From Higher-Spin Gauge Theory to Strings

M.A.Vasiliev

Lebedev Institute, Moscow

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Plan

General introduction

HS symmetries and space-time

Unfolded dynamics and $\mathcal{L}_\infty$

Nonlinear HS theories

From Coxeter HS theories to strings and beyond

Conclusions

Quantum Gravity Challenge

QG effects should matter at ultrahigh energies of Planck scale

$$m_P^2 = \frac{hc}{G} \qquad m_P \sim 10^{19} GeV$$

To be compared with the energies $\sim 10^3 GeV$ available at CERN

Why should we care?

To reach better understanding of the fundamental theory

Thoughtful lesson: special relativity and Einstein formula $E = mc^2$

Strategy

Given little chances to test QGR experimentally how can we hope to understand it?

A unique chance is to conjecture that the regime of ultra high (trans-Planckian) energies exhibits some high symmetries that are spontaneously broken at low energies

The idea is to understand what kind of higher symmetries can be introduced in relativistic theory and to see consequences

HS gauge theory: theory of maximal symmetries beyond the list of usual space-time and inner (super)symmetries, still being consistent with unitary QFT

Fronsdal Fields

Fronsdal fields 1978

All $m = 0$ HS fields are gauge fields

$\varphi_{n_1 \dots n_s}$ is a rank s symmetric tensor obeying $\varphi^k{}_k{}^m{}_{m n_5 \dots n_s} = 0$

Gauge transformation:

$$\delta\varphi_{n_1 \dots n_s} = \partial_{(n_1}\varepsilon_{n_2 \dots n_s)}\,, \qquad \varepsilon^m{}_{m n_3 \dots n_{s-1}} = 0$$

Fronsdal action $S(\varphi)$ implies field equations: $G_{n_1 \dots n_s}(\varphi) = 0$ with

Einstein-like tensor

$$G_{n_1 \dots n_s}(\varphi) := \Box\varphi_{n_1 \dots n_s}(x) - s\partial_{(n_1}\partial^m\varphi_{n_2 \dots n_s m)}(x) + \frac{s(s-1)}{2}\partial_{(n_1}\partial_{n_2}\varphi^m{}_{n_3 \dots n_s m)}(x)$$

No-go and the Role of $(A)dS$

In 60th it was argued (Weinberg, Coleman-Mandula) that HS symmetries cannot be realized in a nontrivial local field theory in Minkowski space

In 70th it was shown by Aragone and Deser that HS gauge symmetries are incompatible with GR if expanding around Minkowski space

Green light: AdS background with $\Lambda \neq 0$ Fradkin, MV, 1987

In agreement with no-go statements the limit $\Lambda \to 0$ is singular

AdS_d is a curved hyperboloid with the radius $R^2 = -\Lambda^{-1}$

AdS/CFT Correspondence

AdS/CFT correspondence: duality between QFT in d dimensions and gravity theories in AdS_{d+1} J Maldacena 1997

That HS gauge theory is formulated in AdS_4 fits naturally the AdS/CFT correspondence: Klebanov and Polyakov conjectured in 2002 that AdS_4 HS theory is dual to $3d$ vectorial conformal σ-models of φ^i, ψ^i_α. KP conjecture was checked by Giombi and Yin in 2009

$$S^B = \frac{k}{4\pi} S_{CS} + \frac{1}{2}\int d^3x D_n\phi_i D^n\phi^i\,, \qquad S^F = \frac{k}{4\pi} S_{CS} + \int d^3x \bar{\psi}_i \gamma^n D_n \psi^i\,, \qquad i = 1, \dots N$$

$3d$ bosonization as a consequence of duality

$$\varphi = \frac{\pi}{2}\lambda_B\,, \qquad \varphi = \frac{\pi}{2}(1-\lambda_F) \qquad (\eta = \exp i\varphi) \qquad \lambda := \frac{N}{k}$$

Unexpected possibility of lab tests of Quantum Gravity

AdS_3/CFT_2 HS correspondence Gaberdiel and Gopakumar (2010)

Analysis of HS holography helps to uncover the origin of AdS/CFT

Global HS Symmetry

Conformal symmetry in d dimensions $= o(d,2) =$ symmetry of the $(d+1)$-dimensional AdS_{d+1} space

Conformal HS symmetry in d dimensions $=$ HS symmetry in AdS_{d+1}

Maximal symmetry of a d-dimensional free conformal field(s):

KG massless equation in Minkowski space

$$\Box C(x) = 0\,, \qquad \Box = \eta^{ab}\frac{\partial^2}{\partial x^a \partial x^b}$$

What are symmetries of KG equation? Shaynkman, MV 2001 $3d$; Eastwood 2002 $\forall d$

The answer is most easily obtained in the first-order unfolded form of the field equations

$3d$ Multispinors

$3d$ **coordinates:** $x^{\alpha\beta} = x^{\beta\alpha} = \tau_n^{\alpha\beta} x^n\,, \qquad \alpha,\beta = 1,2\,,\ \tau_n^{\alpha\beta} = \left(\delta^{\alpha\beta}, {\sigma_1}^{\alpha\beta}, {\sigma_3}^{\alpha\beta}\right)$

Lorentz covariance

$$\det|x^{\alpha\beta}| := \epsilon_{\alpha\beta}\epsilon_{\gamma\delta}x^{\alpha\gamma}x^{\beta\delta} = x^n x_n\,, \qquad \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}\,, \qquad \epsilon_{12} = 1$$

$3d$ **Lorentz algebra:** $o(2,1) \sim sp(2,R) \sim sl_2(R)$. α,β **are spinor indices**

Unfolded Klein-Gordon equations for a scalar $C(x)$

$$\partial_{\alpha\beta}C(x) = C_{\alpha\beta}(x)\,, \qquad \partial_{\alpha\beta}C_{\gamma\delta}(x) = C_{\alpha\beta;\gamma\delta}(x)\,, \qquad \partial_{\alpha\beta} = \frac{\partial}{\partial x^{\alpha\beta}}$$

$$\Box C(x) = 0 \longrightarrow \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\partial_{\alpha\beta}\partial_{\gamma\delta}C(x) = 0$$

implies $C_{\alpha\beta;\gamma\delta}(x)$ **is totally symmetric:** $C_{\alpha\beta;\gamma\delta}(x) = C_{\alpha\beta\gamma\delta}(x)$.

Continuation:

$$dx^{\alpha_1\alpha_2}\partial_{\alpha_1\alpha_2}C_{\beta_1\ldots\beta_n}(x) = dx^{\alpha_1\alpha_2}C_{\alpha_1\alpha_2\beta_1\ldots\beta_n}(x)$$

Totally symmetric $C_{\beta_1\ldots\beta_n}$ **parameterize all on-shell nontrivial higher derivatives of** C.

Spinorial Form of $3d$ Massless Equations

Packing all symmetric multispinors into a generating function of commuting spinor variables y^α

$$C(y|x) = \sum_{n=0}^{\infty} C^{\alpha_1 \ldots \alpha_{2n}}(x) y_{\alpha_1} \ldots y_{\alpha_{2n}}$$

unfolded $3d$ massless equations take the form

$$dx^{\alpha\beta} \left(\frac{\partial}{\partial x^{\alpha\beta}} + \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \right) C(y|x) = 0$$

$3d$ conformal HS algebra is the algebra of various differential operators $\epsilon(y, \frac{\partial}{\partial y})$ obeying $\epsilon(-y, -\frac{\partial}{\partial y}) = \epsilon(y, \frac{\partial}{\partial y})$

$$\delta C(y|x) = \epsilon(y, \frac{\partial}{\partial y}|x) C(y|x)$$

$$\epsilon(y, \frac{\partial}{\partial y}|x) = \exp\left[-x^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta}\right] \epsilon_{gl}(y, \frac{\partial}{\partial y}) \exp\left[x^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta}\right]$$

$\epsilon_{gl}(y, \frac{\partial}{\partial y})$: $3d$ HS symmetry algebra is Weyl algebra with spinor generators

Properties of HS Algebras

Global symmetry of symmetric vacuum of HS theory Fradkin, MV 1986

Let T_s be a homogeneous polynomial of degree $2(s-1)$ in $y^\alpha\,,\frac{\partial}{\partial y^\alpha}$

$$[T_{s_1}\,,T_{s_2}] = T_{s_1+s_2-2} + T_{s_1+s_2-4} + \ldots + T_{|s_1-s_2|+2}$$

Once spin $s>2$ appears, the HS algebra contains an infinite tower of higher spins: $[T_s, T_s]$ gives rise to T_{2s-2} as well as T_2 of $o(3,2) \sim sp(4)$.

Usual symmetries: spin-$s \leq 2$ $u(1) \oplus o(3,2)$: maximal finite-dimensional subalgebra of $hu(1,0|4)$. $u(1)$ is associated with the unit element.

Three series of $4d$ HS algebras: $hu(n,m|4)$, $ho(n,m|4)$, $husp(2n,2m|4)$

Konstein, MV 1988

Particle spectrum contains colorless graviton and colorless scalar

HS Symmetries Versus Riemann Geometry

HS symmetries do not commute with space-time symmetries

$$[T^n, T^{HS}] = T^{HS}\,, \qquad [T^{nm}, T^{HS}] = T^{HS}$$

HS transformations map gravitational fields (metric) to HS fields

$$\delta_{HS}\varphi_{nm} \sim \varphi_{HS}$$

Consequence:

Riemann geometry is not appropriate for HS theory:

concept of local event may become illusive!

Cartan formalism of differential forms preserves coordinate independence without metric. Elaboration of this language in HS theory leads to fundamental structures like L_∞ and A_∞ also suggesting new insights into the nature of space-time including its dimension.

Unfolded dynamics

First-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) \qquad \text{initial values:} \quad q^i(t_0)$$

Unfolded dynamics: multidimensional covariant generalization

$$\frac{\partial}{\partial t} \to \mathsf{d}\,, \qquad q^i(t) \to W^\Omega(x) = dx^{n_1} \wedge \ldots \wedge dx^{n_p} W_{n_1 \ldots n_p}(x)$$

$$\mathbf{dW^\Omega(x) = G^\Omega(W(x))}\,, \qquad \mathbf{d = dx^n \partial_n} \qquad \mathbf{MV} \quad \mathbf{1988}$$

$G^\Omega(W)$: **function of "supercoordinates"** W^Φ

$$G^\Omega(W) = \sum_{n=1}^{\infty} f^\Omega{}_{\Phi_1 \ldots \Phi_n} W^{\Phi_1} \wedge \ldots \wedge W^{\Phi_n}$$

$d > 1$: **Nontrivial compatibility conditions**

$$G^\Phi(W) \wedge \frac{\partial G^\Omega(W)}{\partial W^\Phi} \equiv 0$$

Any solution: FDA Sullivan (1968); D'Auria and Fre (1982)

In HS theory L_∞ algebroid since zero-forms play important role (1988)

Universal Unfolded System as a Q-Manifold

The system is universal if it remains consistent in any space-time dimension d being insensitive to that there is a maximal volume d-form. Universal systems can be rewritten in the form

$$\mathsf{d}F(W) = QF(W)\,, \qquad Q := G^A(W)\frac{\partial}{\partial W^A}$$

Compatibility condition: Q is homological vector field on the target manifold with local coordinates W

$$Q^2 = 0$$

The universal unfolded equation is invariant under the gauge transformation

$$\delta W^\Omega(x) = d\varepsilon^\Omega(x) + \varepsilon^\Phi(x)\frac{\partial G^\Omega(W(x))}{\partial W^\Phi(x)}\,,$$

Generally $G(W)$ defines L_∞ or A_∞ algebroid structure if all fields W^Ω are allowed to be valued in any associative algebra as it happens in HS theories

Vacuum Geometry

$\omega = \omega^\alpha T_\alpha$: h **valued 1-form.**

$$G(\omega) = -\omega \wedge \omega \equiv -\frac{1}{2}\omega^\alpha \wedge \omega^\beta [T_\alpha, T_\beta]$$

the unfolded equation with $W = \omega$ **has the zero-curvature form**

$$\mathrm{d}\omega + \omega \wedge \omega = 0\,.$$

Compatibility condition: Jacobi identity for a Lie algebra h **underlying the** L_∞ **algebra.**

Zero-curvature equations: background geometry in a coordinate independent way.

If h **is Poincare or** AdS **algebra it describes Minkowski or** AdS_d **space-time**

Fluctuations

Free fields - linear expansion. In the unfolded formulation are described by covariant constancy conditions in h-modules where fields are valued.

Free equations

$$DC(x) = 0$$

$$D := \mathsf{d} + [\omega\,, ...]$$

Example: $3d$ **massless fields**

$$dx^{\alpha\beta}\left(\frac{\partial}{\partial x^{\alpha\beta}} + \frac{\partial^2}{\partial y^\alpha \partial y^\beta}\right) C(y|x) = 0$$

Properties

- General applicability
- Manifest (HS) gauge invariance
- Diffeomorphisms invariance
- Interactions: nonlinear deformation of $G^{\Omega}(W)$
- Local degrees of freedom are in 0-forms $C^i(x_0)$ at any $x = x_0$ (as $q(t_0)$)

  infinite-dimensional module dual to the space of single-particle states
- Lie algebra interpretation: Chevalley-Eilenberg cohomology with

  coefficients in infinite-dimensional h-modules
- Independence of ambient space-time

  Geometry is encoded by $G^{\Omega}(W)$

Unfolding and Holographic Duality

Unfolded formulation unifies various dual versions of the same system.

Duality in the same space-time:

ambiguity in what is chosen to be dynamical or auxiliary fields.

Holographic duality between theories in different dimensions:

universal unfolded system admits different space-time interpretations.

Extension of space-time without changing dynamics by letting the differential d and differential forms W to live in a larger space

$$\mathsf{d} = dX^n \frac{\partial}{\partial X^n} \to \tilde{\mathsf{d}} = dX^n \frac{\partial}{\partial X^n} + d\hat{X}^{\hat{n}} \frac{\partial}{\partial \hat{X}^{\hat{n}}}\,, \qquad dX^n W_n \to dX^n W_n + d\hat{X}^{\hat{n}} \hat{W}_{\hat{n}}\,,$$

$\hat{X}^{\hat{n}}$ are additional coordinates

$$\tilde{\mathsf{d}} W^{\Omega}(X, \hat{X}) = G^{\Omega}(W(X, \hat{X}))$$

Two unfolded systems in different space-times are equivalent (dual) if they have the same unfolded form. 2012

Direct way to establish holographic duality between two theories: unfold both to see whether their unfolded formulations coincide.

Particular space-time interpretation of a universal unfolded system, e.g, whether a system is on-shell or off-shell, depends not only on $G^{\Omega}(W)$ but, in the first place, on space-time M^d and chosen vacuum solution $W_0(X)$.

Given unfolded system generates a class of holographically dual theories in different dimensions.

Free Massless Fields in AdS_4

Infinite set of spins $s = 0, 1/2, 1, 3/2, 2\ldots$ **Fermions need doubling of fields**

Doubled Weyl algebra connection: $\omega^{ii}(y,\bar{y}\mid x)\,, \qquad i = 0,1$

Twisted adjoint module: $C^{i1-i}(y,\bar{y}\mid x)$,

$$\bar{\omega}^{ii}(y,\bar{y}\mid x) = \omega^{ii}(\bar{y},y\mid x)\,, \qquad \bar{C}^{i\,1-i}(y,\bar{y}\mid x) = C^{1-i\,i}(\bar{y},y\mid x)$$

$$A(y,\bar{y}\mid x) = i\sum_{n,m=0}^{\infty}\frac{1}{n!m!}y_{\alpha_1}\ldots y_{\alpha_n}\bar{y}_{\dot{\beta}_1}\ldots\bar{y}_{\dot{\beta}_m}A^{\alpha_1\ldots\alpha_n}{}_{,}{}^{\dot{\beta}_1\ldots\dot{\beta}_m}(x)$$

The unfolded system for free massless fields is (CMST) **MV (1989)**

$$\star \qquad R_1^{ii}(y,\bar{y}\mid x) = \eta\,\overline{\mathbf{H}}^{\dot{\alpha}\dot{\beta}}\frac{\partial^2}{\partial\bar{\mathbf{y}}^{\dot{\alpha}}\partial\bar{\mathbf{y}}^{\dot{\beta}}}\;\mathbf{C}^{1-i\,i}(\mathbf{0},\bar{\mathbf{y}}\mid\mathbf{x}) + \bar{\eta}\,\mathbf{H}^{\alpha\beta}\frac{\partial^2}{\partial\mathbf{y}^{\alpha}\partial\mathbf{y}^{\beta}}\;\mathbf{C}^{i\,1-i}(\mathbf{y},\mathbf{0}\mid\mathbf{x})$$

$$\star \qquad \tilde{D}_0 C^{i\,1-i}(y,\bar{y}\mid x) = 0 \qquad \textbf{Chevalley – Eilenberg} \quad \textbf{cohomology}$$

$$R_1(y,\bar{y}\mid x) = D_0^{ad}\omega(y,\bar{y}\mid x) \qquad H^{\alpha\beta} = e^{\alpha}{}_{\dot{\alpha}}\wedge e^{\beta\dot{\alpha}}\,, \quad \overline{H}^{\dot{\alpha}\dot{\beta}} = e_{\alpha}{}^{\dot{\alpha}}\wedge e^{\alpha\dot{\beta}}$$

$$D_0^{ad}\omega = D^L - \lambda e^{\alpha\dot{\beta}}\left(y_\alpha\frac{\partial}{\partial\bar{y}^{\dot{\beta}}} + \frac{\partial}{\partial y^{\alpha}}\bar{y}_{\dot{\beta}}\right)\,, \qquad \tilde{D}_0 = D^L + \lambda e^{\alpha\dot{\beta}}\left(y_\alpha\bar{y}_{\dot{\beta}} + \frac{\partial^2}{\partial\mathbf{y}^{\alpha}\partial\bar{\mathbf{y}}^{\dot{\beta}}}\right)$$

$$D^L = \mathsf{d}_x - \left(\omega^{\alpha\beta}y_\alpha\frac{\partial}{\partial y^{\beta}} + \bar{\omega}^{\dot{\alpha}\dot{\beta}}\bar{y}_{\dot{\alpha}}\frac{\partial}{\partial\bar{y}^{\dot{\beta}}}\right)$$

Interaction Deformation

Goal: To find a nonlinear HS theory such that

(i) in the free field limit it amounts to the Fronsdal theory

(ii) Abelian HS gauge symmetries related to the parameters $\varepsilon^{m_1 \ldots m_{s-1}}$ deform to non-Abelian

For $s = 1, 2$ fundamental Yang-Mills and Einstein theories

HS–Gravity Interaction Problem

Aragone, Deser (1979)

$$\partial_n \to D_n = \partial_n - \Gamma_n \qquad [D_n\,, D_m] = \mathcal{R}_{nm}\ldots$$

Riemann tensor $\mathcal{R}_{nm,kl} \neq 0$ in a curved background.

$$\delta\varphi_{nm\ldots} \to D_n\varepsilon_{m\ldots} \qquad \delta S_s^{cov} = \int \mathcal{R}_{\ldots}(\varepsilon_{\ldots} D\varphi_{\ldots}) \neq 0 \qquad ?!$$

For $s \leq 2$, δS_s^{cov} contains only the Ricci tensor that can be compensated by the variation of the spin two action

$$\delta S^{EH} \sim \int \delta g^{nm} G_{nm}$$

allowing nonlinear gravity and supergravity.

($\delta g^{nm} = (\overline{\psi}^n \gamma^m \varepsilon) + (\overline{\psi}^m \gamma^n \varepsilon)$)

For $s > 2$, full Riemann tensor contributes to δS_s^{cov} : difficult to achieve HS gauge symmetry at the nonlinear level.

Higher Derivatives in HS Interactions

A.Bengtsson, I.Bengtsson, Brink (1983)

Berends, Burgers, van Dam (1984)

$$S = S^2 + S^3 + \dots$$

$$S^3 = \sum_{p,q,r} (D^p\varphi)(D^q\varphi)(D^r\varphi)\rho^{p+q+r+\frac{1}{2}d-3}$$

String: $\rho \sim \sqrt{\alpha'}$

HS Theories: ρ is AdS scale Fradkin, M.V. (1987)

$$[D_n, D_m] \sim \rho^{-2}$$

The $\rho \to \infty$ limit is ill defined at the interaction level both in string theory and in HS theory

Role of AdS Background

Near AdS: expansion in powers of the shifted Riemann tensor $R_{mn,kl} = \mathcal{R}_{mn,kl} - \lambda^2(g_{mk}g_{nl} - g_{ml}g_{nk})$ (which is zero in the AdS space) rather than in powers of the Riemann tensor $\mathcal{R}$

$$S \to S^{cov} + S^{int}, \qquad S^{int} = \sum_{k=0}^{s-1} S_k^{int}$$

The mechanism Fradkin, M.V. (1987)

$$S_k^{int} = \lambda^{-2k} \int_{M^4} \sum_{p+q=2k} D^p(\varphi) D^q(\varphi) R$$

The highest derivative term S_{s-1}^{int} is gauge invariant in the flat limit.

Since

$$[D_n, D_m] \sim \lambda^2 + O(R) \sim O(1) + O(R).$$

$$\delta S^{int}_{s-1} = \lambda^{2(1-k)} \int_{M^4} \sum_{p+q=2s-3} D^p(\varphi) D^q(\varepsilon) R$$

This term compensates δS^{int}_{s-2} modulo terms of order $\lambda^{-(2s-6)}$. The process continues until one is left with the λ-independent terms

$$\delta S^{int} = \int_{M^4} \sum_{p+q=1} D^p(\varphi) D^q(\varepsilon) R$$

which just compensate the variation of the covariantized free action $\delta S^{cov} = \int R_{\dots}(\varepsilon_{\dots} D\varphi_{\dots})$

$$\delta S^{cov} + \delta S^{int} = 0\,.$$

General Properties of HS Interactions

HS interactions contain higher derivatives

Nonanaliticity in Λ via dimensionless combination $\Lambda^{-\frac{1}{2}}\frac{\partial}{\partial x}$

Background HS gauge fields contribute to higher-derivative terms in the evolution equations: evolution is determined by HS fields along with the metric: no geodesic motion in presence of nonzero HS fields

Hence insufficiency of metric in presence of HS fields

HS fields source lower-spin fields (in particular gravity) and vice-versa

Different Approaches to Nonlinear Deformation

Metric-like deformation in terms of Fronsdal fields: to find

$$G(D,\varphi) = G_1(D,\varphi) + G_2(D,\varphi) + G_3(D,\varphi) + \dots$$

$$\delta\varphi = \delta_0(D,\varepsilon) + \delta_1(D,\varphi,\varepsilon) + \delta_2(D,\varphi,\varepsilon) + \dots$$

$$\delta G(D,\varphi) = 0\Big|_{G=0}\,.$$

Unfolded deformation in terms of HS one-forms ω and zero-forms C

$$\mathsf{d}\omega + \omega * \omega = J_1(\omega, C) + J_2(\omega, C) + \dots$$

$$\mathsf{d}C + [\omega, C]_* = H_2(\omega, C) + H_3(\omega, C) + \dots$$

$J_n(\omega, C)$ contain ω^2 and C^n, $H_n(\omega, C)$ contain ω and C^n

Higher derivatives in additional momentum-like components of fields

Relevant Structures for the Unfolded System

Hochschild cohomology of the Weyl algebra

A_∞ ($\mathcal{L}_\infty$) strong homotopy algebroid of the unfolded system

Both approaches is hard to implement. Both are invariant under nonlinear field redefinitions (with higher derivatives in the metric-like version): nonlocality?!

Metric-like approach results from the unfolded one restricted to dynamical Fronsdal components in ω.

Remarkable simplification: generating system that replaces the hard Hochschild cohomology problem by the simple De Rham cohomology in additional variables Z

Fields of the Nonlinear System

Nonlinear HS equations demand doubling of spinors and Klein operator

$$\omega(Y|x) \longrightarrow W(Z;Y;K|x)\,, \qquad C(Y|x) \longrightarrow B(Z;Y;K|x)$$

Some of the nonlinear HS equations determine the dependence on Z_A

in terms of "initial data"

$\omega(Y;K|x) := \sum_{ij=0,1} W^{ij}(0,Y|x)k^i\bar{k}^j \quad i=j \quad k^2 = \bar{k}^2 = 1$

$C(Y;K|x) := \sum_{ij=0,1} B^{ij}(0,Y|x)k^i\bar{k}^j \quad i+j=1$

$S(Z;Y;K|x) = dZ^A S_A(Z;Y;K|x)$ is a connection along Z^A

Topological fields: finite # d.o.f.: tensors

Klein operator k generates chirality automorphisms

$$kf(A) = f(\tilde{A})k\,, \quad A = (a_\alpha\,, \bar{a}_{\dot{\alpha}}): \quad \tilde{A} = (-a_\alpha\,, \bar{a}_{\dot{\alpha}})$$

$$P(Y) = P^{\alpha\dot{\alpha}}y_\alpha\bar{y}_{\dot{\alpha}} \quad \longrightarrow \quad \tilde{P}(Y) = -P(Y)\,, \qquad \tilde{M}(Y) = M(Y)$$

HS Star Product

Weyl algebra A_4 is described in terms of the HS star product

$$(f*g)(Z,Y)=\int dSdT\,\exp iS_AT^A\, f(Z+S,Y+S)g(Z-T,Y+T)$$

$$[Y_A,Y_B]_*=-[Z_A,Z_B]_*=2iC_{AB}\,,$$ $Z-Y\,:\,Z+Y$ normal ordering

Inner Klein operators:

$$\kappa=\exp iz_\alpha y^\alpha\,,\qquad \bar\kappa=\exp i\bar z_{\dot\alpha}\bar y^{\dot\alpha}\,,\qquad \kappa*f=\tilde f*\kappa\,,\qquad \kappa*\kappa=1$$

For Z-independent functions gives usual Moyal product

$$f(Y)*g(Y)=f(Y)\,\exp i[C^{AB}\frac{\overleftarrow{\partial}}{\partial Y^A}\frac{\overrightarrow{\partial}}{\partial Y^B}]\,g(Y)$$

Nonlocality of the Moyal product induces the space-time nonlocality of HS theory

Nonlinear HS Equations

1992

$$\mathrm{d}W + W * W = 0$$

$$\mathrm{d}B + W * B - B * W = 0$$

$$\mathrm{d}S + W * S + S * W = 0$$

$$\mathbf{S} * \mathbf{B} - \mathbf{B} * \mathbf{S} = \mathbf{0}$$

$$\mathbf{S} * \mathbf{S} = \mathbf{i}(\mathbf{d}\mathbf{Z}^{\mathbf{A}}\mathbf{d}\mathbf{Z}_{\mathbf{A}} + \eta \mathbf{d}\mathbf{z}^{\alpha}\mathbf{d}\mathbf{z}_{\alpha}\mathbf{B} * \mathbf{k} * \kappa + \bar{\eta}\mathbf{d}\bar{\mathbf{z}}^{\dot{\alpha}}\mathbf{d}\bar{\mathbf{z}}_{\dot{\alpha}}\mathbf{B} * \bar{\mathbf{k}} * \bar{\kappa})$$

Dynamical content is located in the x-independent twistor sector

The non-zero curvature has the form of Z_2-Cherednik algebra guaranteeing formal compatibility

Perturbative Analysis

Vacuum solution

$$B_0 = 0\,, \qquad S_0 = dZ^A Z_A\,, \qquad W_0 = \frac{1}{2}\omega_0^{AB}(x) Y_A Y_B$$

$$\mathrm{d}W_0 + W_0 * W_0 = 0$$

$\omega_0^{AB}(x)$: **describes** AdS_4.

First-order fluctuations

$$B_1 = C(Y)\,, \qquad S = S_0 + S_1\,, \qquad W = W_0(Y) + W_1(Y) + W_0(Y)C(Y)$$

$$[S_0\,, f]_\star = -2i\mathrm{d}_Z f\,, \qquad \mathrm{d}_Z = dZ^A \frac{\partial}{\partial Z^A}$$

Reconstruction of Z^A Variables

Perturbatively, equations containing S have the form

$$\mathsf{d}_Z U_n(Z;Y|dZ) = V[U_{<n}](Z;Y|dZ) \qquad \mathsf{d}_Z V[U_{<n}](Z;Y|dZ) = 0$$

can be solved as

$$U_n(Z;Y|dZ) = \mathsf{d}^*_Z V[U_{<n}](Z;Y|dZ) + \mathbf{h(Y)} + \mathsf{d}_Z \epsilon(Z;Y|dZ)$$

For instance

$$\mathsf{d}^*_Z V(Z;Y|dZ) = Z^A \frac{\partial}{dZ^A} \int_0^1 \frac{dt}{t} V(tZ;Y|tdZ)$$

Alternative d^*_Z that differ by d_Z-closed forms can also be used.

Proper choice of boundary conditions in Z-variables is most important in the context of locality beyond the free field level! Definition of the minimally nonlocal Hochschild complex is the hot topic nowadays.

Nontrivial space-time equations on $\omega(Y|x)$ and $C(Y|x)$ are in the sector of d_Z-cohomology.

Central On-Shell Theorem is reproduced in the lowest order

HS Gauge Theory Versus String Theory

Important feature: $(A)dS$ background with $\Lambda \neq 0$ Fradkin, MV, 1987

HS theories: $\Lambda \neq 0$, $m = 0$, symmetric fields $s = 0, 1, 2, \ldots \infty$

First Regge trajectory

String Theory: $\Lambda = 0$, $m \neq 0$ except for a few zero modes

Infinite set of Regge trajectories

What is a HS symmetry of a string-like extension of HS theory?

MV 2012, Gaberdiel and Gopakumar 2014-2018

String Theory as spontaneously broken HS theory?! $(s > 2, m > 0)$

Difficulty of the Naive Extension

Free field analysis: realization of the HS algebra as Weyl algebra

$$[y_\alpha\,, y_\beta]_* = 2i\varepsilon_{\alpha\beta}\,, \qquad [\bar{y}_{\dot\alpha}\,, \bar{y}_{\dot\beta}]_* = 2i\varepsilon_{\dot\alpha\dot\beta}$$

AdS_4 algebra $sp(4) \sim o(3,2)$

Naive way to extend the spectrum $y_\alpha \to y_\alpha^n$ does not lead to physically acceptable HS theories

Let hs_1 be a HS algebra with the single set of oscillators

The Fock hs_1-module F_1 describes free boundary conformal fields

$$D|0\rangle = h_1|0\rangle$$

The lowest weight representations of the naively extended algebras hs_p built from p copies of oscillators have too high weights

$$h_p = ph_1$$

$F_1 \otimes F_1$ = massless fields in the bulk Flato, Fronsdal (1978)

For $p > 1$ $F_p \otimes F_p$ has no room for gravity (massless spin-two)

Framed Oscillator Algebras

The problem is resolved in the framed oscillator algebras replacing usual oscillator algebra

$$[y^n_\alpha , y^m_\beta]_* = 2i\delta^{nm}\epsilon_{\alpha\beta}I \,,$$

where I is the unit element by

$$[y^n_\alpha , y^m_\beta]_* = 2i\delta^{nm}\epsilon_{\alpha\beta}I_n$$

"Units" I_n are assigned to each specie of the oscillators forming a set of commutative central idempotents

$$I_iI_j = I_jI_i \,, \qquad I_iI_i = I_i$$

This allows us to consider Fock modules F_i obeying

$$I_jF_i = \delta_{ij}F_i$$

equivalent to those of the single-oscillator case

Coxeter Groups and Cherednik Algebras

A rank-p Coxeter group $\mathcal{C}$ is generated by reflections with respect to a system of root vectors $\{v_a\}$ in a p-dimensional Euclidean vector space V.

An elementary reflection associated with the root vector v_a

$$R_{v_a} x^i = x^i - 2 v_a^i \frac{(v_a, x)}{(v_a, v_a)}, \qquad R_{v_a}^2 = I$$

Cherednik deformation of the semidirect product of the oscillator algebra with the group algebra of $\mathcal{C}$ is

$$[q_\alpha^n, q_\beta^m] = -i\epsilon_{\alpha\beta} \left(2\delta^{nm} + \sum_{v \in \mathcal{R}} \nu(v) \frac{v^n v^m}{(v,v)} k_v \right), \qquad k_v q_\alpha^n = R_v{}^n{}_m q_\alpha^m k_v$$

q_α^n $(\alpha = 1, 2,\ n = 1; \ldots, p)$

Coupling constants $\nu(v)$ are invariants of $\mathcal{C}$ being constant on the conjugacy classes of root vectors under the action of $\mathcal{C}$.

Double commutator of q_α^n respects Jacobi identities.

B_p–Coxeter System

Important case of the Coxeter root system is B_p with the roots

$$R_1 = \{\pm e^n \qquad 1 \le n \le p\}\,, \qquad R_2 = \{\pm e^n \pm e^m \quad 1 \le n < m \le p\}\,.$$

Apart from permutations B_p contains reflections of basis axes $v^n_\pm = e^n$.

R_1 and R_2 form two conjugacy classes of B_p.

The Coxeter group of $3d$ HS theory is $A_1 \sim B_1$.

B_2 underlies the string-like HS models.

The fact of fundamental importance for HS theories is that for any Coxeter root system the generators

$$t_{\alpha\beta} := \frac{i}{4}\sum_{n=1}^{p}\{q^n_\alpha\,, q^n_\beta\}$$

obey the $sp(2)$ commutation relations properly rotating all indices α

$$[t_{\alpha\beta}\,, q^n_\gamma] = \epsilon_{\beta\gamma} q^n_\alpha + \epsilon_{\alpha\gamma} q^n_\beta$$

Framed Cherednik Systems

A_{p-1} **system.** **In addition to** $q_{\alpha n}$ **and** k_{nm}, $n,m=1,\dots p$ **introduce** I_n

$$I_n I_m = I_m I_n\,, \qquad I_n I_n = I_n\,, \qquad I_n q_{\alpha n} = q_{\alpha n} I_n = q_{\alpha n}\,, \qquad I_n q_{\alpha m} = q_{\alpha m} I_n\,.$$

In presence of I_n **the deformed oscillator relations respecting Jacobi**

$$[q_{\alpha\, n}\,, q_{\beta\, m}] = -i\epsilon_{\alpha\beta}\Big(\delta_{nm}\Big(2I_n + \nu\sum_{l=1}^{p}\widehat{k}_{ln}\Big) - \nu\widehat{k}_{nm}\Big)\,, \qquad \widehat{k}_{nm} = I_n I_m k_{nm}\,.$$

$\widehat{k}_{nm}$ **obey all relations of** S_p **except for involutivity replaced by**

$$\widehat{k}_{nm}\widehat{k}_{nm} = I_n I_m\,.$$

$$I_l\widehat{k}_{nm} = \widehat{k}_{nm} I_l \quad \forall\, l,n,m\,, \qquad I_n\widehat{k}_{nm} = I_m\widehat{k}_{nm} = \widehat{k}_{nm}\,.$$

General framed Cherednik algebra

$$[q^n_\alpha\,, q^m_\beta] = -i\epsilon_{\alpha\beta}\left(2\delta^{nm}I_n + \sum_{v\in\mathcal{R}}\nu(v)\frac{v^n v^m}{(v,v)}\widehat{k}_v\right)\,, \qquad \widehat{k}_v := k_v\prod I_{i_1(v)}\dots I_{i_k(v)}$$

Framed Cherednik algebra still possesses inner $sp(2)$ **automorphisms**

$$t_{\alpha\beta} := \frac{i}{4}\sum_{n=1}^{p}\{q^n_\alpha\,, q^n_\beta\}I_n$$

Framed Star Product

x-dependent fields W, S and B depend on p sets of variables Y^n_A, Z^n_A $(A = 1, \dots M)$, I_n, anticommuting differentials dZ^A_n $(n = 1, \dots p)$ and Klein-like operators $\widehat{k}_v$ associated with all roots of $\mathcal{C}$. Coxeter HS field equations are formulated in terms of the star product

$$(f*g)(Z;Y;I) = \frac{1}{(2\pi)^{pM}} \int d^{pM}S\, d^{pM}T \exp\,[iS^A_n T^B_m \delta^{nm} C_{AB}]\, f(Z_i+I_iS_i; Y_i+I_iS_i; I)g$$

$$I_n * Y^n_A = Y^n_A * I_n = Y^n_A\,, \qquad I_n * Z^n_A = Z^n_A * I_n = Z^n_A\,, \qquad I_n * I_n = I_n$$

Implying

$$[Y^n_A, Y^m_B]_* = -[Z^n_A, Z^m_B]_* = 2iC_{AB}\delta^{nm} I_n\,, \qquad [Y^n_A, Z^m_B]_* = 0\,.$$

This star product admits inner Coxeter-Klein operators

$$\exp i \frac{v^n v^m Z_{\alpha n} Y^{\alpha}{}_m}{(v,v)}$$

Coxeter HS Equations

Unfolded equations for 1804.06520 $\mathcal{C}$-HS theories remain the same except

$$iS * S = dZ^{An} dZ_{An} + \sum_i \sum_{v\in\mathcal{R}_i} F_{i*}(B) \frac{dZ_n^{\alpha} v^n dZ_{\alpha\, m} v^m}{(v,v)} * \kappa_v$$

κ_v are generators of $\mathcal{C}$ acting trivially on all elements except for $dZ_{\alpha n}$

$$\kappa_v * dZ_{\alpha}^{n} = R_v{}^n{}_m dZ_{\alpha}^{m} * \kappa_v$$

$F_{i*}(B)$ is any star-product function of the zero-form B on the conjugacy classes R_i of $\mathcal{C}$. In the important case of the Coxeter group B_p

$$S*S = dZ_{An} dZ^{An} + \sum_{v\in\mathcal{R}_1} F_{1*}(B) \frac{dZ_n^{\alpha} v^n dZ_{\alpha\, m} v^m}{(v,v)} * \kappa_v + \sum_{v\in\mathcal{R}_2} F_{2*}(B) \frac{dZ_n^{\alpha} v^n dZ_{\alpha\, m} v^m}{(v,v)} * \kappa_v$$

with arbitrary $F_{1*}(B)$ and $F_{2*}(B)$ responsible for the

HS and stringy/tensorial features, respectively

$F_{2*}(B) \neq 0$ for $p \geq 2$.

The framed construction leads to a proper massless spectrum.

Extensions

W, S and B can be valued in any associative algebra A: A_∞ structure.

Multi-particle extensions are associated with the semi-simple Coxeter groups. The simplest option with $\mathcal{C} = B_p^{\mathcal{N}}$ is the product of $\mathcal{N}$ of B_p systems

$$B_p^{\mathcal{N}} := \underbrace{B_p \times B_p \times \dots}_{\mathcal{N}} .$$

The limit $\mathcal{N} \to \infty$ along with the graded symmetrization of the product factors expressing the spin-statistics gives the (graded symmetric) multi-particle algebra $M(h(\mathcal{C}))$ of the HS algebra $h(\mathcal{C})$

$M(h(\mathcal{C})) = U(h(\mathcal{C}))$: Hopf algebra.

Extension to Higher Forms and Invariant Functionals

$W + S \to \mathcal{W}$: forms of all odd degrees

$B \to \mathcal{B}$: forms of all even degrees

$\mathcal{L}$ are Lagrangian-type closed forms generating invariants

$$\mathcal{W} * \mathcal{W} = -i\Big(dZ^{An} dZ_{An} + F_*(\mathcal{B}, \gamma_i)\Big) + \mathcal{L}(x) I\,, \tag{1}$$

$$[\mathcal{W}\,, \mathcal{B}]_* = 0\,, \tag{2}$$

where

$$\gamma_i = \sum_{v \in \mathcal{R}_i} \frac{dZ^\alpha_n v^n dZ_{\alpha\, m} v^m}{(v, v)} * \widehat{\kappa}_v\,. \tag{3}$$

are central.

Resulting theory describes also higher differential forms

Algebraically, this constructions leads to an interesting generalization of Cherednik algebras

Klein Operators and Single-Trace operators

Enlargement of the field spectra of the rank-$p>1$ Coxeter HS models:

$C(Y^n_\alpha; k_v)$ **depend on p copies of oscillators Y^n_α and Klein operators k_v**

Qualitative agreement with enlargement of the boundary operators in tensorial boundary models.

Klein operators of Coxeter reflections permute master field arguments

At $p=2$ the star product of two master fields $C(Y_1,Y_2|x)k_{12}$ gives

$$(C(Y_1,Y_2|x)k_{12})*(C(Y_1,Y_2|x)k_{12}) = C(Y_1,Y_2|x)*C(Y_2,Y_1|x)\,.$$

$p=2$ **system: strings of fields with repeatedly permuted arguments**

$$C^n_{string} := \underbrace{C(Y_1,Y_2|x)*C(Y_2,Y_1|x)*C(Y_1,Y_2|x)\ldots}_{n}\,.$$

are analogous of the single-trace operators in AdS/CFT.

$C(Y_1,Y_2|x)$ **and** $C(Y_1,Y_2|x)*C(Y_2,Y_1|x)$: **single-trace-like**

$C(Y_1,Y_2|x)*C(Y_1,Y_2|x)$: **double-trace-like.**

From Coxeter HS Theory to Strings and Tensor Models

The spectrum of the B_2 HS model is analogous to that of String Theory with the infinite set of Regge trajectories.

B_p–HS models with $p \geq 2$ have two coupling constants.
F_{1*} is analogous to that of the B_1-HS theory.
F_{2*} first appears in the rank-two stringy model and, containing the Klein operators that permute different Y-variables, generates single-trace-like strings of operators and their tensor generalizations.

To establish relation with usual string theory in flat space the limit $F_{2*}/F_{1*} \to \infty$ is most interesting.

Idempotent Extension

Let A be an associative algebra with the star product and a set of idempotents

$$\pi_i * \pi_i = \pi_i \,, \qquad \pi_i \in A \,.$$

$$a_i{}^j \in A_i{}^j : \quad a_i{}^j = \pi_i * a * \pi_j \,, \quad a \in A \,.$$

The matrix-like composition law

$$(a * b)_i{}^j = \sum_k a_i{}^k * b_k{}^j$$

A is the algebra of functions of dx, dZ, Z, Y, k_v, x

π_i: Z-independent Fock idempotents of the star-product algebra.

The set of idempotents π_i has to be $\mathcal{C}$-invariant

The idempotent-extended $\mathcal{C}$-HS equations have the same form with the replacement of $A \to A_{\{\pi\}}$.

Vector-Like Models

Fock idempotent in the $4d$ HS theory

$$\pi_i^{star} = 4 I_i \exp y_{i\alpha} \bar{y}_i^{\alpha}$$

$$(y_{i\alpha} - i\bar{y}_{i\alpha}) * \pi_i^{star} = 0\,, \qquad \pi_i^{star} * (y_{i\alpha} + i\bar{y}_{i\alpha}) = 0\,.$$

For HS fields carrying matrix indices

$$\pi_i = \pi_i^{star} \pi_i^{color}\,, \qquad \pi_i^{color} = \delta_1^u \delta_v^1\,.$$

${A_0}^i$**-module describes $3d$ conformal fields= $4d$ singletons:**

Idempotent realization of Klebanov-Polyakov AdS_4/CFT_3

vector model HS holography checked by Giombi and Yin in 2009

Idempotent extensions of the Coxeter HS systems describe

lower-dimensional brane-like objects.

$N=4$ SUSY

$4d$ conformal massless fields are valued in the Fock module π 2002

$$a_\alpha * \pi = 0\,, \qquad \bar{b}^{\dot\beta} * \pi = 0\,, \qquad \phi_i * \pi = 0\,, \qquad \pi * \bar{a}^{\dot\alpha} = 0\,, \ldots$$

$$[a_\alpha, b^\beta]_* = \delta^\beta_\alpha\,, \qquad [\bar{a}_{\dot\gamma}, \bar{b}^{\dot\beta}]_* = \delta^{\dot\beta}_{\dot\gamma}\,, \qquad \{\phi_i, \bar\phi^j\}_* = \delta_i{}^j\,,$$

$i, j = 1, \ldots N$. Bilinears: $su(2,2;N)$. Clifford oscillators: color $Mat_{2^{2N}}$.

B_2-HS theory contains $y_{i\alpha}$, $\bar{y}_{i\dot\alpha}$. Vacuum π is defined by ϕ_i, $\bar\phi^j$ and

$$a_\alpha = y_{1\alpha} + i y_{2\alpha}\,, \quad b_\alpha = \frac{1}{4i}(y_{1\alpha} - i y_{2\alpha})\,, \quad \bar{a}_{\dot\alpha} = \bar{y}_{1\dot\alpha} - i\bar{y}_{2\dot\alpha}\,, \quad \bar{b}_{\dot\alpha} = \frac{1}{4i}(\bar{y}_{1\dot\alpha} + i\bar{y}_{2\dot\alpha})$$

$4d$ massless conformal fields are valued in the Fock modules.

Reflection $Y_1^A \leftrightarrow Y_2^A$ maps π to the opposite idempotent $\tilde\pi$

$$b_\alpha * \tilde\pi = 0\,, \qquad \bar{a}^{\dot\beta} * \tilde\pi = 0\,, \qquad \bar\phi^i * \tilde\pi = 0\,, \qquad \tilde\pi * \bar{b}^{\dot\alpha} = 0\,, \ldots$$

Both π and $\tilde\pi$ have to be present. Elements $\pi * a * \tilde\pi$ are ill defined: at $N=0$, $\pi * \tilde\pi = \infty$. Bosons and fermions contribute with opposite signs. The compensation occurs at $N=4$ when $\#_B = \#_F$. $N=4$ SYM is the only $N=4$ massless conformal system with spins $s \le 1$.

Conclusion

Unfolded dynamics is based on L_∞ and A_∞ structures and
unifies dynamical systems living in space-times of different dimensions

HS gauge theories contain gravity along with infinite towers of other fields with various spins including ordinary matter fields.
Main principle: formal consistency & massless fields in the spectrum

Coxeter HS theories extend minimal HS theories to String-like B_2 models
$\mathcal{N} = 4$ **SYM is argued to be a natural dual of the B_2–HS model**

B_p**–Coxeter HS theories have two coupling constants and are formulated in AdS: being different from the genuine String Theory in flat space**

Multi-particle states of a lower-dimensional model = elementary states in a higher-dimensional (particularly, $10d$ model)

The original $3d$ and $4d$ spinorial theories: branes in the $10d$ theory

Problems on the Agenda

Locality: to find a (minimally non)local HS L_∞ algebroid:

appropriate class of contracting homotopies in the d_Z complex

Gelfond, MV 1805.11941, Didenko, Gelfond, Korybut, MV 1807.00001

Spontaneous breaking of HS symmetries in the Coxeter HS models to find relation of HS theories to String Theory