When is Coalescing as fast as Meeting?

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joint work with Frederik Mallmann-Trenn (SFU/ENS Paris) & Varun Kanade (Oxford)

03/08/2017

Outline

Introduction

Interlude: Complete Graph

Relating Coalescing-Time to the Mixing and Meeting Time

Conclusion

Random Walk Notation

- P transition matrix of a lazy walk on an undirected, connected graph G

$$p_{u,v} = \begin{cases} \frac{1}{2} & \text{if } u = v, \\ \frac{1}{2\deg(u)} & \text{if } \{u,v\} \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

- π with $\pi_v = \frac{\deg(v)}{2|E|}$ is the stationary distribution

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Fundamental Quantities

- mixing time: $t_{\mathrm{mix}}(\frac{1}{e}) = \min\{t \in \mathbb{N} \colon \forall u \in V \colon \frac{1}{2}\sum_{v \in V} \left|p^t_{u,v} - \pi_v\right| \leq \frac{1}{e}\}$

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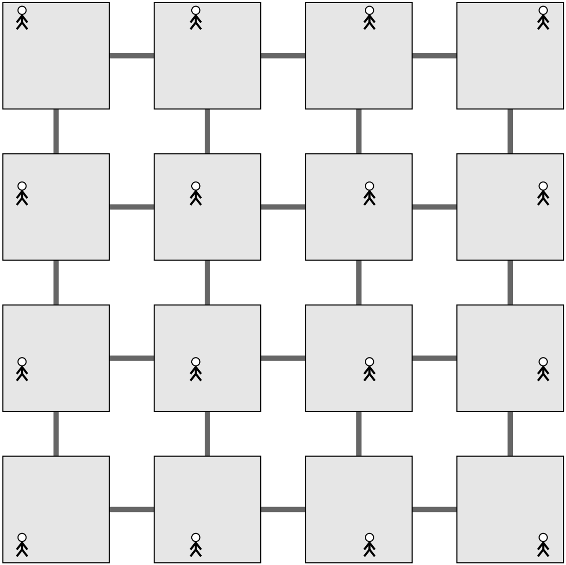
Focus of this talk

- meeting time: $t_{\mathrm{meet}} = \max_{u,v \in V} \mathbf{E}_{u,v}\left[\min\{t \colon X_t = Y_t\}\right]$
- coalescing time: $t_{\mathrm{coal}} = \mathbf{E}_{1,2,\ldots,n}\left[\ldots\right]$

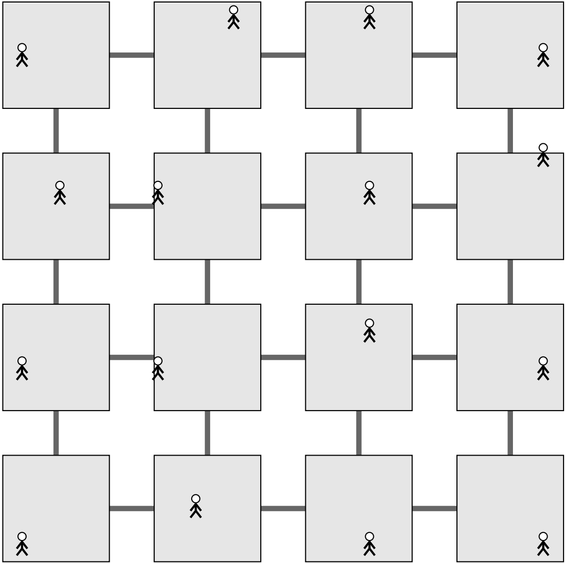
Coalescing Random Walks (Example)



Time: 0

Particles: 16

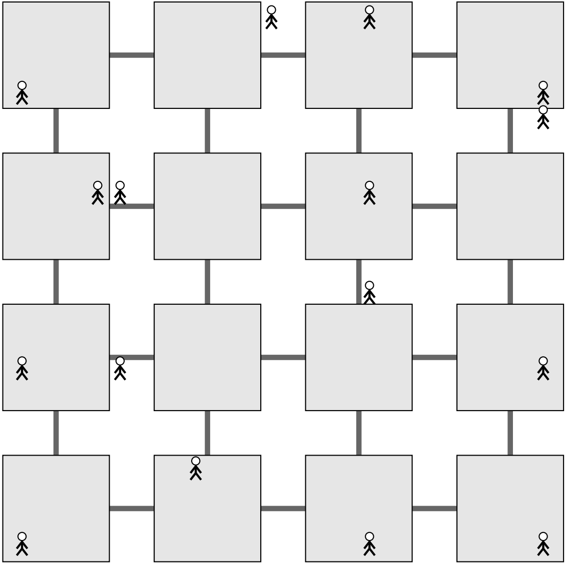
Coalescing Random Walks (Example)



Time: 0.25

Particles: 16

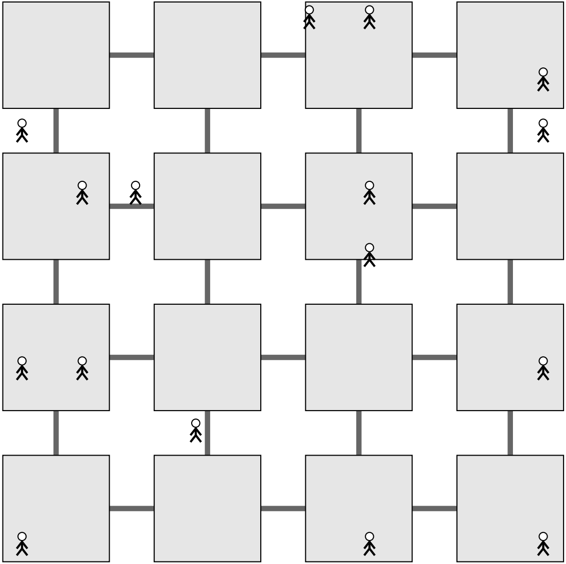
Coalescing Random Walks (Example)



Time: 0.5

Particles: 16

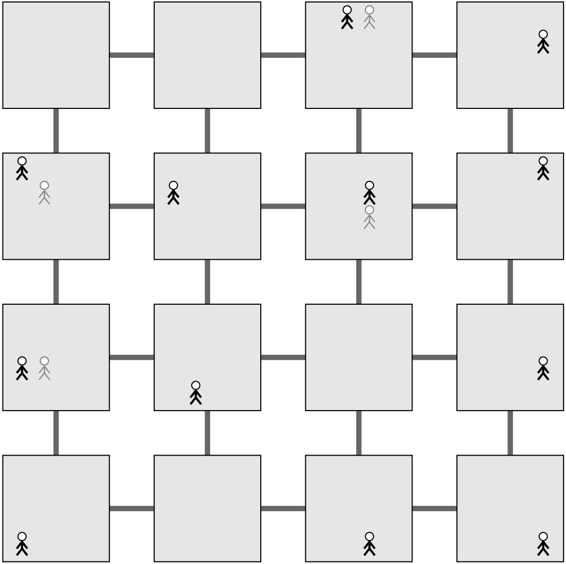
Coalescing Random Walks (Example)



Time: 0.75

Particles: 16

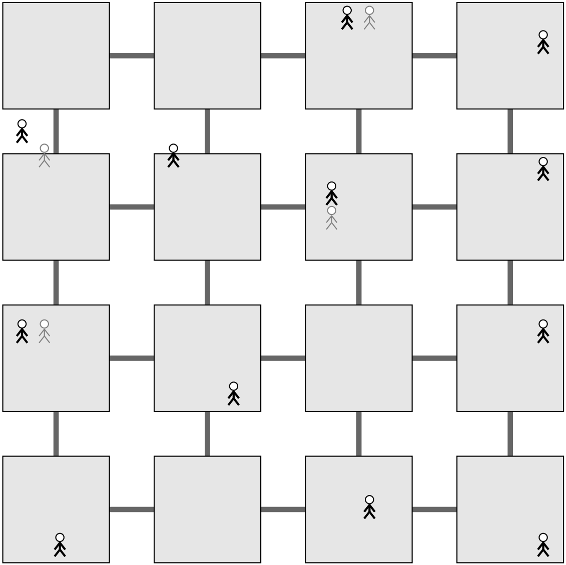
Coalescing Random Walks (Example)



Time: 1

Particles: 12

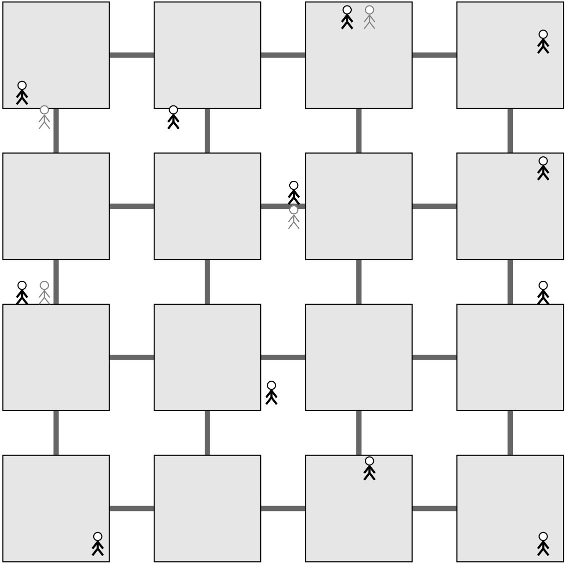
Coalescing Random Walks (Example)



Time: 1.25

Particles: 12

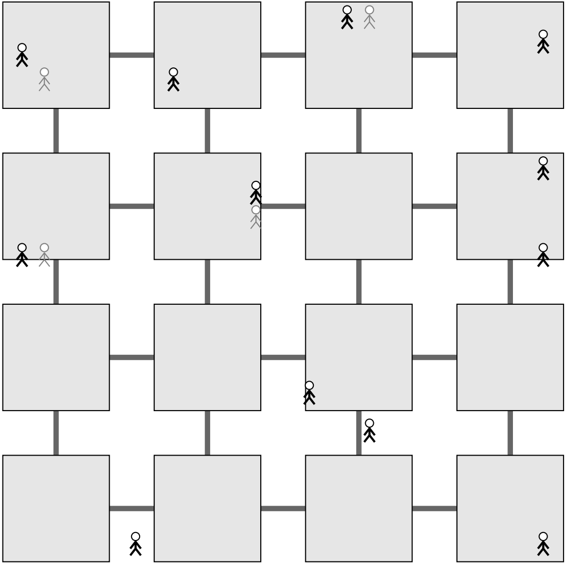
Coalescing Random Walks (Example)



Time: 1.5

Particles: 12

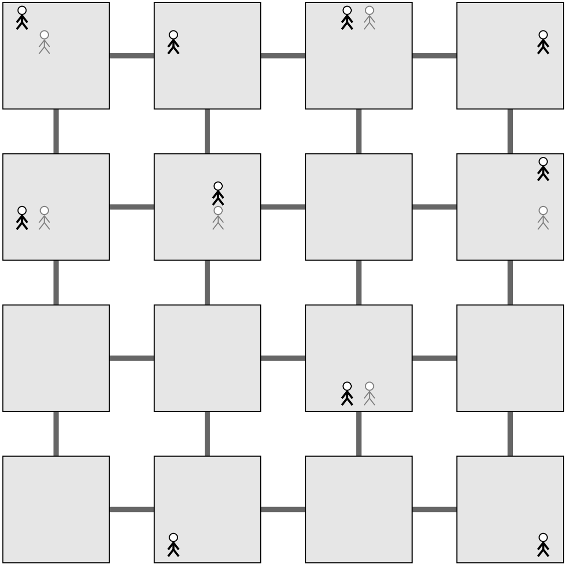
Coalescing Random Walks (Example)



Time: 1.75

Particles: 12

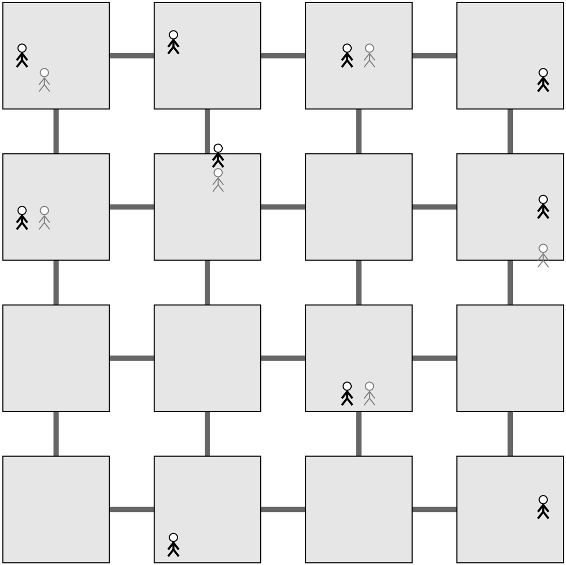
Coalescing Random Walks (Example)



Time: 2

Particles: 10

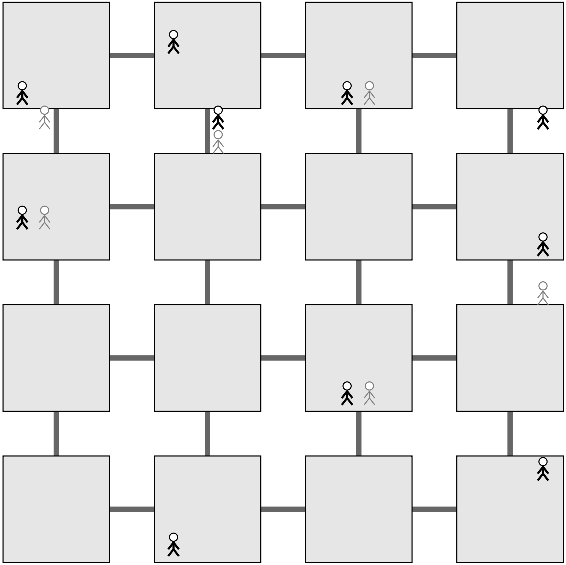
Coalescing Random Walks (Example)



Time: 2.25

Particles: 10

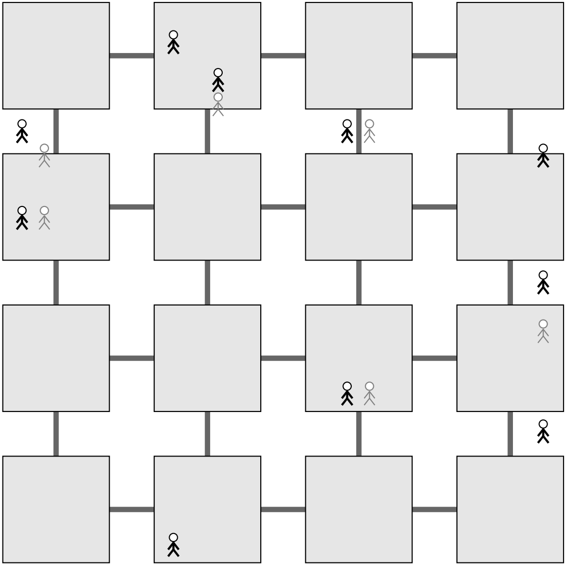
Coalescing Random Walks (Example)



Time: 2.5

Particles: 10

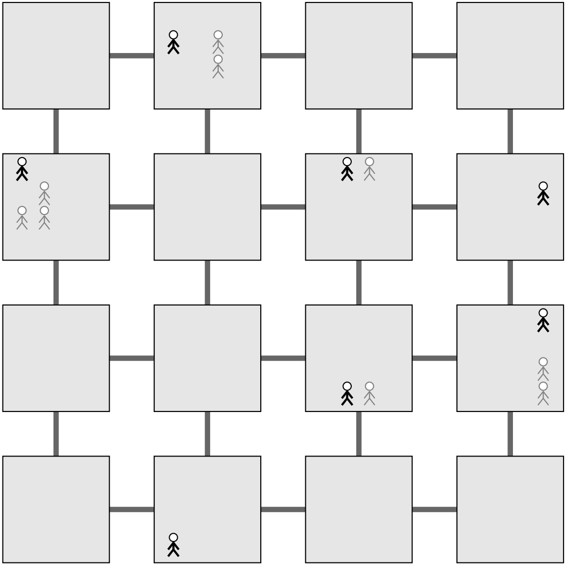
Coalescing Random Walks (Example)



Time: 2.75

Particles: 10

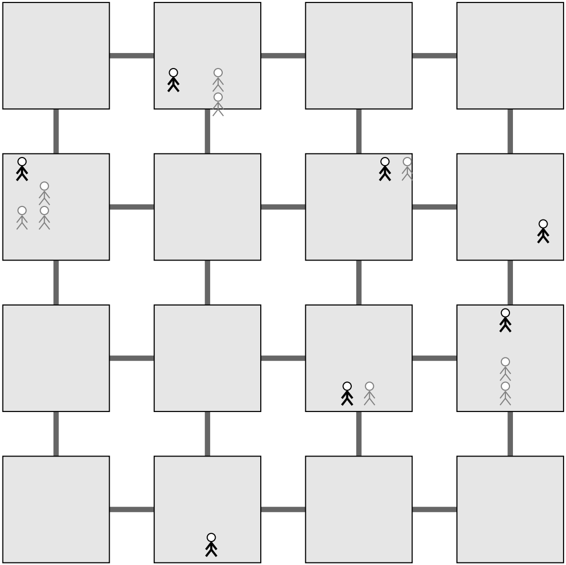
Coalescing Random Walks (Example)



Time: 3

Particles: 7

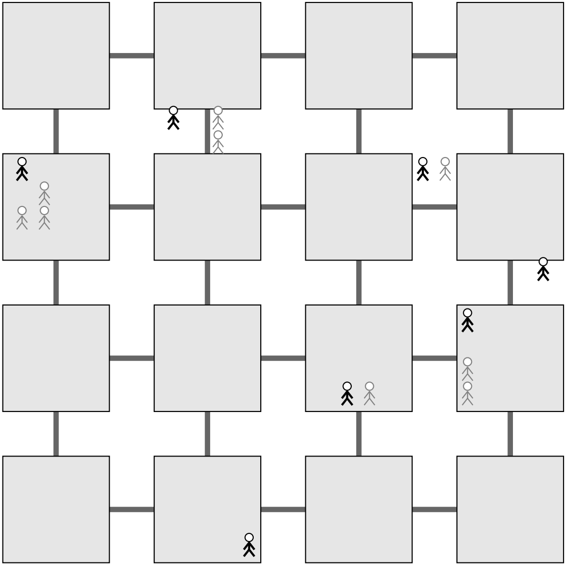
Coalescing Random Walks (Example)



Time: 3.25

Particles: 7

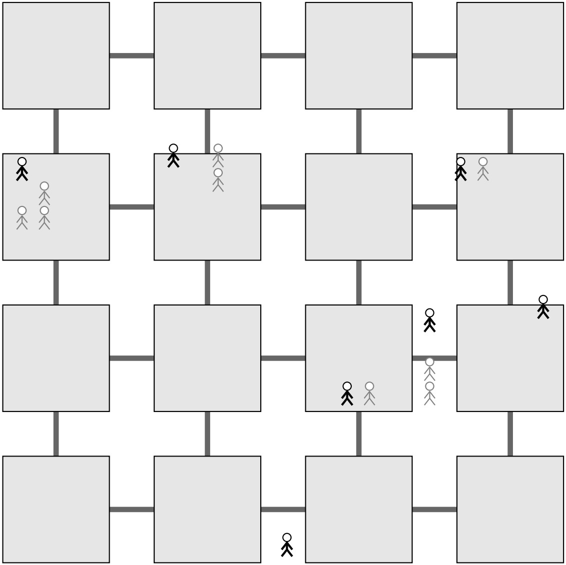
Coalescing Random Walks (Example)



Time: 3.5

Particles: 7

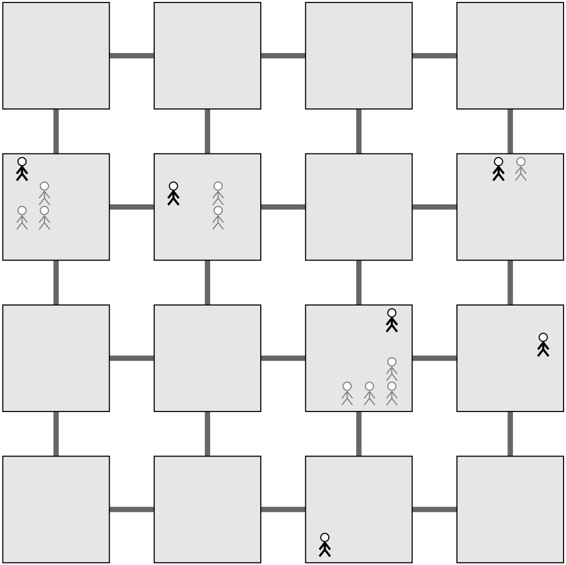
Coalescing Random Walks (Example)



Time: 3.75

Particles: 7

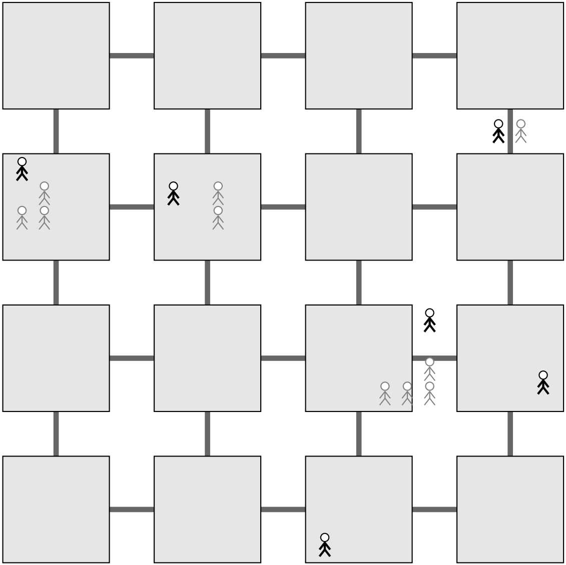
Coalescing Random Walks (Example)



Time: 4

Particles: 6

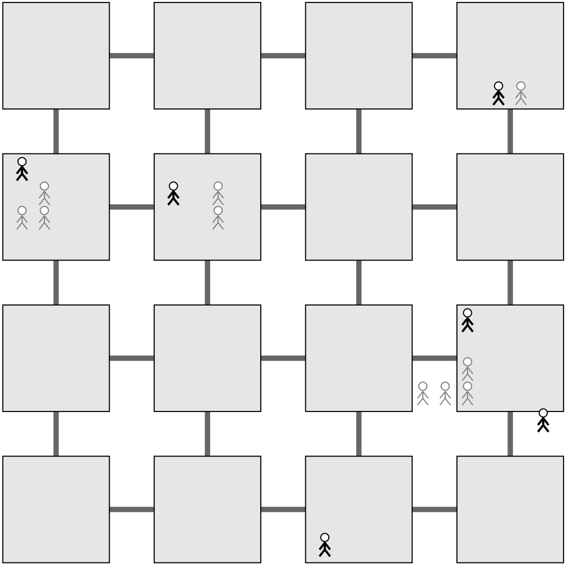
Coalescing Random Walks (Example)



Time: 4.25

Particles: 6

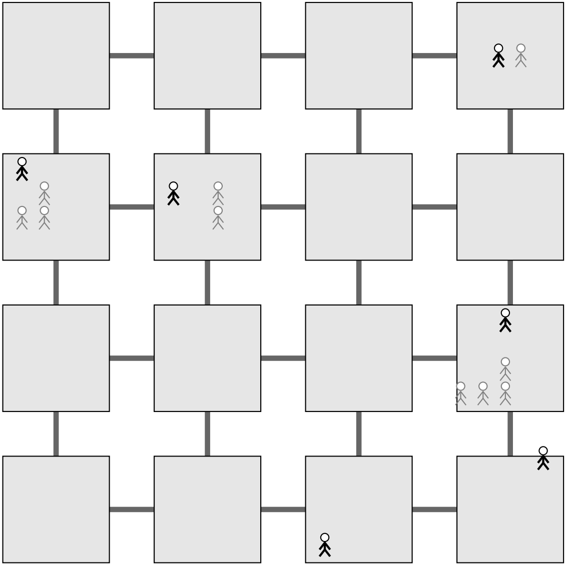
Coalescing Random Walks (Example)



Time: 4.5

Particles: 6

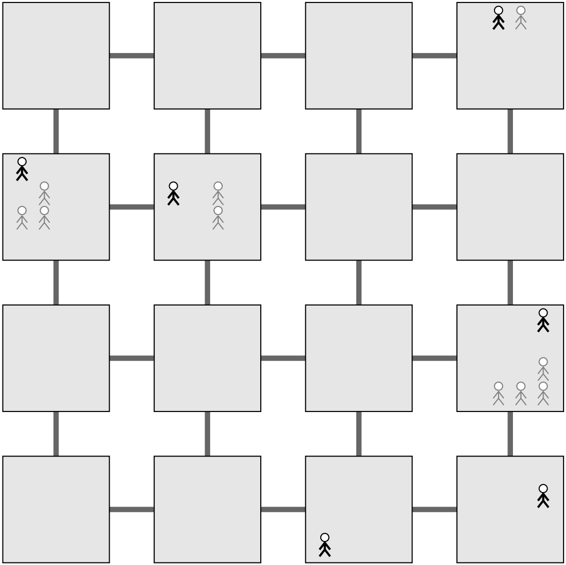
Coalescing Random Walks (Example)



Time: 4.75

Particles: 6

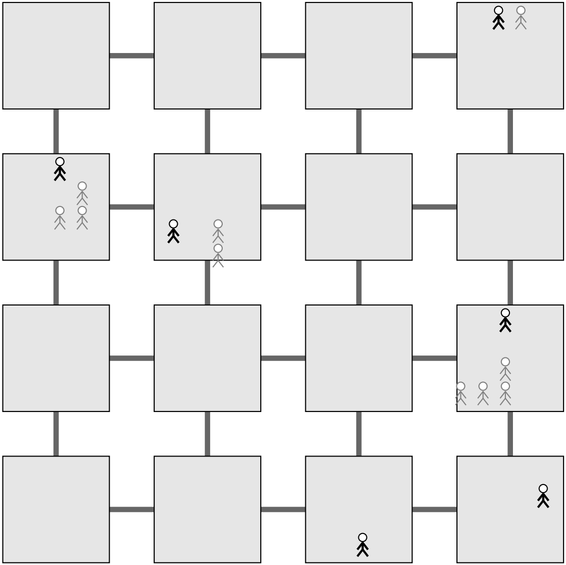
Coalescing Random Walks (Example)



Time: 5

Particles: 6

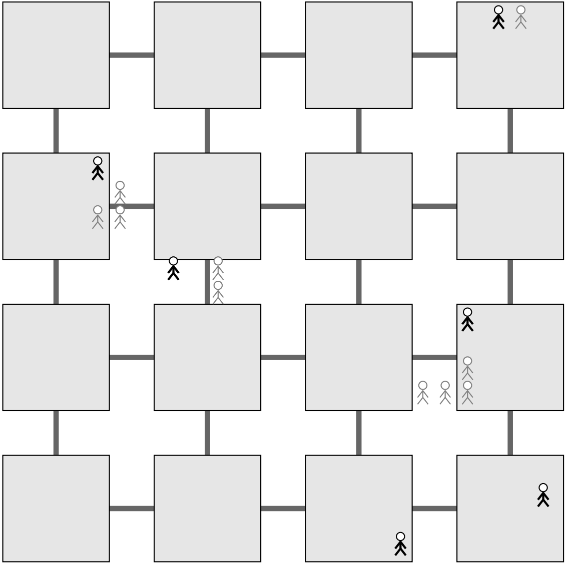
Coalescing Random Walks (Example)



Time: 5.25

Particles: 6

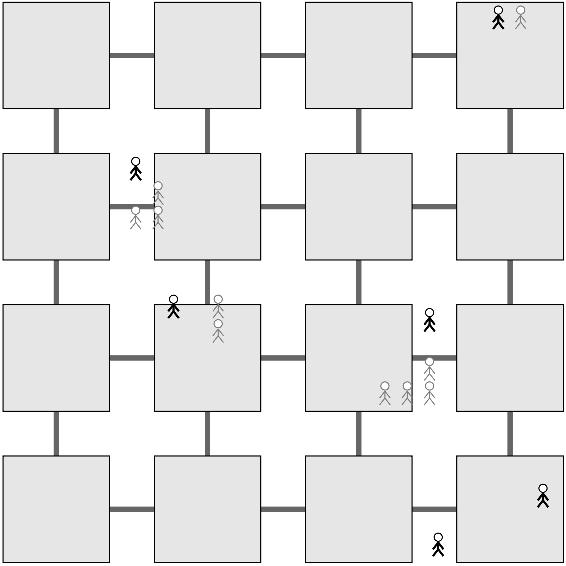
Coalescing Random Walks (Example)



Time: 5.5

Particles: 6

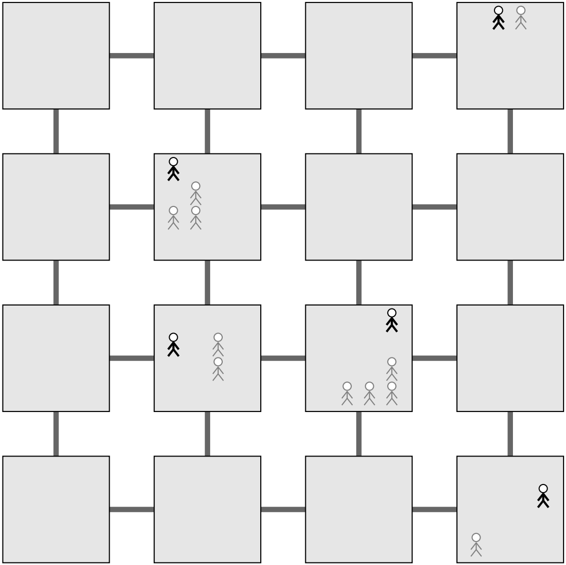
Coalescing Random Walks (Example)



Time: 5.75

Particles: 6

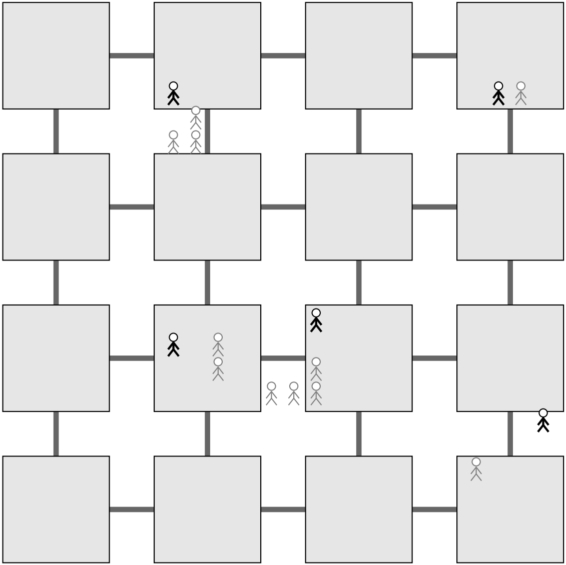
Coalescing Random Walks (Example)



Time: 6

Particles: 5

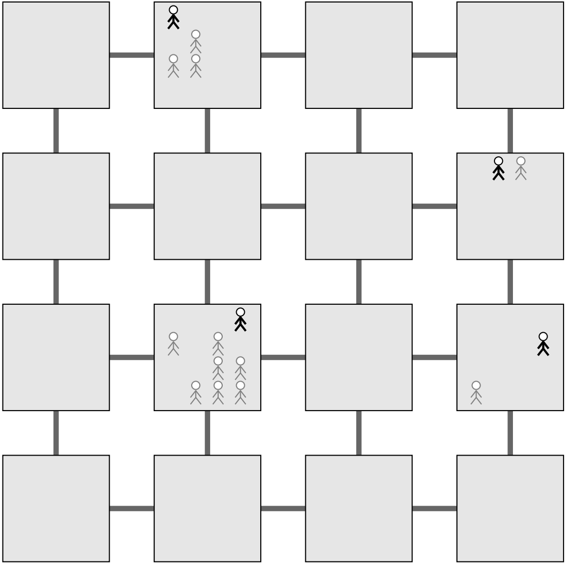
Coalescing Random Walks (Example)



Time: 6.5

Particles: 5

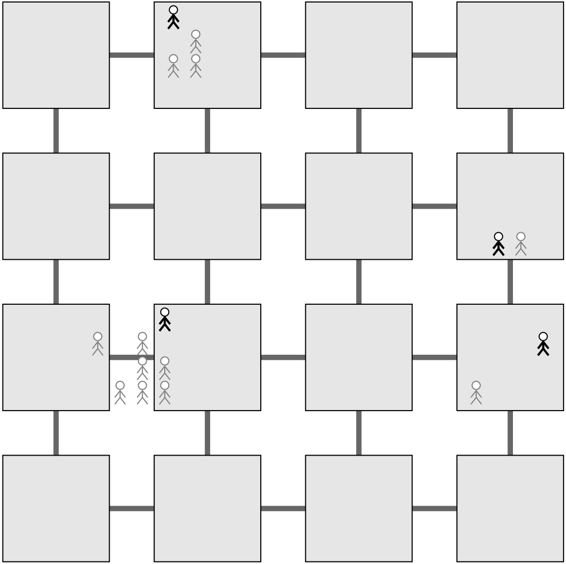
Coalescing Random Walks (Example)



Time: 7

Particles: 4

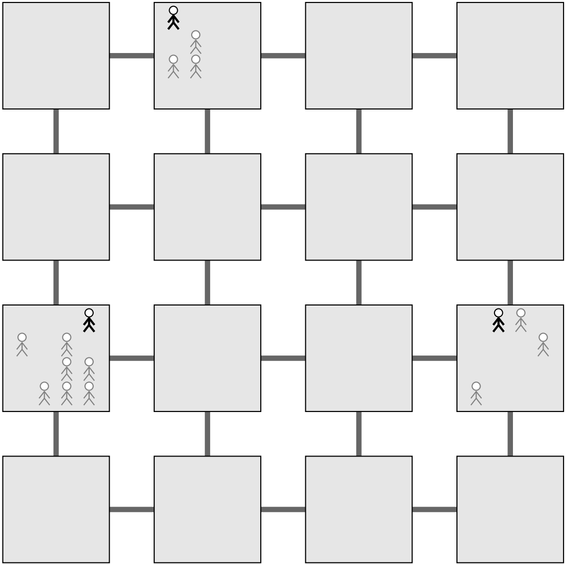
Coalescing Random Walks (Example)



Time: 7.5

Particles: 4

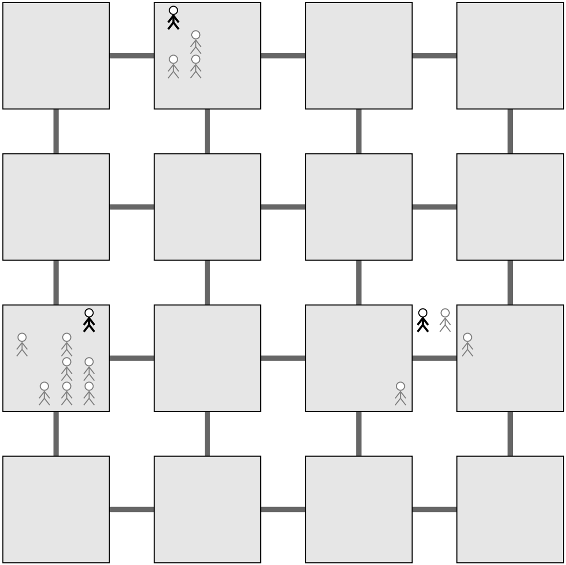
Coalescing Random Walks (Example)



Time: 8

Particles: 3

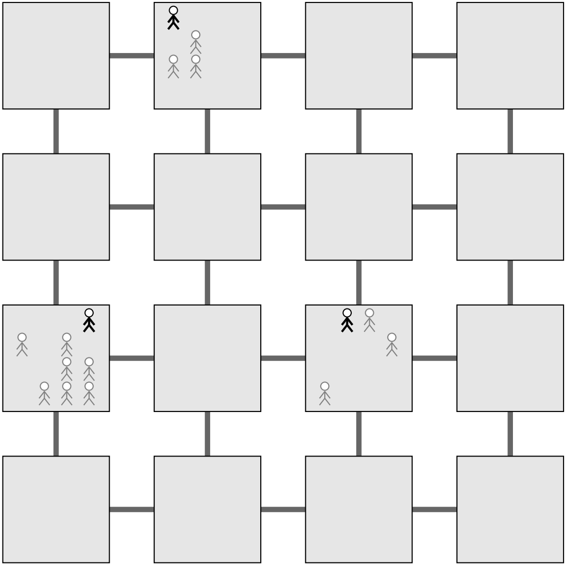
Coalescing Random Walks (Example)



Time: 8.5

Particles: 3

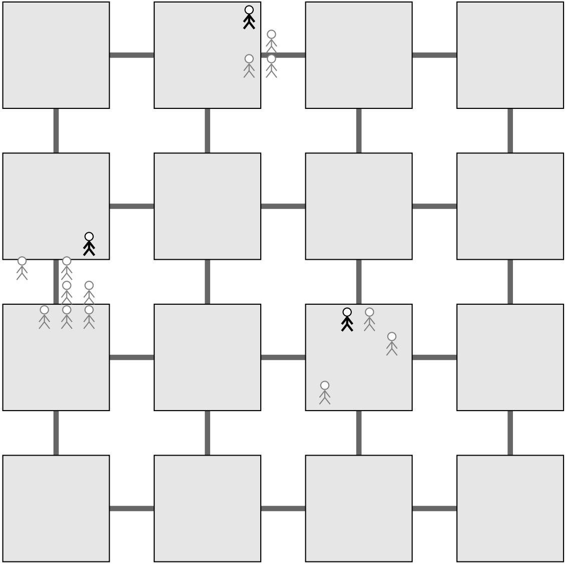
Coalescing Random Walks (Example)



Time: 9

Particles: 3

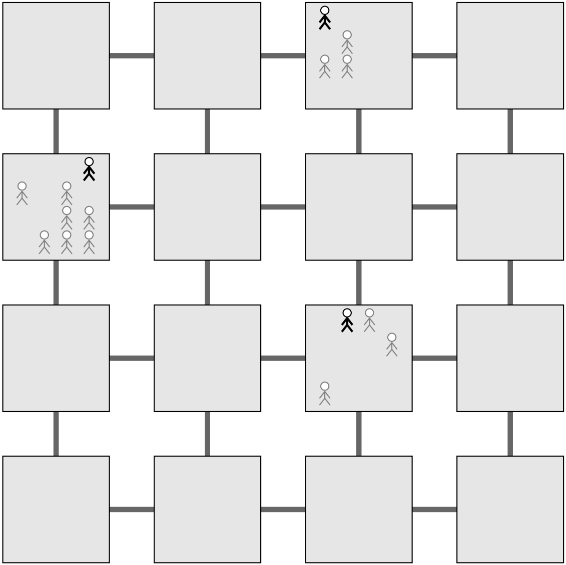
Coalescing Random Walks (Example)



Time: 9.5

Particles: 3

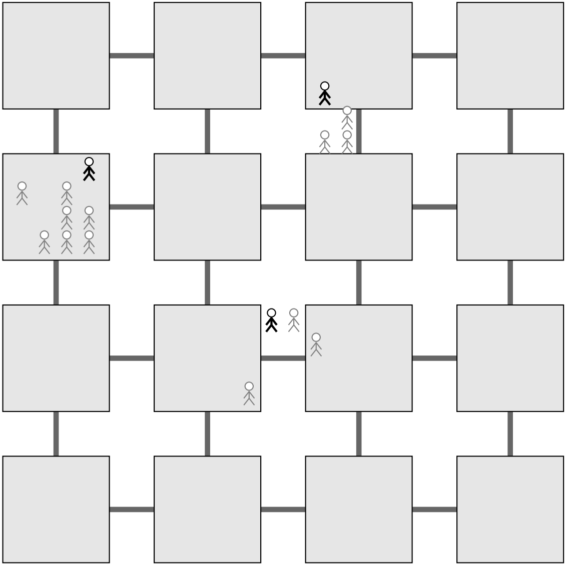
Coalescing Random Walks (Example)



Time: 10

Particles: 3

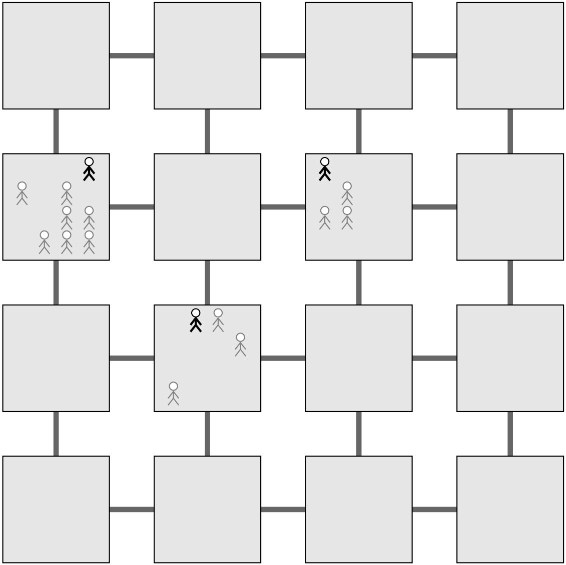
Coalescing Random Walks (Example)



Time: 10.5

Particles: 3

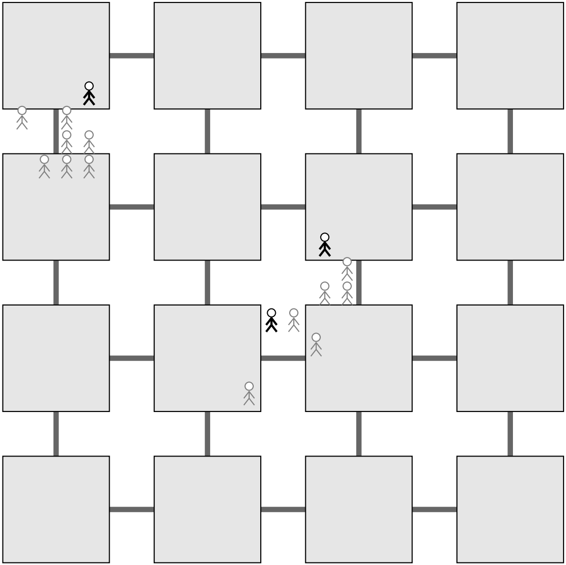
Coalescing Random Walks (Example)



Time: 11

Particles: 3

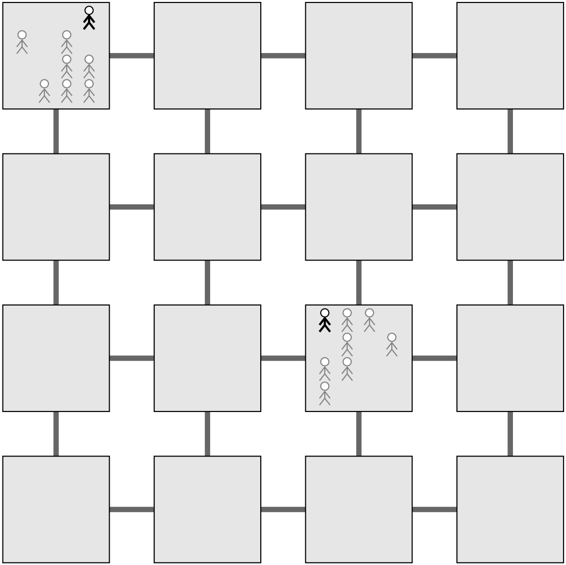
Coalescing Random Walks (Example)



Time: 11.5

Particles: 3

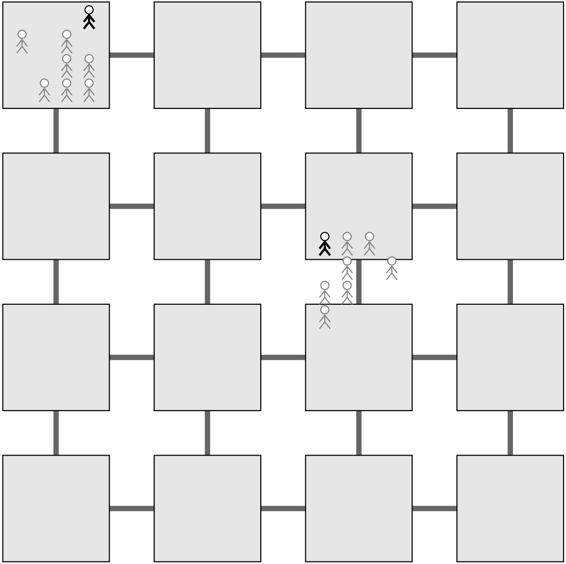
Coalescing Random Walks (Example)



Time: 12

Particles: 2

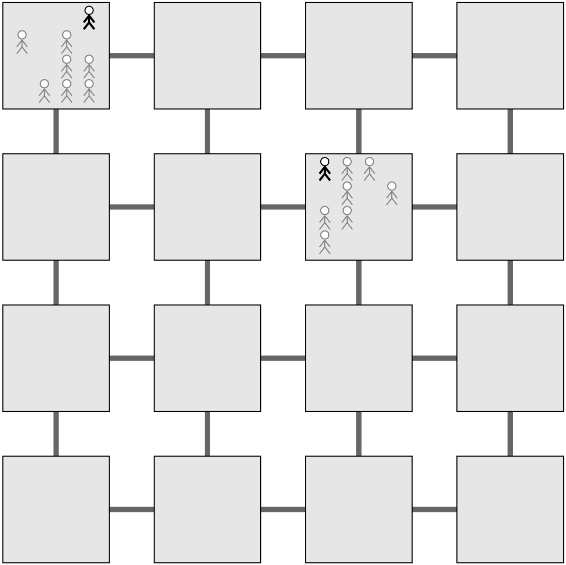
Coalescing Random Walks (Example)



Time: 12.5

Particles: 2

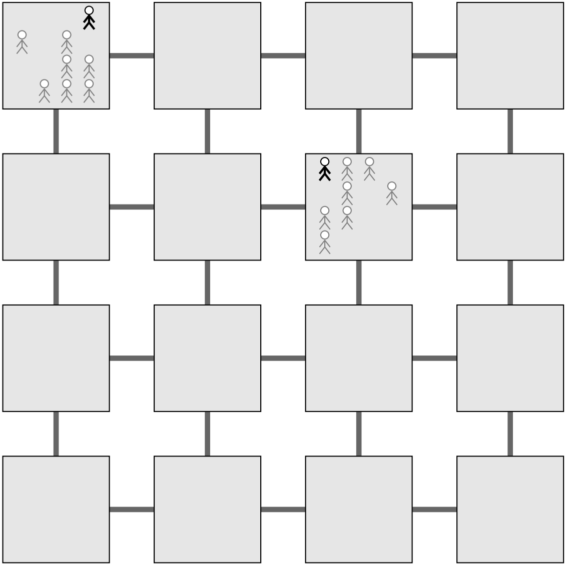
Coalescing Random Walks (Example)



Time: 13

Particles: 2

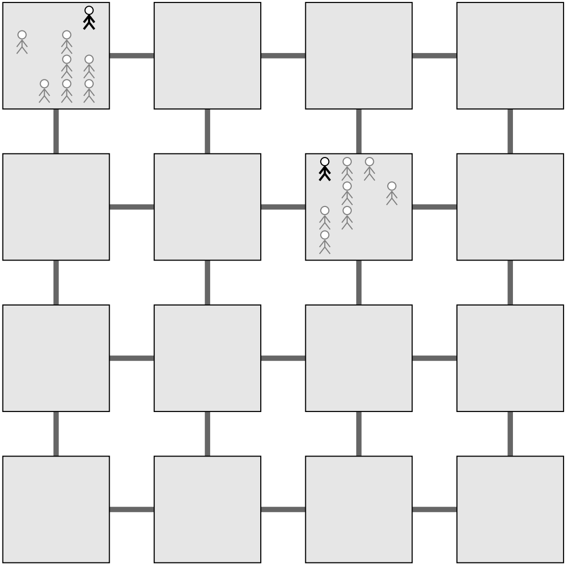
Coalescing Random Walks (Example)



Time: 13.5

Particles: 2

Coalescing Random Walks (Example)



Time: 14

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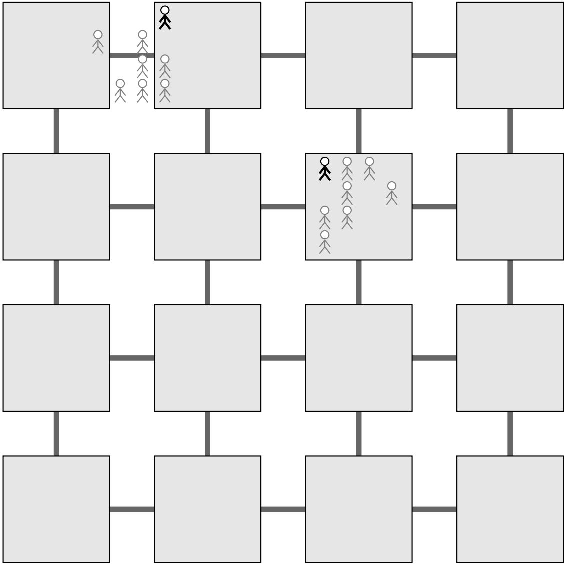
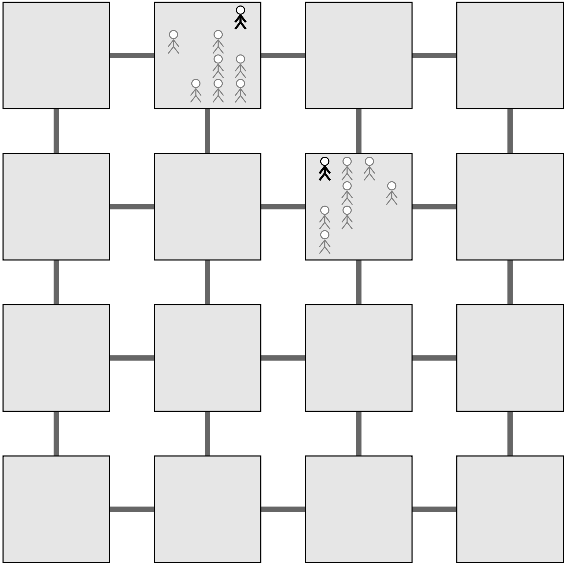
Coalescing Random Walks (Example)



Time: 14.5

Particles: 2

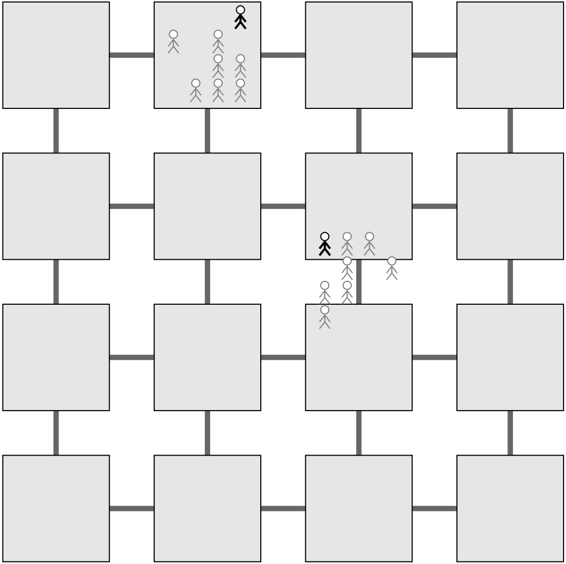
Coalescing Random Walks (Example)



Time: 15

Particles: 2

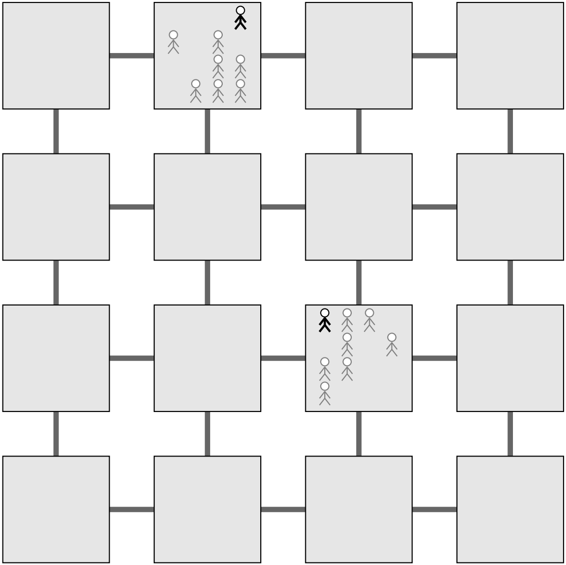
Coalescing Random Walks (Example)



Time: 15.5

Particles: 2

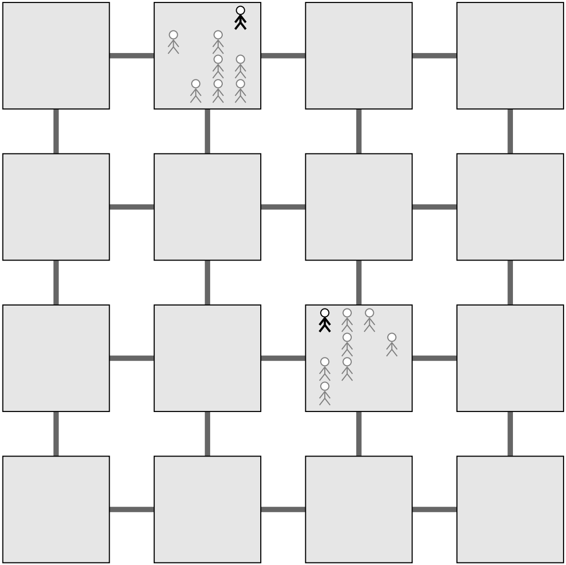
Coalescing Random Walks (Example)



Time: 16

Particles: 2

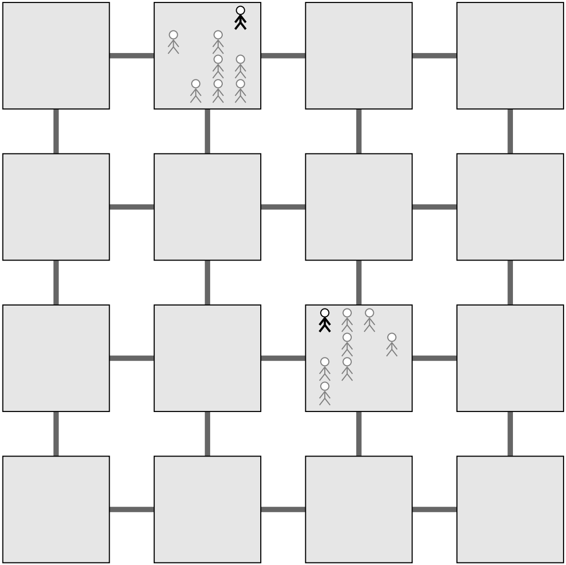
Coalescing Random Walks (Example)



Time: 16.5

Particles: 2

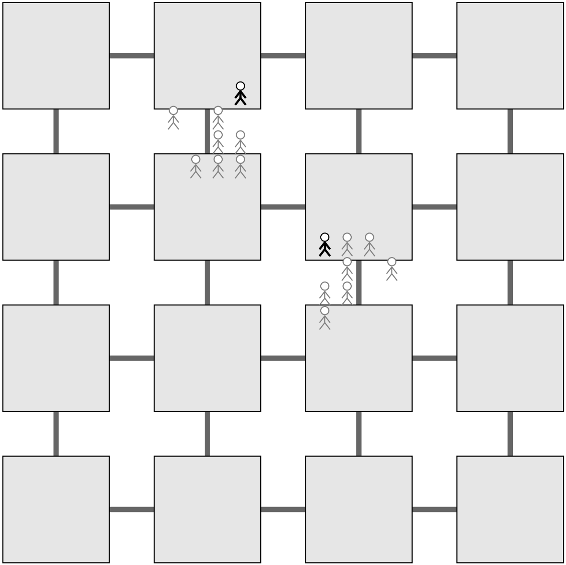
Coalescing Random Walks (Example)



Time: 17

Particles: 2

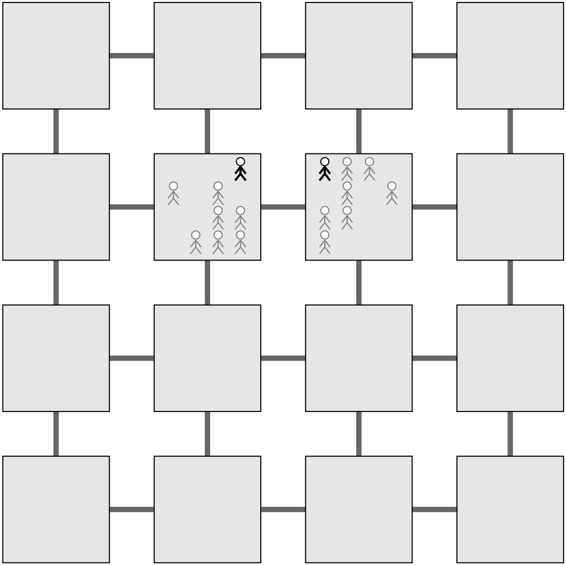
Coalescing Random Walks (Example)



Time: 17.5

Particles: 2

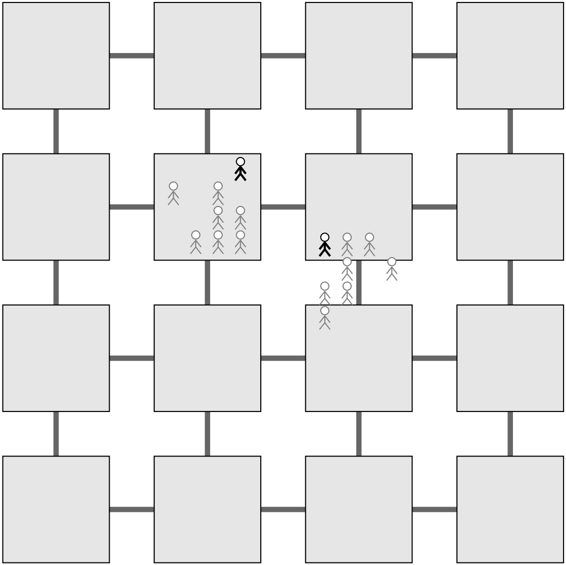
Coalescing Random Walks (Example)



Time: 18

Particles: 2

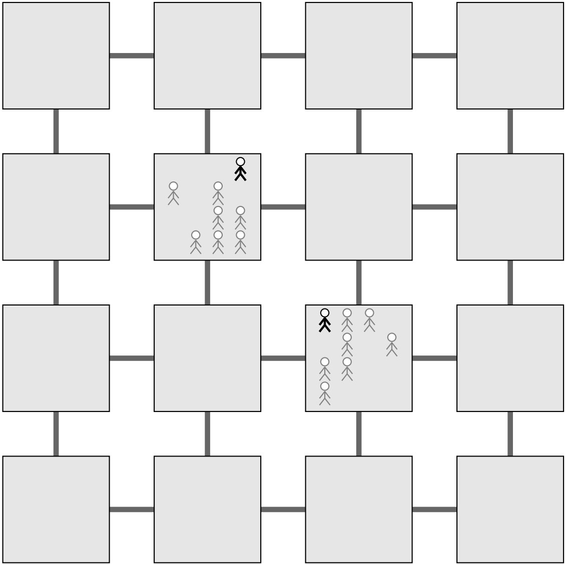
Coalescing Random Walks (Example)



Time: 18.5

Particles: 2

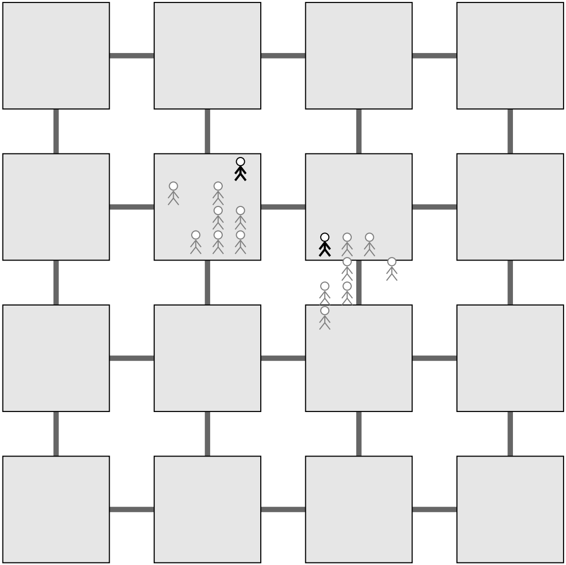
Coalescing Random Walks (Example)



Time: 19

Particles: 2

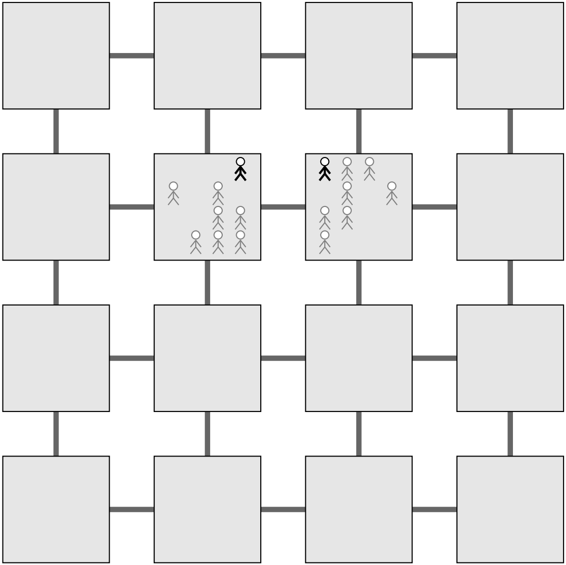
Coalescing Random Walks (Example)



Time: 19.5

Particles: 2

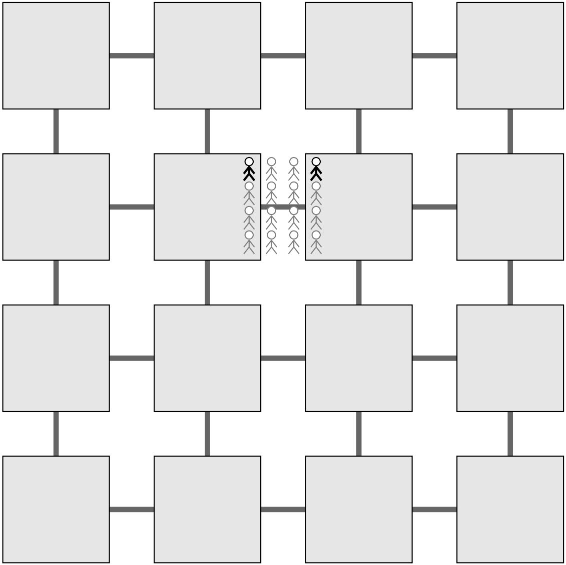
Coalescing Random Walks (Example)



Time: 20

Particles: 2

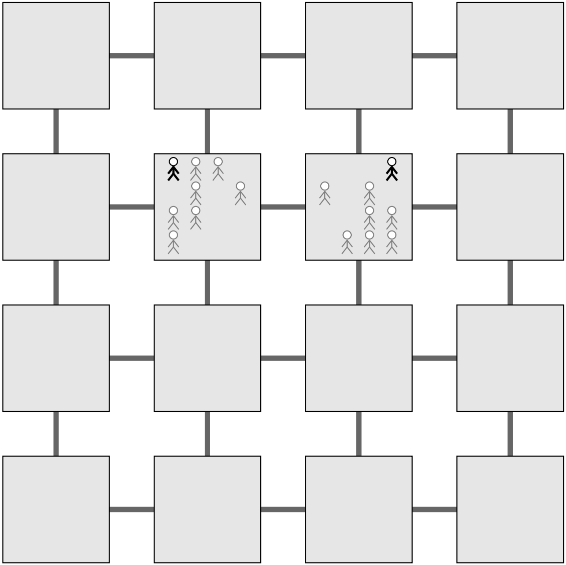
Coalescing Random Walks (Example)



Time: 20.5

Particles: 2

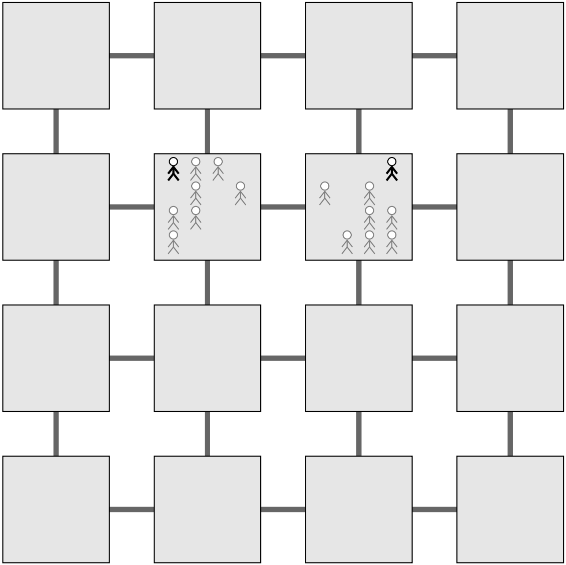
Coalescing Random Walks (Example)



Time: 21

Particles: 2

Coalescing Random Walks (Example)



Time: 21.5

Particles: 2

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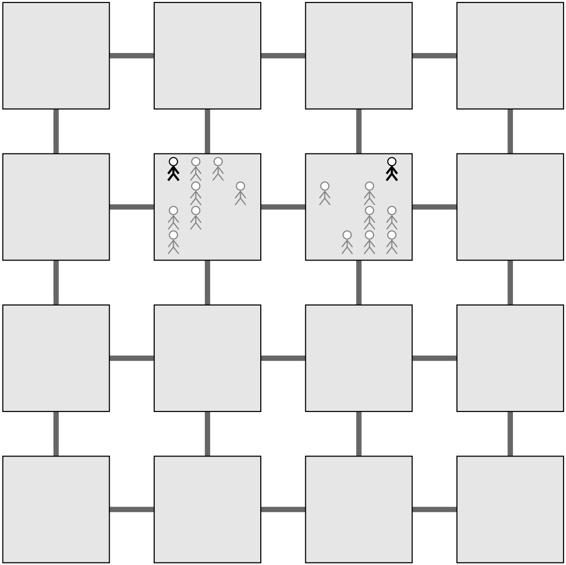
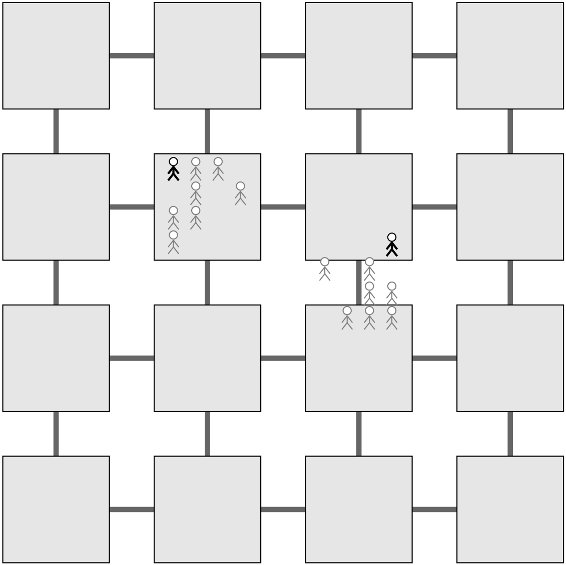


Time: 22

Particles: 2

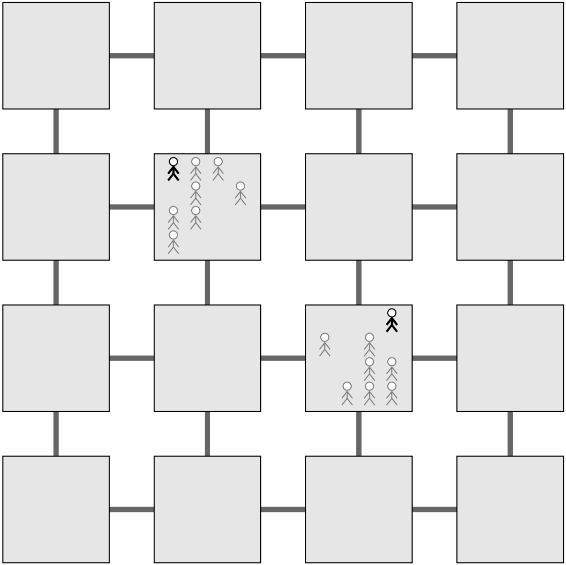
Coalescing Random Walks (Example)



Time: 22.5

Particles: 2

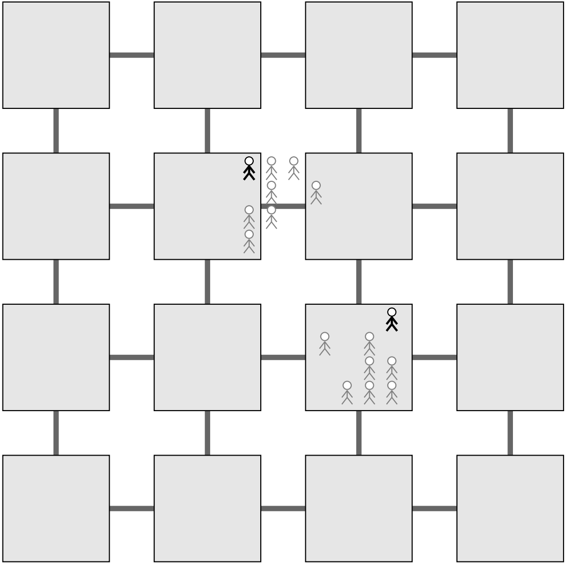
Coalescing Random Walks (Example)



Time: 23

Particles: 2

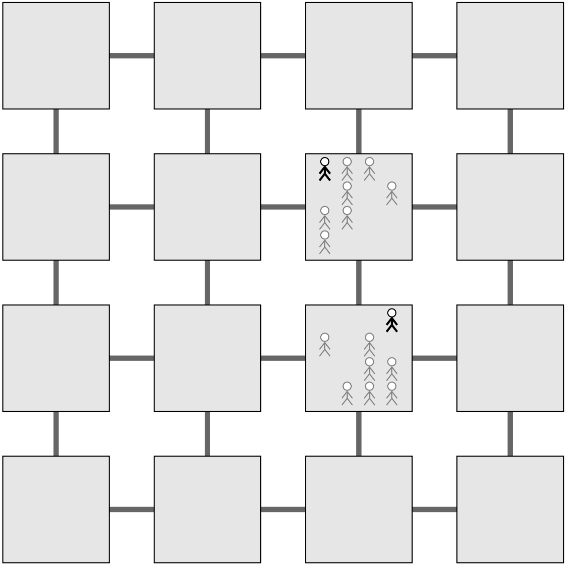
Coalescing Random Walks (Example)



Time: 23.5

Particles: 2

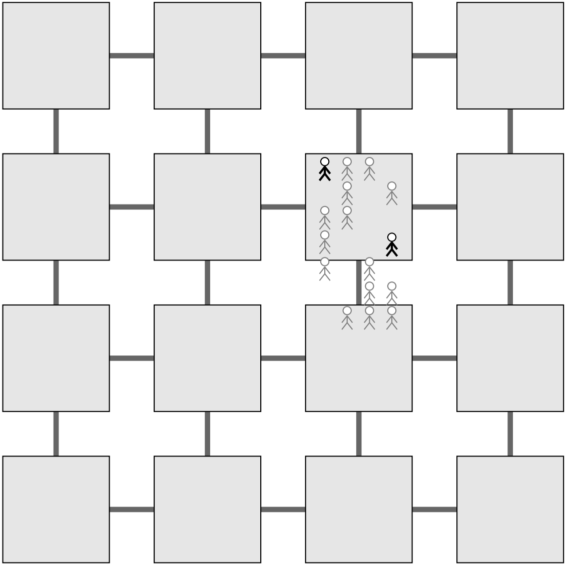
Coalescing Random Walks (Example)



Time: 24

Particles: 2

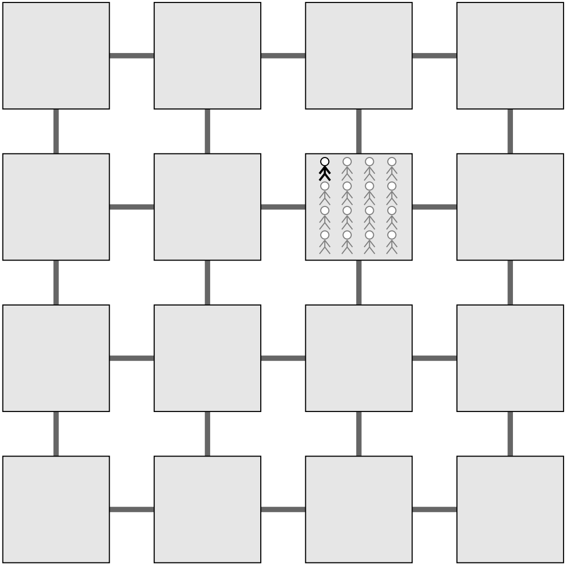
Coalescing Random Walks (Example)



Time: 24.5

Particles: 2

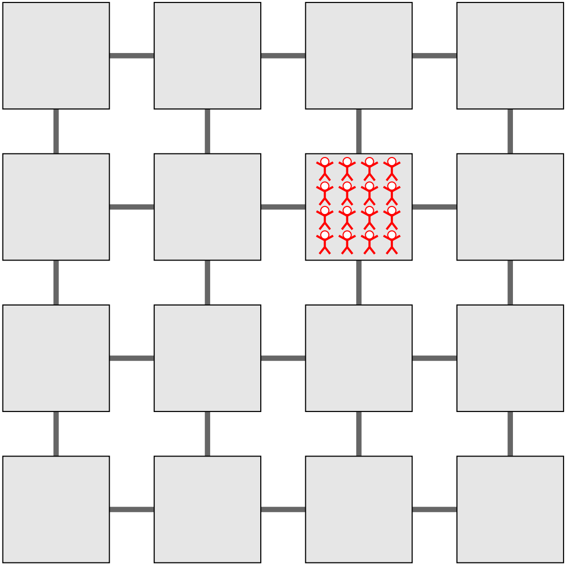
Coalescing Random Walks (Example)



Time: 25

Particles: 1

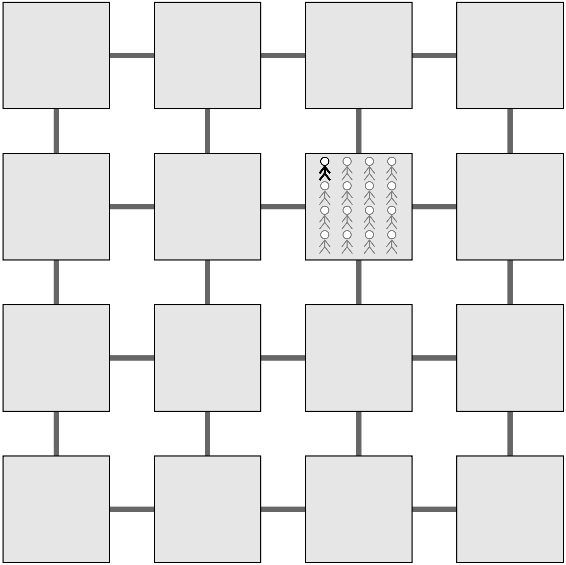
Coalescing Random Walks (Example)



Time: 25

Particles: 1

Coalescing Random Walks (Example)



Time: 25

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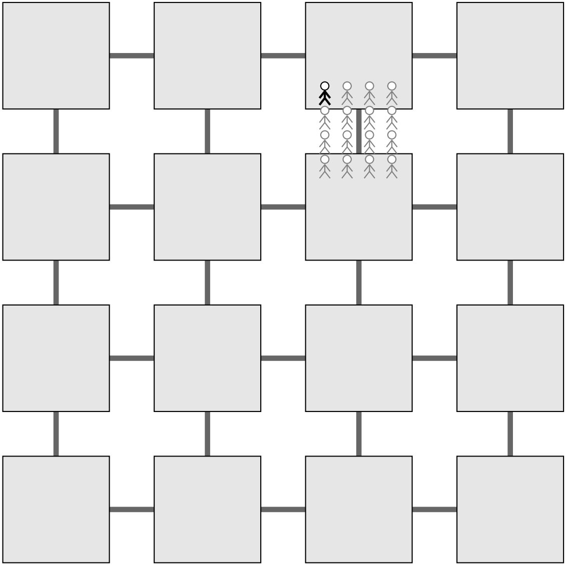

Time: 25.5

Particles: 1

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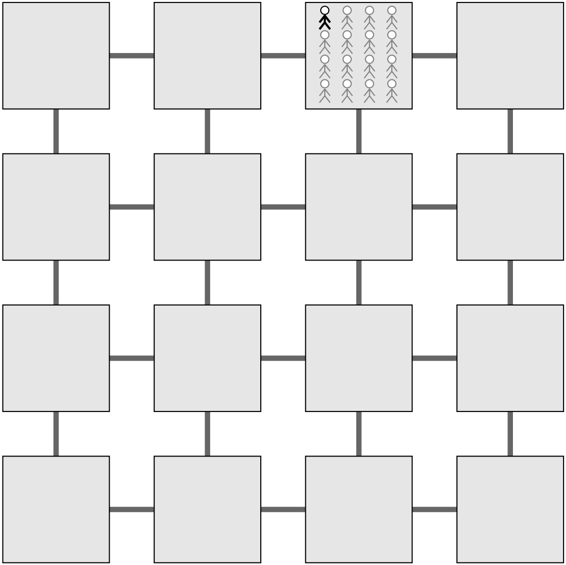

Time: 26

Particles: 1

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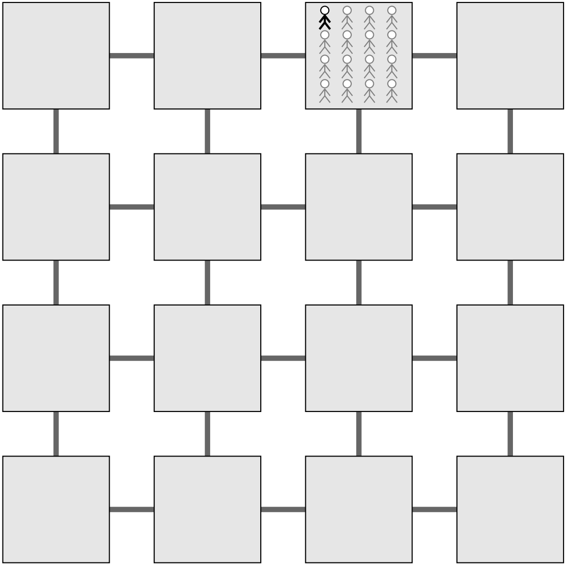

Time: 26.5

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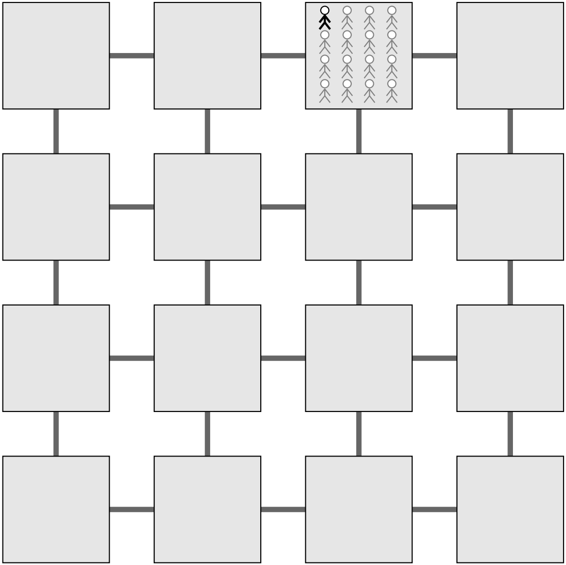

Time: 27

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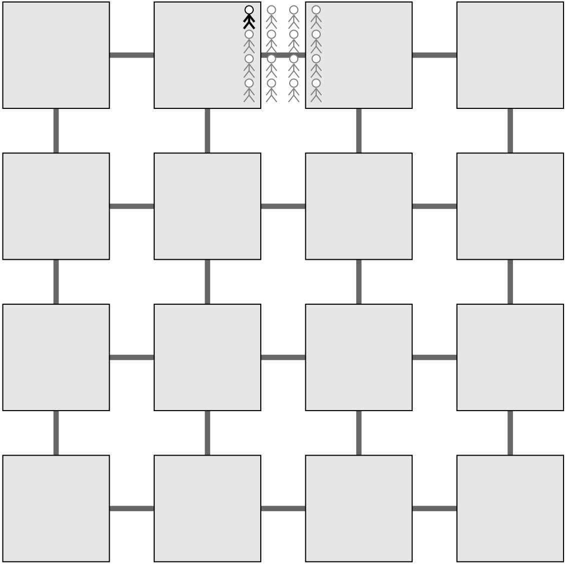

Time: 27.5

Particles: 1

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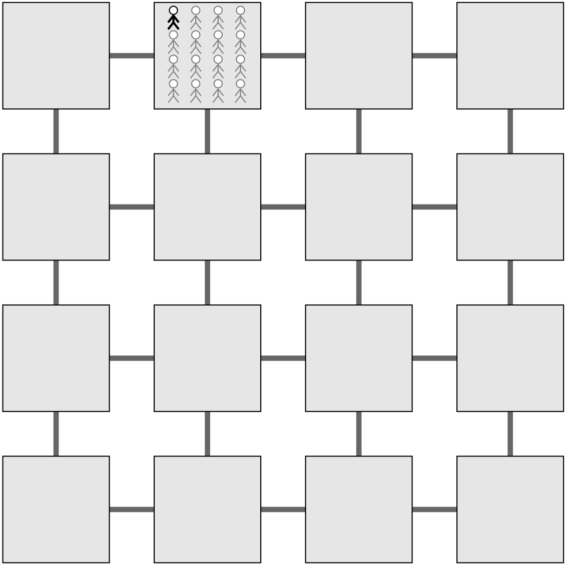

Time: 28

Particles: 1

Motivation: Voter Model

Voter Model

- Given a graph $G = (V, E)$ with n nodes, each with a different opinion
- At each round, each node "pulls" w.p. 1/2 the opinion of a random neighbor, otherwise keeps his current opinion.

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Time to reach consensus = Time for n coalescing particles to merge.

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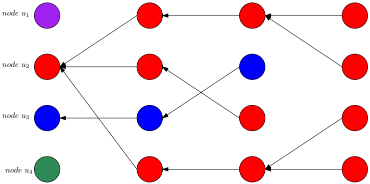

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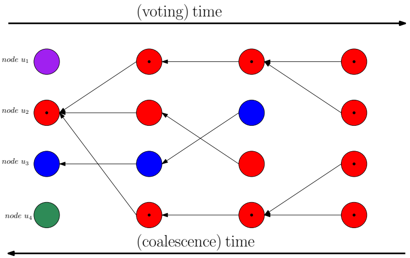

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For the discrete-time variant:

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- For any graph $t_{\text{coal}} \lesssim \frac{1}{\Phi} \cdot \frac{|E|}{\delta}$, where δ is the minimum degree *[Berenbrink, Giakkoupis, Kermarrec and Mallmann-Trenn, ICALP'16]*

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For the continuous-time variant:

- For any graph, $t_{\mathrm{coal}} \lesssim t_{\mathrm{hit}}$ *[Oliveira, TAMS'12]*
- (simplified) For graphs with $t_{\mathrm{mix}} \ll n$, t_{coal} behaves like on a clique *[Oliveira, Ann. Prob.'12]*

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- For many graphs, $t_{\mathrm{coal}} \asymp t_{\mathrm{meet}}$ or even $t_{\mathrm{coal}} \asymp n$ (if G is regular)
- Under the premise that t_{mix} and t_{meet} are "simpler" quantities, when does $t_{\mathrm{coal}} \asymp t_{\mathrm{meet}}$ hold?

Outline

Warm-Up: Complete Graph

For the continuous-time variant:

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Waiting times are i.i.d. exponentials with mean 1.

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For the discrete-time variant:

Answer "should be" $(\frac{8}{3} + o(1)) \cdot n$ for lazy random walks (loop probability 1/2)

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Theorem (Upper Bound)

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Application to "Real World" Graph Models

If the max-degree satisfies $\Delta \lesssim n/\log^3 n$ and $t_{\mathrm{mix}} \lesssim \log n$, then $t_{\mathrm{coal}} \asymp t_{\mathrm{meet}}$.

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Unfortunately we are not able to determine t_{meet}
(it is conceivable though that $t_{\text{meet}} \asymp 1/\|\pi\|_2^2$)

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Proof is quite technical, and we will only glance over one challenging part.

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$$\mathbf{Pr}\left[\bigcup_{\ell=1}^{j} \mathrm{int}(X, Y^{\ell}, \tau) \right] \geq 1 - (1-p)^j \qquad ??$$

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This is of course wrong, since the events are not independent!

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- Suppose $\mathbf{Pr}\left[\,(X_t)_{t=0}^{\tau} \in C_2\,\right] \geq \frac{p}{3}$. Then a p-fraction of all walks have a "good" trajectory that is hit by a stationary walk with probability at least $\sqrt{p}$ ...

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$$C_1 := \{(x_0, \ldots, x_\tau) \in \mathcal{T}_\tau \colon\ \mathbf{Pr}\left[\,\mathrm{int}(x, Y, \tau)\,\right] \geq \frac{p}{3}\}$$

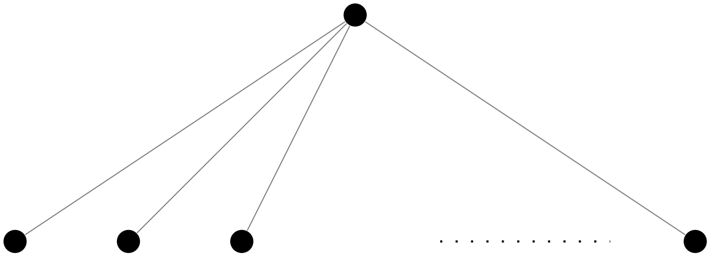
$$C_2 := \{(x_0, \ldots, x_\tau) \in \mathcal{T}_\tau \colon\ \mathbf{Pr}\left[\,\mathrm{int}(x, Y, \tau)\,\right] \geq \sqrt{p}\}.$$

clique (vertex-transitive graphs)

"asymmetric" graphs with core

- Then, $\mathbf{Pr}\left[\,(X_t)_{t=0}^{\tau} \in C_1\,\right] \geq \frac{\sqrt{p}}{3}$ or $\mathbf{Pr}\left[\,(X_t)_{t=0}^{\tau} \in C_2\,\right] \geq \frac{p}{3}$.
- Suppose $\mathbf{Pr}\left[\,(X_t)_{t=0}^{\tau} \in C_2\,\right] \geq \frac{p}{3}$. Then a p-fraction of all walks have a "good" trajectory that is hit by a stationary walk with probability at least $\sqrt{p}$ ...
- (Issue: Random walks coalesce and could therefore have terminated earlier!)

A Graph Demonstrating Tightness



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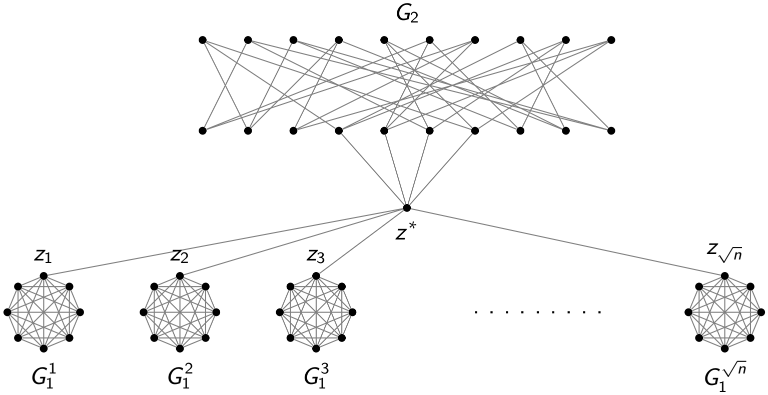

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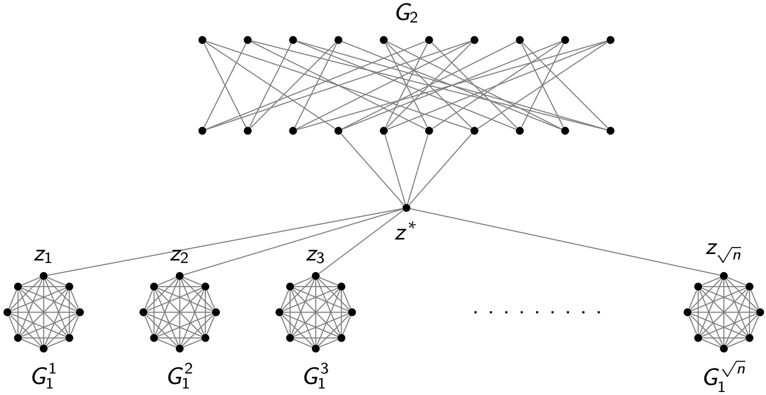
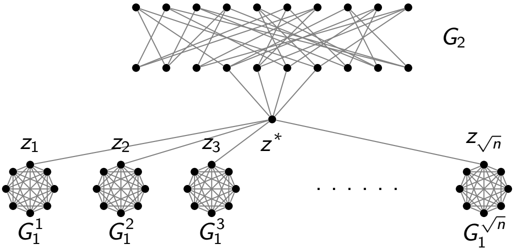


- G_1^i, $1 \le i \le \sqrt{n}$ are cliques over $\sqrt{n}$ nodes, where $\alpha = t_{\text{meet}}/t_{\text{mix}}$
- G_2 is a $\sqrt{n}$-regular Ramanujan graph on $n/\sqrt{\alpha}$ nodes
- Node z^* is connected to one designated node in each G_1^i and to $\sqrt{n/\alpha}$ distinct nodes in G_2

Intuition of the Construction



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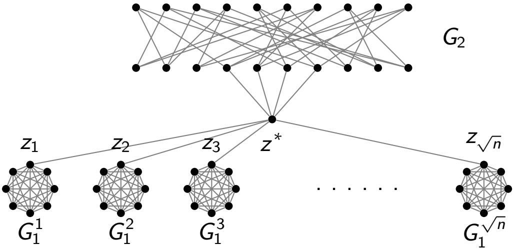

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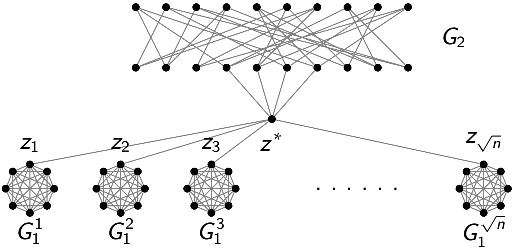

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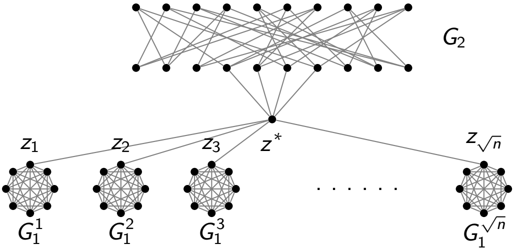

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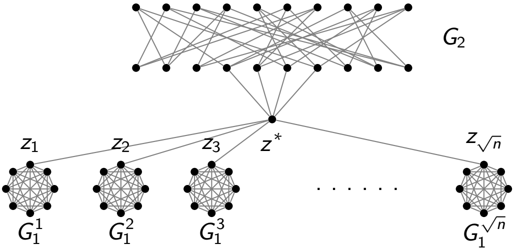

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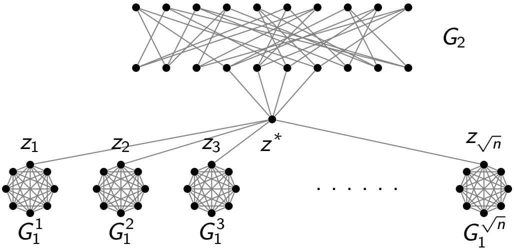

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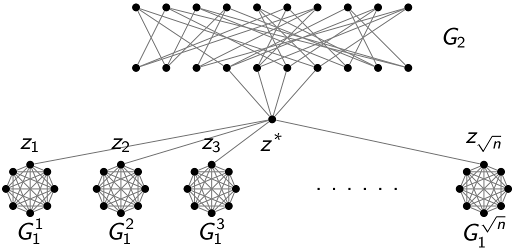

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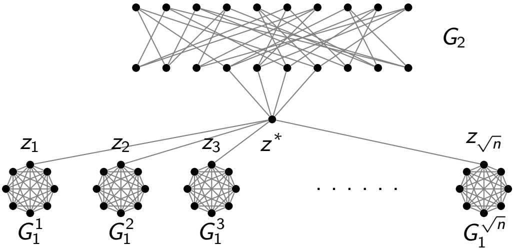

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For the example $t_{\mathsf{mix}} \asymp \sqrt{n}$, $t_{\mathsf{meet}} \asymp \alpha\sqrt{n}$ and $t_{\mathsf{coal}} \gtrsim \sqrt{\alpha \cdot n} \log n$:

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- However, for any vertex-transitive graph, $t_{\mathrm{coal}} \asymp t_{\mathrm{meet}} (\asymp t_{\mathrm{hit}})$

Outline

Application to Concrete Networks

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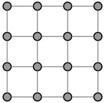
1D Grid

$t_{\text{mix}} \asymp n^2$

$t_{\text{hit}} \asymp t_{\text{meet}} \asymp n^2$

$t_{\text{coal}} \asymp n^2$ (✓)

2D Grid



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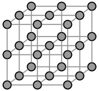

$t_{\text{mix}} \asymp n^{2/3}$

$t_{\text{hit}} \asymp t_{\text{meet}} \asymp n$

$t_{\text{coal}} \asymp n$ ✓

Application to Concrete Networks

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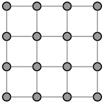

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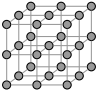

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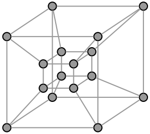
Hypercube

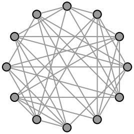


$t_{\mathrm{mix}} \asymp \log n \log\log n$

$t_{\mathrm{hit}} \asymp t_{\mathrm{meet}} \asymp n$

$t_{\mathrm{coal}} \asymp n$ $\checkmark$

Expander Graph



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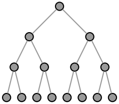
Binary Tree



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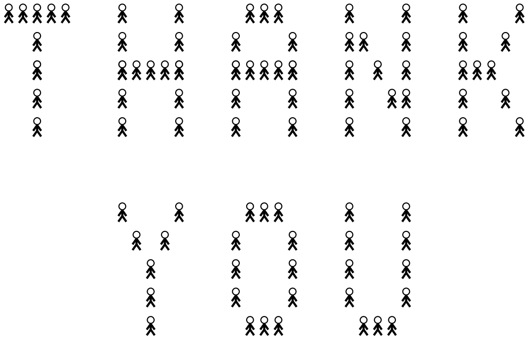
- Can we prove $t_{\text{coal}} \lesssim t_{\text{hit}}$ for all graphs?
- Is it true that $t_{\text{coal}}^{(\text{disc})} \asymp t_{\text{coal}}^{(\text{cont})}$ for any graph?
- Reduce the number of walks to some threshold $\kappa \in [1, n]$.
  Conjecture:
  - For any (regular) graph, no. walks can be reduced to $\sqrt{n}$ in $O(n)$ time.
  - More generally, it takes $O((n/\kappa)^2)$ time to go from n to κ.

The End



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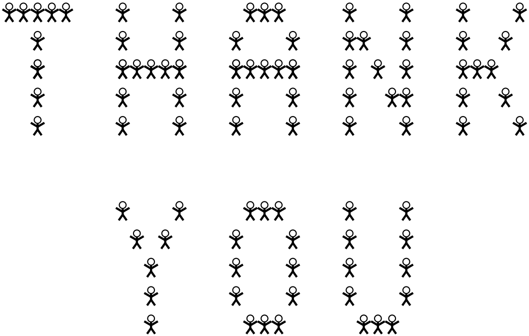

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Comments on the Cat-and-Mouse Game:

- Easier to deal with in the sense there is only one random object (the cat!)
- Clearly, $t_{\text{meet}} \lesssim t_{\text{cat-mouse}}$ and $t_{\text{hit}} \lesssim t_{\text{cat-mouse}}$.
  But do we have $t_{\text{cat-mouse}} \asymp t_{\text{hit}}$?