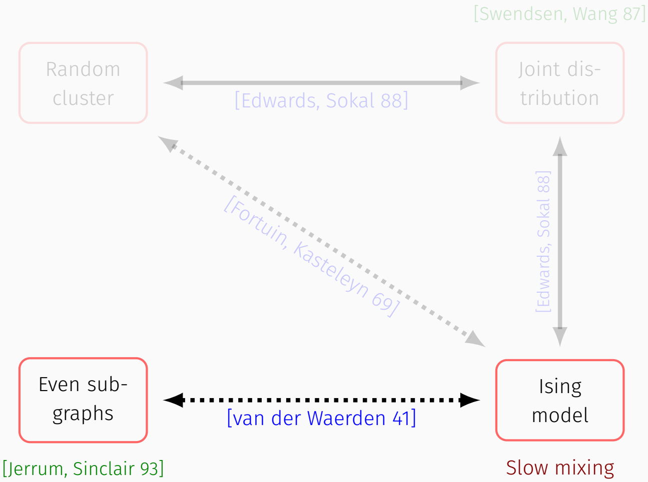
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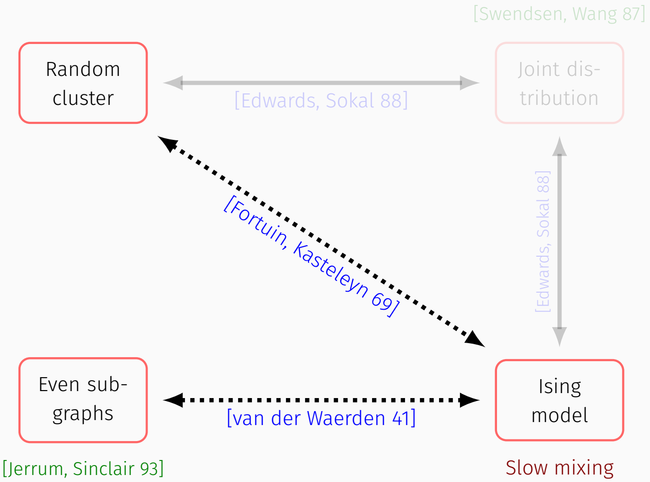

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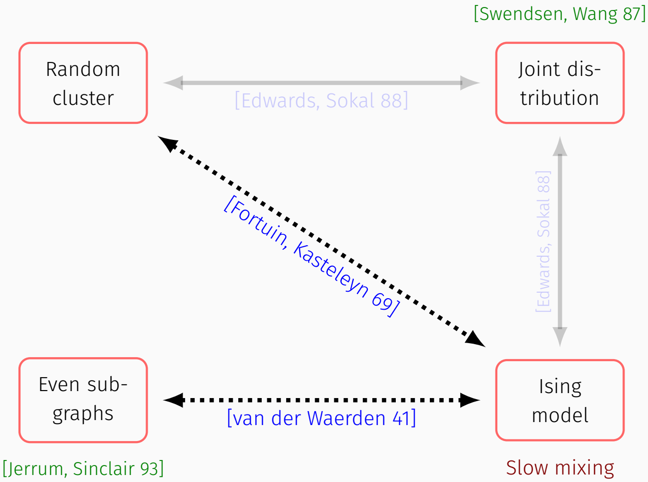
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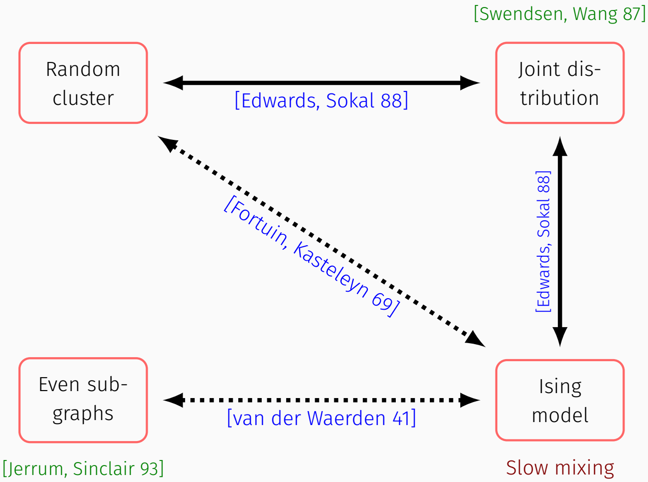

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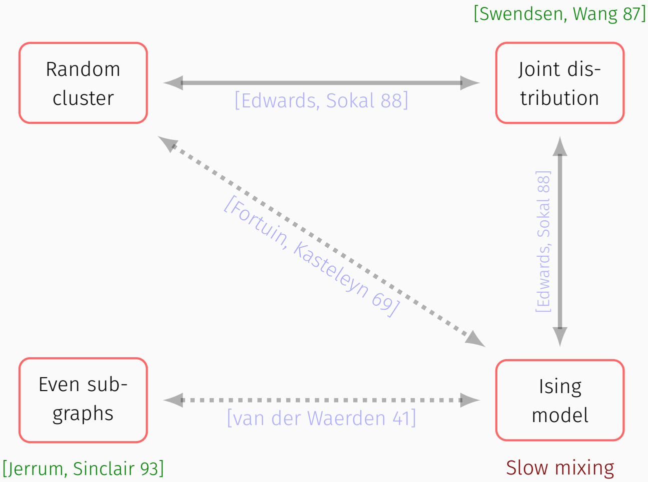
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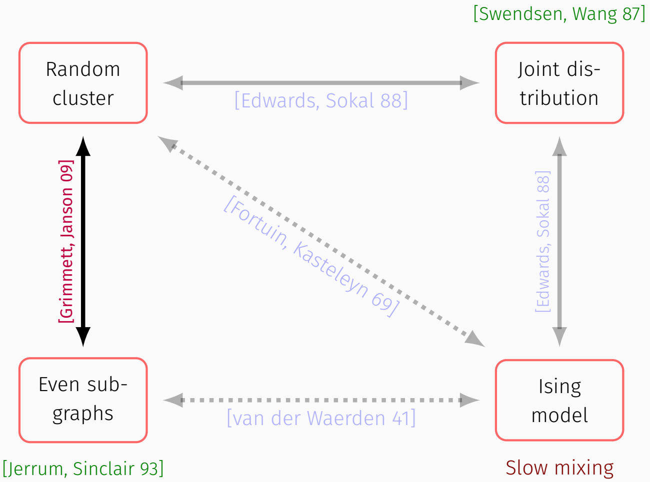
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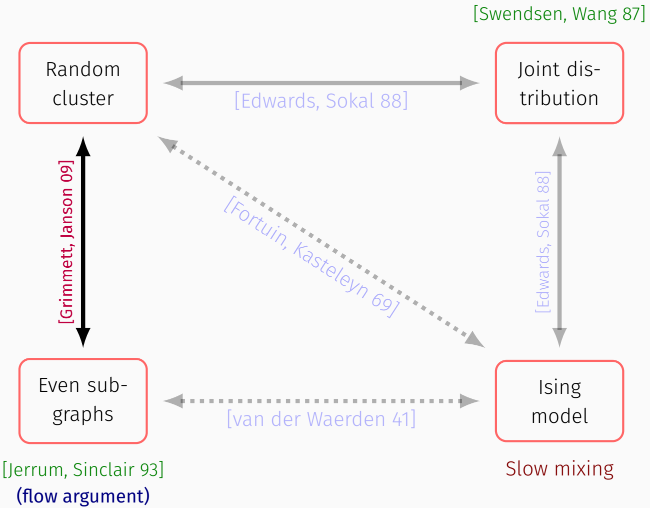
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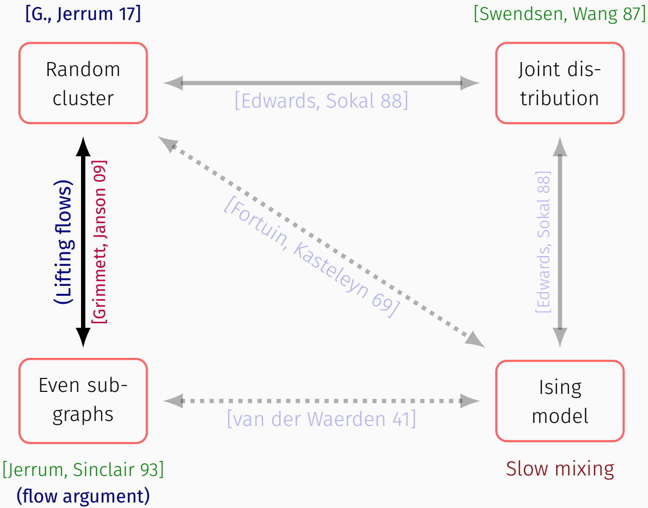

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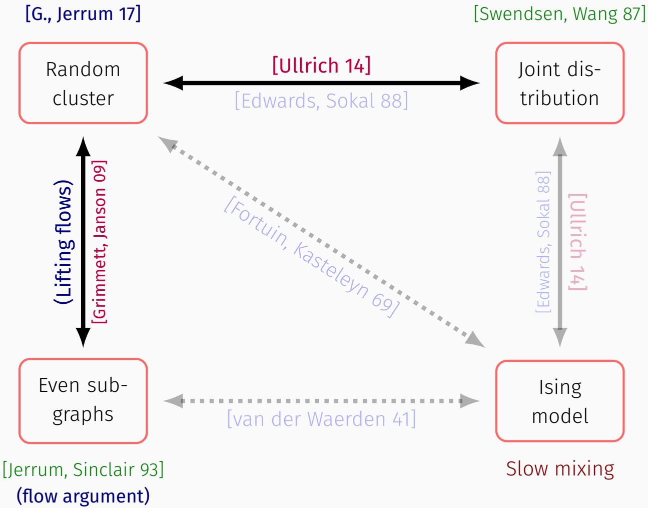

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Grimmett-Janson coupling

Given a graph G, draw a random even subgraph $S \subseteq E$ with $p \leqslant \frac{1}{2}$:

$$\Pr(S = s) = \pi_{even}(s).$$

Then we add every edge $e \notin S$ with probability $p' = \frac{p}{1-p}$.

Call this subgraph R.

Theorem (Grimmett, Janson 09)

$$\Pr(R = r) = \pi_{RC;\,2p,2}(r).$$

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The Proof

Congestion and flows

$T_{rel} \leqslant$ congestion of any flow [Sinclair 92].

For any two states x and y, we construct a random path from x to y.

The random variable Z_k:

1. Random independent initial and final states I and F.
2. A random path γ from I to F.
3. Z_k is the kth state of γ.

The quantity $\max_k \frac{\Pr(Z_k = z)}{\pi(z)}$ is polynomially related to the congestion.

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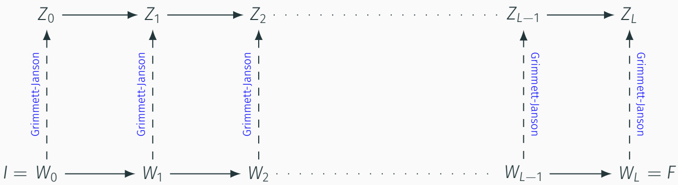
Lifting flows

In an ideal world ...

- Suppose we have canonical paths Γ_{even} for even subgraphs with low congestion (similar to [Jerrum, Sinclair 93]).
- Then use Grimmett-Janson to lift Γ_{even} to a flow for random cluster.



In the real world ...

Two issues:

1. We do not have good canonical paths for even subgraphs — Jerrum-Sinclair chain moves among all subgraphs!

   Patch 1: modify Jerrum-Sinclair to even/near-even subgraphs, and extend Grimmett-Janson for near-even.

2. Grimmett-Janson adds indepdendent edges — Z_i and Z_{i+1} are not adjacent states! They may differ by a lot of edges.

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Random cluster Z_0 Z_{2m}

- Goal: low congestion flows.
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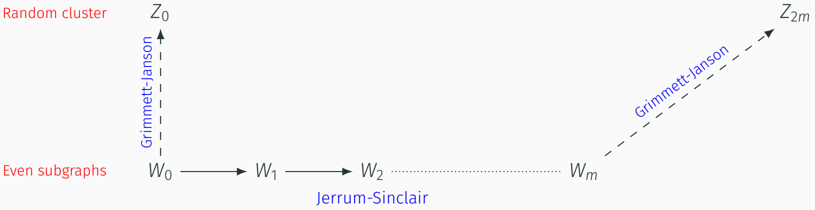

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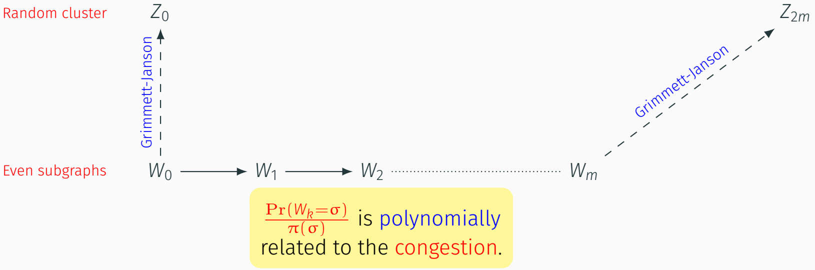

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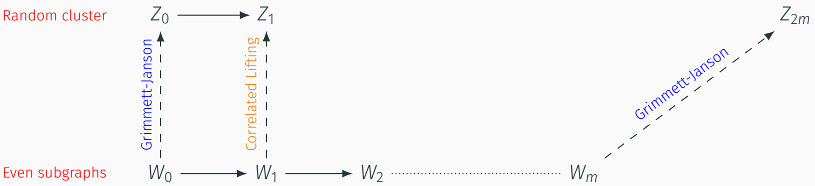

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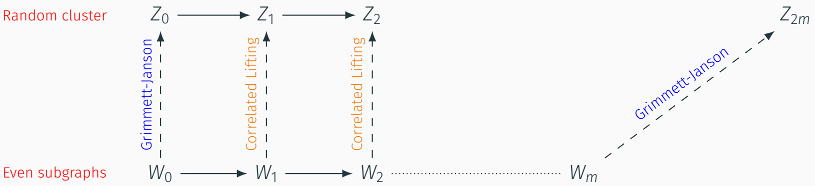

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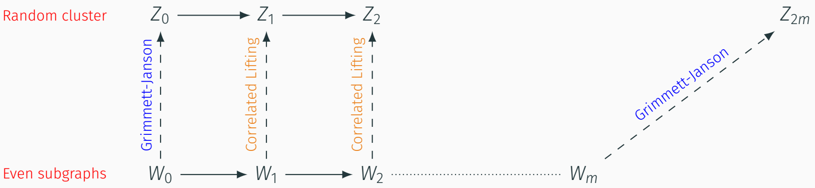
Lifting flows

Z_k is close to RC.
$\Rightarrow$ low congestion



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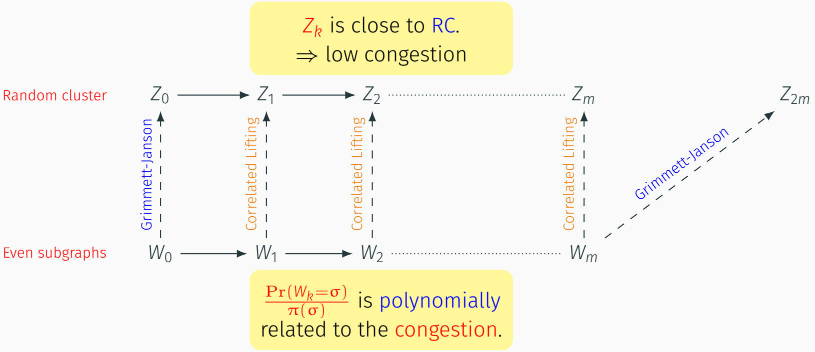

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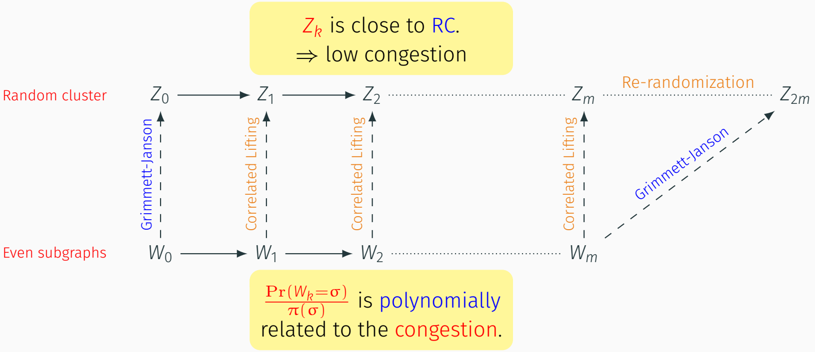

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- Re-randomization to remove correlations between Z_0 and Z_m.

Recap

Theorem

$$\text{At } q = 2, \quad T_{rel}(P_{RC}) \leqslant 8n^4m^2.$$

- $q = 2$ tighter mixing time bound? $O(n^{1/4})$?
- $1 < q < 2$ (monotone) fast mixing?
- $0 \leqslant q < 1$ (e.g. #Forests) fast mixing???

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Thank You!

`arxiv.org/abs/1605.00139`