Higher Gauge Theory from Twistor Space

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Outline

Introduction and Motivation

Problem

One of the big challenges in M-theory is the formulation of the $\mathcal{N} = (2, 0)$ theory. This a chiral superconformal gauge theory in six dimensions with maximal $\mathcal{N} = (2, 0)$ supersymmetry. At the linearised level, we have:

- A potential 2-form B with curvature 3-form $H = \mathrm{d}B$ such that $H = \star_6 H$
- Five scalars ϕ^{IJ} such that $\Box\phi^{IJ} = 0$
- Four Weyl fermions ψ^I such that $\mathcal{D}\psi^I = 0$

Problem: How can this be promoted to an interacting non-Abelian theory?

Proposal: Combine twistor theory and categorified principal bundles.

Higher Gauge Algebras

NQ-Manifolds—Higher Gauge Algebras

- An NQ-manifold is a non-negatively graded manifold quipped with a nil-quadratic degree-one vector field Q.
- An NQ-manifold concentrated in degree one is a Lie algebra. Indeed, let ξ^α be local coordinates. The most general degree-one vector field Q is of the form

$$Q := \xi^\alpha \xi^\beta f_{\alpha\beta}{}^\gamma \frac{\partial}{\partial \xi^\gamma} ,$$

  with $f_{\alpha\beta}{}^\gamma$ constant. Then $Q^2 = 0$ is equivalent to requiring $f_{\alpha\beta}{}^\gamma$ to satisfy Jacobi. Thus, we obtain a Lie algebra with Q as its Chevalley–Eilenberg differential.

NQ-Manifolds—Higher Gauge Algebras

- An NQ-manifold in degree zero and one is a Lie algebroid. Indeed, such a manifold must be of the form $E[1] \to X$. Let (x^i, ξ^α) be local coordinates so that

$$Q := \xi^\alpha \rho^i_\alpha \frac{\partial}{\partial x^i} + \xi^\alpha \xi^\beta f_{\alpha\beta}{}^\gamma \frac{\partial}{\partial \xi^\gamma}~.$$

  Now $f_{\alpha\beta}{}^\gamma \in C^\infty(X)$ are structure functions of a Lie bracket $[-,-]$ on $\Gamma(E)$ and the $\rho^i_\alpha \in C^\infty(X)$ encode a map $\rho : E \to TX$. Then $Q^2 = 0$ implies that the $f_{\alpha\beta}{}^\gamma$ satisfy Jacobi, ρ is a Lie algebra homomorphism, and $[s_1, fs_2] = (\rho(s_1)f)s_2 + f[s_1, s_2]$ for all $f \in C^\infty(M)$ and $s_{1,2} \in \Gamma(E)$. Hence, this describes a Lie algebroid with Q as its Chevalley–Eilenberg differential.

- A k-term L_∞-algebroid is an NQ-manifold concentrated in degrees $0, 1, \ldots, k$. When concentrated in degrees $1, \ldots, k$ we call it a k-term L_∞-algebra.

NQ-Manifolds—Higher Gauge Algebras

- For $k = 1, 2$, let (ξ^α, η^i) be local coordinates. Then,

$$Q := \xi^\alpha \xi^\beta f_{\alpha\beta}{}^\gamma \frac{\partial}{\partial \xi^\gamma} + f_i{}^\alpha \eta^i \frac{\partial}{\partial \xi^\alpha} + f_{i\alpha}{}^j \xi^\alpha \eta_j \frac{\partial}{\partial \eta^i} + f_{\alpha\beta\gamma}{}^i \xi^\alpha \xi^\beta \xi^\gamma \frac{\partial}{\partial \eta^i} ,$$

where $f_{\alpha\beta}{}^\gamma$, $f_i{}^\alpha$, $f_{i\alpha}{}^j$, and $f_{\alpha\beta\gamma}{}^i$ are constants. Letting $\mathfrak{w}$ be a vector space with basis w_α and $\mathfrak{v}$ a vector space with basis v_i we may thus write

$$\mu_1(v_i) := f_i{}^\alpha w_\alpha , \quad \mu_2(w_\alpha, w_\beta) := f_{\alpha\beta}{}^\gamma w_\gamma ,$$
$$\mu_2(v_i, w_\alpha) := f_{i\alpha}{}^j v_j , \quad \mu_3(w_\alpha, w_\beta, w_\gamma) := f_{\alpha\beta\gamma}{}^i v_i ,$$

i.e. we obtain a 2-term complex $\mathfrak{v} \xrightarrow{\mu_1} \mathfrak{w}$ with

$$\mu_2 : \mathfrak{w} \wedge \mathfrak{w} \to \mathfrak{w} , \quad \mu_2 : \mathfrak{v} \wedge \mathfrak{w} \to \mathfrak{v} , \quad \mu_3 : \mathfrak{w} \wedge \mathfrak{w} \wedge \mathfrak{w} \to \mathfrak{v}$$

and Q^2 yields higher homotopy Jacobi identities e.g.

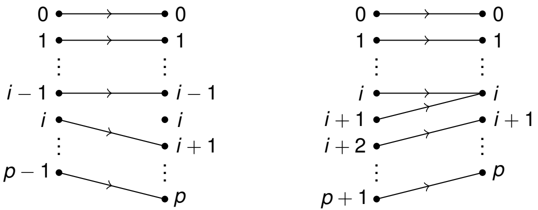
$$\mu_1(\mu_3(W_1, W_2, W_3)) = \mu_2(\mu_2(W_1, W_2), W_3) + \mu_2(\mu_2(W_3, W_1), W_2) + \mu_2(\mu_2(W_2, W_3), W_1)$$

Higher Gauge Group(oid)s

Simplex Category

- The simplex category Δ is the category that has finite totally ordered sets $[p] := \{0, 1, \ldots, p\}$ as objects and order-preserving maps as morphisms.
- The objects of Δ have a geometric realisation as standard topological simplices.
- The morphisms of Δ are generated by the coface maps, ϕ_i^p, and codegeneracy maps, δ_i^p, defined by

$$\phi_i^p : [p-1] \to [p] \qquad\qquad \delta_i^p : [p+1] \to [p]$$



Simplicial Sets and Manifolds

- A simplicial set (manifold) is a functor $\mathscr{X} : \Delta^{\mathrm{op}} \to \mathsf{Set}\ (\mathsf{Mfd})$.
- Hence, $\mathscr{X} = \bigcup_p \mathscr{X}_p$ and $\mathscr{X}_p := \mathscr{X}([p])$ is the set of simplicial p-simplices; the elements of $\mathscr{X}_0$ are the vertices of $\mathscr{X}$. We obtain the face maps, $\mathsf{f}_i^p := \mathscr{X}(\phi_i^p) : \mathscr{X}_p \to \mathscr{X}_{p-1}$, and the degeneracy maps, $\mathsf{d}_i^p := \mathscr{X}(\delta_i^p) : \mathscr{X}_p \to \mathscr{X}_{p+1}$ subject to the simplicial identities

$$\mathsf{f}_i \circ \mathsf{f}_j = \mathsf{f}_{j-1} \circ \mathsf{f}_i \ \text{ for } \ i < j\,, \quad \mathsf{d}_i \circ \mathsf{d}_j = \mathsf{d}_{j+1} \circ \mathsf{d}_i \ \text{ for } \ i \leq j\,,$$
$$\mathsf{f}_i \circ \mathsf{d}_j = \mathsf{d}_{j-1} \circ \mathsf{f}_i \ \text{ for } \ i < j\,, \quad \mathsf{f}_i \circ \mathsf{d}_j = \mathsf{d}_j \circ \mathsf{f}_{i-1} \ \text{ for } \ i > j+1\,,$$
$$\mathsf{f}_i \circ \mathsf{d}_i = \mathrm{id} = \mathsf{f}_{i+1} \circ \mathsf{d}_i\ .$$

- Depict simplicial sets by writing arrows for the face maps,

$$\Big\{ \cdots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \mathscr{X}_2 \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \mathscr{X}_1 \rightrightarrows \mathscr{X}_0 \Big\}\,.$$

- For a ordinary set X write $\Big\{ \cdots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} X \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} X \rightrightarrows X \Big\}$ with all face and degeneracy maps identities. Such a set is called a simplicially constant simplicial set.

Simplicial Maps and Homotopies

- A simplicial map between simplicial sets is a natural transformation between the defining functors.
- The standard simplicial p-simplex, Δ^p, is the simplicial set $\mathrm{hom}_\Delta(-,[p]) : \Delta^{\mathrm{op}} \to \mathsf{Set}$.
- For any simplicial set $\mathscr{X} = \bigcup_p \mathscr{X}_p$, one can show that $\mathscr{X}_p \cong \mathrm{hom}_{\mathsf{sSet}}(\Delta^p, \mathscr{X})$.
- For two simplicial sets $\mathscr{X}$ and $\mathscr{Y}$, a simplicial homotopy between two simplicial maps $g, \tilde{g} : \mathscr{X} \to \mathscr{Y}$ is a simplicial map $h : \mathscr{X} \times \Delta^1 \to \mathscr{Y}$ that renders

$$\begin{tikzcd}
\mathscr{X} \times \Delta^0 \cong \mathscr{X} \arrow[d, "\mathrm{id}\times\phi^1_1"'] \arrow[drr, "g"] & & \\
\mathscr{X} \times \Delta^1 \arrow[rr, "h"] & & \mathscr{Y} \\
\mathscr{X} \times \Delta^0 \cong \mathscr{X} \arrow[u, "\mathrm{id}\times\phi^1_0"] \arrow[urr, "\tilde{g}"'] & &
\end{tikzcd}$$

commutative. Here, ϕ^1_0 and ϕ^1_1 are the coface maps.

Kan Simplicial Sets and Manifolds

- For each i, the (p, i)-horn Λ^p_i of Δ^p is the simplicial subset of Δ^p given by all faces of Δ^p except for the i-th one. The (p, i)-horns of a simplicial set $\mathscr{X}$ is the set $\mathrm{hom}_{\mathsf{sSet}}(\Lambda^p_i, \mathscr{X})$.
- The horns Λ^p_i of Δ^p can always be filled (i.e. completed) to Δ^p. For a simplicial set $\mathscr{X}$ this is, in general, not the case.
- A Kan simplicial set is a simplicial set such that any horn $\lambda : \Lambda^p_i \to \mathscr{X}$ can be filled, that is,

$$\begin{tikzcd} \Lambda^p_i \arrow[r, "\lambda"] \arrow[d, hook] & \mathscr{X} \\ \Delta^p \arrow[ur, "\delta"] & \end{tikzcd}$$

  is commutative. Put differently, the natural restrictions

$$\mathscr{X}_p \cong \mathrm{hom}_{\mathsf{sSet}}(\Delta^p, \mathscr{X}) \to \mathrm{hom}_{\mathsf{sSet}}(\Lambda^p_i, \mathscr{X})$$

  are surjective. For a simplicial manifold, these are surjective submersions.

Quasi-Groupoids

- Let $\mathscr{X}$ and $\mathscr{Y}$ be two simplicial sets. Consider the relation $g \sim \tilde{g}$ on the set of all simplicial maps between $\mathscr{X}$ and $\mathscr{Y}$ defined by saying that g is related to $\tilde{g}$ whenever there exists a simplicial homotopy from g to $\tilde{g}$. If $\mathscr{Y}$ is a Kan simplicial set then this is an equivalence relation.
- A quasi-groupoid is a Kan simplicial set. A Lie quasi-groupoid is a Kan simplicial manifold. A Lie k-quasi-groupoid is a Lie quasi-groupoid for which the (p, i)-horns can be filled uniquely for $p > k$, $i \in \{0, \ldots, p\}$.
- Every Lie k-quasi-groupoid differentiates to a k-term L_∞-algebroid following a method due to Ševera in which the algebroid is given as the 1-jet of the quasi-groupoid.

Examples $k = 1$

Let $f : Y \to X$ be a surjective submersion between two manifolds Y and X. Consider

$$Y \times_X Y := \{(y_1, y_2) \in Y \times Y \,|\, f(y_1) = f(y_2)\} \,.$$

- The Čech groupoid $\check{\mathcal{C}}(Y \to X)$ is the Lie groupoid $Y \times_X Y \rightrightarrows Y$ with pairs $(y_1, y_2) \in Y \times_X Y$ as its morphisms and

$$\mathsf{s}(y_1, y_2) := y_2 \,, \quad \mathsf{t}(y_1, y_2) := y_1 \,, \quad \mathrm{id}_y := (y, y) \,,$$
$$(y_1, y_2) \circ (y_2, y_3) := (y_1, y_3) \,.$$

- The Čech nerve of the Čech groupoid $\check{\mathcal{C}}(Y \to X)$ is the Lie 1-quasi-groupoid

$$N(\check{\mathcal{C}}(Y \to X)) := \{\cdots \substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow} Y \times_X Y \times_X Y \substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow} Y \times_X Y \rightrightarrows Y\} \,,$$

with face and degeneracy maps given by

$$\mathsf{f}^p_i(y_0, \ldots, y_p) := (y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_p) \,,$$
$$\mathsf{d}^p_i(y_0, \ldots, y_p) := (y_0, \ldots, y_{i-1}, y_i, y_i, \ldots, y_p) \,.$$

Let G be a Lie group.

- The delooping BG is the Lie groupoid $G \rightrightarrows *$, where the source and target maps are trivial, $\mathrm{id}_* = \mathbb{1}_G$, and the composition is group multiplication in G.
- The nerve $N(BG)$ of the delooping BG is the Lie 1-quasi-groupoid

$$N(BG) := \{ \cdots \substack{\textstyle\rightarrow\\[-0.3em]\textstyle\rightarrow\\[-0.3em]\textstyle\rightarrow\\[-0.3em]\textstyle\rightarrow} G \times G \substack{\textstyle\rightarrow\\[-0.3em]\textstyle\rightarrow\\[-0.3em]\textstyle\rightarrow} G \rightrightarrows * \}$$

with the obvious face and degeneracy maps.

Higher Principal Bundles

Principal Bundles

- For G a Lie group, a principal G-bundle over a manifold X subordinate to a surjective submersion $Y \to X$ is a simplicial map $g : N(\check{\mathcal{C}}(Y \to X)) \to N(BG)$.
- Take an ordinary cover $\bigcup_a \{(x, a) | x \in U_a\} \to X$ so that the set of morphisms of the corresponding Čech groupoid is $\bigcup_{a,b} \{(x, a, b) | x \in U_a \cap U_b\}$ with the composition $(x, a, b) \circ (x, b, c) = (x, a, c)$.
- Hence, a simplicial map $g : N(\check{\mathcal{C}}(Y \to X)) \to N(BG)$ consists of

$$g_a(x) := g^0(x, a) = * \ , \quad g_{ab}(x) := g^1(x, a, b) \in G \ ,$$
$$g_{abc}(x) := g^2(x, a, b, c) = (g^1_{abc}(x), g^2_{abc}(x)) \in G \times G \ , \quad \text{etc.}$$

and as it commutes with the face and degeneracy maps,

$$g^1_{abc}(x) = g_{ab}(x) \ , \quad g^1_{abc}(x) g^2_{abc}(x) = g_{ac}(x) \ , \quad g^2_{abc}(x) = g_{bc}(x) \ .$$

Higher Groupoid Bundles

- Since, in addition, homotopies yield equivalent bundles, we give the following definition ...
- For $\mathscr{G}$ a Lie quasi-groupoid, a Lie quasi-groupoid bundle or principal $\mathscr{G}$-bundle over X subordinate to a surjective submersion $Y \to X$ is a simplicial map $g : N(\check{C}(Y \to X)) \to \mathscr{G}$. Two such principal $\mathscr{G}$-bundles $g, \tilde{g} : N(\check{C}(Y \to X)) \to \mathscr{G}$ are called equivalent if and only if there is a simplicial homotopy between g and $\tilde{g}$.
- This can be generalised to higher bases spaces i.e. base spaces which are Kan simplicial manifolds.

Higher non-Abelian Deligne Cohomology

- Generalising the above construction, we can also infer the connective structure on such principal $\mathscr{G}$-bundles as well its patching transformations. The latter follow from computing the 1-jet of the simplicial manifold $\underline{\mathrm{hom}}(\Delta^1, \mathscr{G})$ appearing in

$$\mathrm{hom}_{\mathsf{sSMfd}}(\mathscr{X} \times \Delta^1, \mathscr{G}) \cong \mathrm{hom}_{\mathsf{sSMfd}}(\mathscr{X}, \underline{\mathrm{hom}}(\Delta^1, \mathscr{G}))$$

- Let $\mathscr{G}$ be a Lie 2-quasi group with the induced 2-term L_∞ algebra $\mathfrak{v} \xrightarrow{\mu_1} \mathfrak{w}$. Let $\bigcup_a \{(x,a)|x \in U_a\} \to X$. A Deligne cocycle describing a principal $\mathscr{G}$-bundle with connective structure consists of the transition functions $\{g_{ab}, g_{abc}, \Lambda_{ab}\}$ with $\Lambda_{ab} \in \Omega^1(U_a \cap U_b) \otimes \mathfrak{w}$ and the connective structure $\{A_a, B_a\} \in \Omega^1(U_a) \otimes \mathfrak{w} \oplus \Omega^2(U_a) \otimes \mathfrak{v}$ with curvatures

$$\mathcal{F}_a := \mathrm{d}A_a + \tfrac{1}{2}\mu_2(A_a, A_a) - \mu_1(B_a) \ ,$$
$$H_a := \mathrm{d}B_a + \mu_2(A_a, B_a) + \tfrac{1}{3!}\mu_3(A_a, A_a, A_a) \ .$$

$6D$ Self-Dual Higher Gauge Theory

Twistor Space

- Consider $\mathcal{N} = (2,0)$ superspace $M := \mathbb{C}^{6|16}$ with coordinates (x^{AB}, η_I^A) with $A, B, \ldots = 1, \ldots, 4$ and $I, J, \ldots = 1, \ldots, 4$. Then,

  $$P_{AB} := \partial_{AB} \ , \quad D_A^I := \partial_A^I - 2\Omega^{IJ}\eta_J^B \partial_{AB}$$

  have the non-vanishing (anti-)commutation relations

  $$\{D_A^I, D_B^J\} = -4\Omega^{IJ} P_{AB} \ .$$

- Define the correspondence space F to be $F := \mathbb{C}^{4|16} \times \mathbb{P}^3$ with coordinates $(x^{AB}, \eta_I^A, \lambda_A)$.
- Introduce a rank-3|12 distribution $\langle V^A, V^{I\,AB} \rangle \hookrightarrow TF$ by $V^A := \lambda_B \partial^{AB}$ and $V^{I\,AB} := \frac{1}{2}\varepsilon^{ABCD}\lambda_C D_D^I$ which is integrable. Hence, we have foliation $P := F/\langle V^A, V^{I\,AB} \rangle$.

Twistor Space

- On P, we may use coordinates (z^A, η_I, λ_A) with $z^A\lambda_A = \Omega^{IJ}\eta_I\eta_J$ and thus

$$\begin{tikzcd} & F \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ P & & M \end{tikzcd}$$

  with π_2 being the trivial projection and

$$\pi_1 : (x^{AB}, \eta_I^A, \lambda_A) \;\mapsto\; (z^A, \eta_I, \lambda_A) = \\ = ((x^{AB} + \Omega^{IJ}\eta_I^A\eta_J^B)\lambda_B, \eta_I^A\lambda_A, \lambda_A)$$

- A point $x \in M$ corresponds to a $\mathbb{P}^3$ in P, while a point $p \in P$ corresponds to a 3|12-superplane

$$x^{AB} = x_0^{AB} + \varepsilon^{ABCD}\mu_C\lambda_D + 2\Omega^{IJ}\varepsilon^{CDE[A}\lambda_C\theta_{IDE}\eta_{0J}^{B]} ,$$
$$\eta_I^A = \eta_{0I}^A + \varepsilon^{ABCD}\lambda_B\theta_{ICD} .$$

Let $\mathscr{G}$ be a Lie 2-quasi-group. There is a bijection between equivalence classes

(i) of holomorphic M-trivial principal $\mathscr{G}$-bundles on P and

(ii) of solutions to the constraint system on the chiral superspace M

$$
\begin{aligned}
F_A{}^B &= \mu_1(B_A{}^B)\ , \quad F_{AB}{}^I_C = \mu_1(B_{AB}{}^I_C)\ , \quad F^{IJ}_{AB} = \mu(B^{IJ}_{AB})\ ,\\
H^{AB} &= 0\ ,\\
H_A{}^{BI}{}_C &= \delta^B_C\psi^I_A - \tfrac{1}{4}\delta^B_A\psi^I_C\ ,\\
H_{AB}{}^{IJ}_{CD} &= \varepsilon_{ABCD}\phi^{IJ}\ , \quad \text{with} \quad \phi^{IJ}\Omega_{IJ} = 0\\
H^{IJK}_{ABC} &= 0\ .
\end{aligned}
$$

This is a quasi-isomorphism of L_∞-algebras.

$4D$ Super Yang–Mills Theory

Ambitwistor Space

- Consider $M := \mathbb{C}^{4|12}$ with coordinates $(x^{\alpha\dot\alpha}, \theta^{i\alpha}, \eta_i^{\dot\alpha})$ where $\alpha, \dot\alpha, \ldots = 1, 2$ and $i, j, \ldots = 1, \ldots, 3$. Then,

$$P_{\alpha\dot\alpha} := \partial_{\alpha\dot\alpha}\ , \quad D_{i\alpha} := \partial_{i\alpha} + \eta_i^{\dot\alpha}\partial_{\alpha\dot\alpha}\ , \quad D^i_{\dot\alpha} := \partial^i_{\dot\alpha} + \theta^{i\alpha}\partial_{\alpha\dot\alpha}$$

  have the non-vanishing (anti-)commutation relations

$$\{D_{i\alpha}, D^j_{\dot\alpha}\} = 2\delta_i^j P_{\alpha\dot\alpha}\ .$$

- Define $F := \mathbb{C}^{4|12} \times \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $(x^{\alpha\dot\alpha}, \theta^{i\alpha}, \eta_i^{\dot\alpha}, \mu_\alpha, \lambda_{\dot\alpha})$.
- Introduce a rank-$1|6$ distribution $\langle V, V_i, V^i\rangle \hookrightarrow TF$ by $V := \mu^\alpha\lambda^{\dot\alpha}\partial_{\alpha\dot\alpha}$, $V_i := \mu^\alpha D_{i\alpha}$, and $V^i := \lambda^{\dot\alpha} D^i_{\dot\alpha}$ which is integrable. Hence, we have foliation $L := F/\langle V, V_i, V^i\rangle$.

Ambitwistor Space

- On L, we may use coordinates $(z^\alpha, w^{\dot\alpha}, \theta^i, \eta_i, \mu_\alpha, \lambda_{\dot\alpha})$ with $z^\alpha \mu_\alpha - w^{\dot\alpha}\lambda_{\dot\alpha} = 2\theta^i \eta_i$ and thus

$$\begin{tikzcd} & F \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ L & & M \end{tikzcd}$$

  with π_2 being the trivial projection and

$$\pi_1 : (x^{\alpha\dot\alpha}, \theta^{i\alpha}, \eta_i^{\dot\alpha}, \mu_\alpha, \lambda_{\dot\alpha}) \;\mapsto\; (z^\alpha, w^{\dot\alpha}, \theta^i, \eta_i, \mu_\alpha, \lambda_{\dot\alpha}) =$$
$$= ((x^{\alpha\dot\alpha} - \theta^{i\alpha}\eta_i^{\dot\alpha})\lambda_{\dot\alpha}, (x^{\alpha\dot\alpha} + \theta^{i\alpha}\eta_i^{\dot\alpha})\mu_\alpha, \theta^{i\alpha}\mu_\alpha, \eta_i^{\dot\alpha}\lambda_{\dot\alpha}, \mu_\alpha, \lambda_{\dot\alpha})$$

- A point $x \in M$ corresponds to a $\mathbb{P}^1 \times \mathbb{P}^1$ in L, while a point $p \in L$ corresponds to a $1|6$-superline

$$x^{\alpha\dot\alpha} = x_0^{\alpha\dot\alpha} + t\mu^\alpha\lambda^{\dot\alpha} + t^i\mu^\alpha\eta_i^{\dot\alpha} - t_i\theta^{i\alpha}\lambda^{\dot\alpha}\ ,$$
$$\theta^{i\alpha} = \theta_0^{i\alpha} + t^i\mu^\alpha\ , \quad \eta_i^{\dot\alpha} = \eta_{0\,i}^{\dot\alpha} + t_i\lambda^{\dot\alpha}\ .$$

Penrose–Ward Transform: $L \stackrel{\pi_1}{\leftarrow} F \stackrel{\pi_2}{\rightarrow} M$

Due to Witten and Isenberg–Yasskin–Green we have the following result. Let G be a Lie group. There is a bijection between equivalence classes

(i) of holomorphic M-trivial principal G-bundles on L and

(ii) of solutions to the constraint system of maximally supersymmetric Yang–Mills theory on M

$$F_{i\alpha j\beta} = \epsilon_{\alpha\beta}\epsilon_{ijk}\phi^k \ , \quad F^{ij}_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\beta}\epsilon^{ijk}\phi_k \ , \quad F_{i\alpha}{}^{j}_{\dot\beta} \ = \ 0 \ .$$

To prove this theorem, one makes use of the Čech description of holomorphic principal bundles. This is an intrinsically on-shell approach as the holomorphicity of the bundles encodes the equations of motion. How do we go off-shell?

Dolbeault Approach and Higher Gauge Theory

- To go off-shell, we make use of the Dolbeault approach. In particular, a holomorphic principal G-bundle can be described by a smooth principal G-bundle equipped with a $(0,1)$-connection locally given by a $\mathrm{Lie}(G)$-valued $(0,1)$-form $A^{0,1}$ subject to

$$F^{0,2} \;=\; \bar{\partial} A^{0,1} + \tfrac{1}{2}[A^{0,1}, A^{0,1}] = 0\ .$$

- For a three-dimensional Calabi–Yau manifold, this equation is variational as it follows from the holomorphic Chern–Simons action functional

$$S := \int \Omega^{3,0} \wedge \mathrm{tr}\left\{ A^{0,1} \wedge \bar{\partial} A^{0,1} + \tfrac{2}{3} A^{0,1} \wedge A^{0,1} \wedge A^{0,1} \right\}.$$

- Ambitwistor space is a Calabi–Yau supermanifold, however, its bosonic part is five-dimensional, and so we cannot use this action functional.

Dolbeault Approach and Higher Gauge Theory

- We propose to consider higher holomorphic Chern–Simons theory which we can motivate from string field theory of the B type topological sigma model on higher-dimensional Calabi–Yau spaces.
- Let $\mathscr{G}$ be a Lie 3-quasi-group. Consider a smooth principal $\mathscr{G}$-bundle equipped with Lie($\mathscr{G}$)-valued $(0,p|0)$-forms $A^{0,1|0}$, $B^{0,2|0}$, and $C^{0,3|0}$ with

$$
\begin{aligned}
S := \int \Omega^{5|6,0} \wedge \Big\{ & \langle A^{0,1|0}, \bar\partial C^{0,3|0}\rangle + \langle B^{0,2|0}, \mu_1(C^{0,3|0})\rangle + \\
& + \tfrac{1}{2}\langle B^{0,2|0}, \bar\partial B^{0,2|0}\rangle + \tfrac{1}{2}\langle A^{0,1|0}, \mu_2(A^{0,1|0}, C^{0,3|0})\rangle + \\
& + \tfrac{1}{2}\langle A^{0,1|0}, \mu_2(B^{0,2|0}, B^{0,2|0})\rangle + \\
& + \tfrac{1}{3!}\langle A^{0,1|0}, \mu_3(A^{0,1|0}, A^{0,1|0}, B^{0,2|0})\rangle + \\
& + \tfrac{1}{5!}\langle A^{0,1|0}, \mu_4(A^{0,1|0}, A^{0,1|0}, A^{0,1|0}, A^{0,1|0})\rangle \Big\}\,,
\end{aligned}
$$

where the fermionic integration is in the sense of Berezin.

Dolbeault Approach and Higher Gauge Theory

- The corresponding equations of motion are

$$\bar\partial A^{0,1|0} + \tfrac{1}{2}\mu_2(A^{0,1|0}, A^{0,1|0}) + \mu_1(B^{0,2|0}) = 0\ ,$$

$$\begin{gathered}\bar\partial B^{0,2|0} + \mu_2(A^{0,1|0}, B^{0,2|0}) + \\ + \tfrac{1}{3!}\mu_3(A^{0,1|0}, A^{0,1|0}, A^{0,1|0}) + \mu_1(C^{0,3|0}) = 0\ ,\end{gathered}$$

$$\begin{gathered}\bar\partial C^{0,3|0} + \mu_2(A^{0,1|0}, C^{0,3|0}) + \tfrac{1}{2}\mu_2(B^{0,2|0}, B^{0,2|0}) + \\ + \tfrac{1}{2}\mu_3(A^{0,1|0}, A^{0,1|0}, B^{0,2|0}) + \tfrac{1}{4!}\mu_4(A^{0,1|0}, A^{0,1|0}, A^{0,1|0}, A^{0,1|0}) = 0\ .\end{gathered}$$

- Every L_∞-algebra is quasi-isomorphic to an L_∞-algebra which has $\mu_1 = 0$ (Minimal Model Theorem). For this algebra, the first equation turns into

$$\bar\partial A^{0,1|0} + \tfrac{1}{2}\mu_2(A^{0,1|0}, A^{0,1|0}) = 0\ ,$$

and by means of the Penrose–Ward transform this will correspond to maximally supersymmetric Yang–Mills theory in four dimensions.

Conclusions and Outlook

Summary

In general, we have seen that the area of twistor geometry and categorified principal bundles can be fruitfully combined to formulate self-dual higher gauge theory in six dimensions.

The advantage of twistor geometry is that the e.o.m. and the gauge transformations follow directly from complex algebraic data on twistor space.

Furthermore, we have seen that higher gauge theory enables us to write down a twistor action principle for maximally supersymmetric Yang–Mills theory in four dimensions.

Many open questions remain, such as the choice of higher gauge group, the explicit constructions of higher bundles, including the dimensional reductions.

Thank You!