SIRSN and abstract scattering processes

LMS-EPSRC Durham Symposium on Markov processes, Mixing Times, and Cutoff

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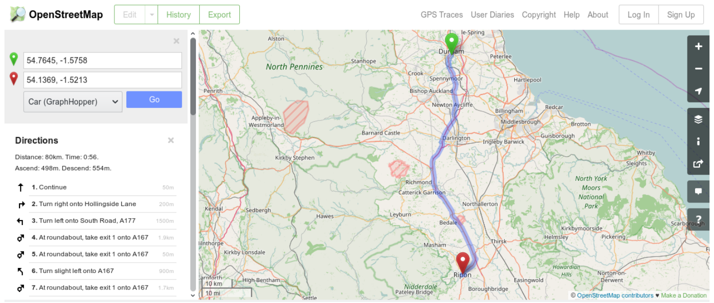
Online maps: from Durham to Ripon

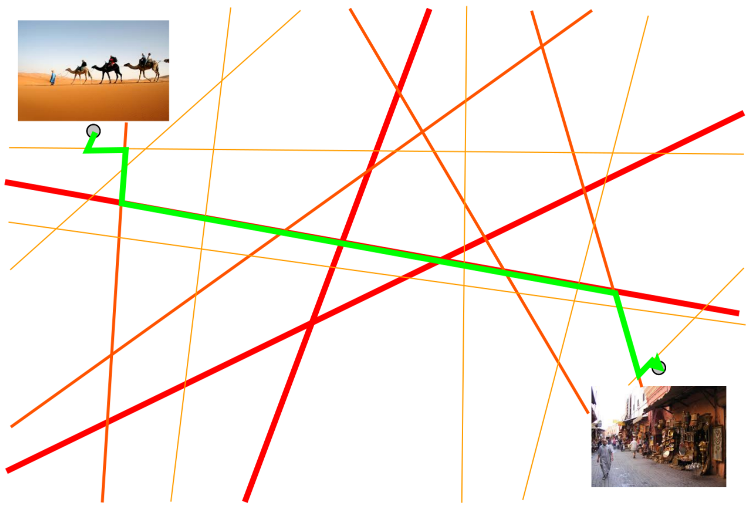


https://www.openstreetmap.org/

Tsukayama, Washington Post, January 30, 2013:
Estimated annual impact of online maps: $1.6tn.
Estimated growth 30% per year (smartphones!).

SIRSN – a very short introduction

A journey to the Soukh



$\frac{\gamma-1}{2} v^{-\gamma} \, \mathrm{d}v \quad \mathrm{d}r \quad \mathrm{d}\theta$ for $\gamma > 2$.

SIRSN axioms

What would a scale-invariant (random) network look like?
(Aldous, 2014; Aldous and Ganesan, 2013)
Input: set of nodes $x_1, \ldots, x_n$;
Output:
random network $N(x_1, \ldots, x_n)$ connecting nodes.

(1) Scale-invariance: for each Euclidean similarity λ,
$\mathcal{L}(N(\lambda x_1, \ldots, \lambda x_n)) = \mathcal{L}(\lambda N(x_1, \ldots, x_n))$.

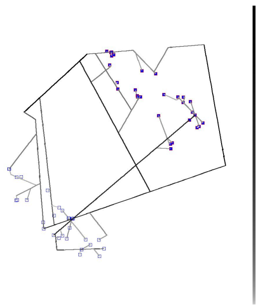
(2) If D_1 is length of fastest route between two points at unit distance apart then $\mathbb{E}[D_1] < \infty$.

(3) Weak SIRSN property: the network connecting points of (independent) unit intensity Poisson point process has finite average length per unit area.
(Strong) SIRSN property: the network connecting points of *dense* Poisson point process has finite average length per unit area of "long-range" part of network (more than distance 1 from source or destination).

Results and a question

- First SIRSN construction based on randomized dyadic lattice: Aldous (2014).
- Improper speed-marked Poisson line process Π produces random map and pre-SIRSN: WSK (2017)

  ...which is actually a strong SIRSN: Kahn (2016).



- But can the Π-geodesics pause *en route*?
- Hard question, so seek answer for "randomly broken" Π-geodesics?

Random wandering on a SIRSN

Options: (thanks to Banerjee, Croydon for helpful conversations!)

1. Random walk, connecting between successive jumps using Π-geodesics.

   (Con: Always stopping and starting!)
2. Brownian motion on the line pattern.

   (Con: Relationship with speed limits is not explicit.)
3. Rayleigh random flight:
   - Proceed at top speed along current line;
   - Switch to intersecting lines depending on relative speeds;
   - Choose new direction equi-probably.
4. This is SIRSN-RRF: a "randomly broken Π-geodesic".

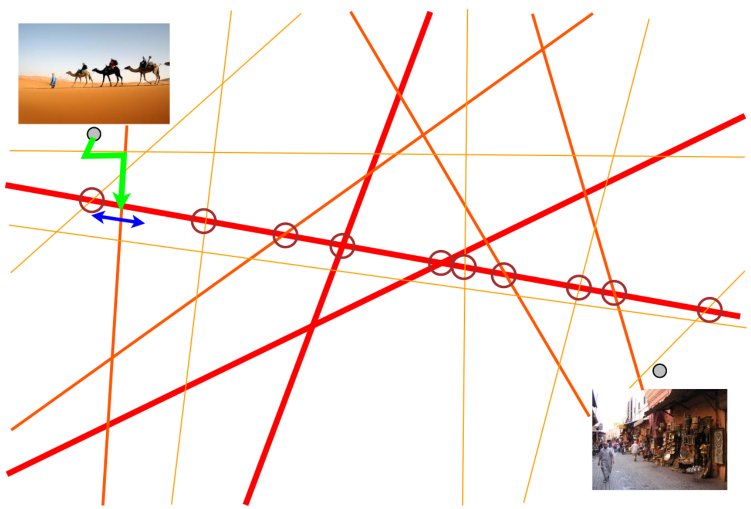
   Can it be made to be speed-neighbourhood-recurrent?

Scattering – an abstract approach (I)

- How to define a SIRSN-RRF?
  - Proceed at top speed along current line;
  - Switch to intersecting lines depending on relative speeds;
  - Choose new direction equi-probably.
- Suffices to sample process when it changes speed.
- Since Π is improper line process, opportunities to change speed are dense along each line!
- Convenient to take an abstract view:

  Advantages of axiomatic method "same as the advantages of theft over honest toil" (Russell, 1919, p.71).

Scattering – an abstract approach (sketch)



Scattering – an abstract approach (II)

Generalization (possibly of wider interest?):

1. A discrete-time Markov chain on countable state-space is an abstract scattering process if $p_{ab} = \omega_{ab} s_b$ for (symmetric) transmission probabilities ω_{ab} and scattering probabilities s_b (set $\omega_{a,\tilde{a}} = 0$, $s_a > 0$ for all a).
2. (Consider a matrix (ω_{ab}): if row-vectors $(\omega_{a\cdot})$ all lie in ℓ^1 then the Hahn-Banach theorem can be used to characterize whether (ω_{ab}) is transmission matrix.)
3. Require dynamically reversibility for measure π:
   Involution $a \leftrightarrow \tilde{a}$ preserving measure π,
   with $\pi_a p_{a,\tilde{b}} = \pi_b p_{b,\tilde{a}}$.
4. Relate $\pi_a / s_{\tilde{a}}$ to scatter-equivalence classes ("lines"): if there is a chain $a = b_0, b_1, \ldots, b_n = c$ with $\omega_{b_{m-1},\tilde{b}_m} > 0$, then $\pi_a / s_{\tilde{a}} = \pi_c / s_{\tilde{c}}$.

Scattering – an abstract approach (III)

5. Adopt "Metropolis-Hastings recipe": divide state-space into equivalence classes using ω,
   - set $\pi_a = \min\{\kappa, \kappa'\}$ where we choose κ, κ' as positive constants belonging to classes of a and $\tilde{a}$,
   - set scattering probability $s_a = \min\{1, \kappa/\kappa'\}$ (dynamic reversibility is then automatic!).
6. In case of a suitable total ordering $\prec$ for each "line", transmission probabilities are functions of scattering probabilities.
   For each $a \prec b$, there are $\omega_{a,\pm}$ summing to 1 with

$$p_{a,\tilde{b}} \;=\; \omega_{a,\tilde{b}} s_{\tilde{b}} \;=\; \omega_{a,+} \left(\prod_{a \prec c \prec b} (1 - s_{\tilde{c}}) \right) s_{\tilde{b}} \,,$$

   and similar for $p_{b,\tilde{a}}$ using $\omega_{b,-}$.
   All follows from choice of the class constants κ (and say equiprobable choice of direction $\omega_{a,\pm} = \frac{1}{2}$).

Application

Define the SIRSN-RRF, sampled at changes in direction, by specifying equilibrium probabilities at intersections of lines.

1. Scaling invariance: $\pi_{(\ell_1,\ell_2)} = \min\{v_1^\alpha, v_2^\alpha\}$, parameter α.
2. Scattering probability: $s_{(\ell_1,\ell_2)} = \min\{1, (v_2/v_1)^\alpha\}$.
3. Dynamical reversibility: non-symmetric Dirichlet form.
4. Apply Campbell-Slivnyak-Mecke theorem (twice!) to identify (translated, rotated, scaled) "environment viewed from particle" *via* reduced non-symmetric Dirichlet form.
5. Resulting log-relative-speed-changes X_1, X_2, ... form a stationary process.

Dynamical reversibility

Let $f(x, \Pi)$ be bounded, measurable, x an intersection of lines $\mathcal{L}_1$, $\mathcal{L}_2$ in Π.
For convenience set $\widetilde{f}(\mathcal{L}_1, \mathcal{L}_2; \Pi) = f(\mathcal{L}_2, \mathcal{L}_1; \Pi)$.
Consider the non-symmetric form

$$B(f,g) = \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\sum\sum_{\mathcal{L}_1 \neq \mathcal{L}_2 \in \Pi} \widetilde{f}(\widetilde{Z}_0; \Pi) \times g(Z_1; \Pi) \times \pi_x \;\middle|\; Z_0 = x = (\mathcal{L}_1, \mathcal{L}_2) \right] \;\middle|\; \Pi \right] \right].$$

Using Campbell-Mecke-Slivnyak theory twice, this can be reduced (taking out translations, rotations, scale-changes) to the study of

$$\mathbb{E}\Big[\sum_{\mathcal{L}_3 \in \Pi} f^{(2)}(\mathcal{L}_1^*; \Pi \cup \{\mathcal{L}_1^*\}) s_{(\tilde{\mathcal{L}}_0, \mathcal{L}_1^*)} \left(\prod_{\mathcal{L} \in \Pi:\ \mathcal{L} \text{ separates origin from } \tilde{\mathcal{L}}_0 \cap \mathcal{L}_3} (1 - s_{(\tilde{\mathcal{L}}_0, \mathcal{L})}) \right) \times s_{(\tilde{\mathcal{L}}_0, \mathcal{L}_3)} g^{(2)}(\mathcal{L}_3; \Pi \cup \{\mathcal{L}_1^*\}) \Big]$$

Read off equilibrium distribution from reduced non-symmetric form: at critical $\alpha = 2(\gamma - 1)$, typical log-relative-speed-change X has stationary symmetric Laplace distribution.

Outline of remainder of argument

- Adapt Kozlov (1985, Section 2) to show X is ergodic (depends on nice properties of Poisson line process!).
- Critical case $\mathbb{E}[X] = 0$, i.e. $\alpha = 2(\gamma - 1)$: apply continuum adaptation of Kesten-Spitzer-Whitman range theorem (Spitzer, 1976, Page 38) to show log-speed process $\sum X$ is neighbourhood recurrent.
- In this critical case $\alpha = 2(\gamma - 1)$, SIRSN-RRF provides a "randomly-broken Π-geodesic" which avoids slowing down to zero speed (or speeding up to infinite speed).

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Conclusion

- Critical case ($\alpha = 2(\gamma - 1)$): SIRSN-RRF speed is neighbourhood-recurrent.
- Sub-critical case ($\alpha < 2(\gamma - 1)$): SIRSN-RRF converges to a random limiting point in the plane (trapped by cells of tessellation of high-speed lines).
- Super-critical case ($\alpha > 2(\gamma - 1)$): SIRSN-RRF disappears off to infinity (consider high-speed tessellation).
- So a critical "randomly-broken Π-geodesic" does not halt *en route*. What about Π-geodesics themselves?

THANK YOU!

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