The Non-Abelian Self-Dual String and a 6d Superconformal Field Theory

Christian Sämann

School of Mathematical and Computer Sciences
Heriot-Watt University, Edinburgh

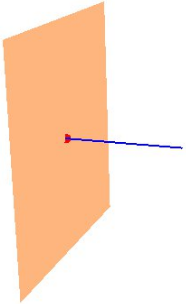
LMS/EPSRC Symposium, Durham, 15.8.2018

Based on:

- CS & L Schmidt, arXiv:1705.02353
- CS & L Schmidt, arXiv:1712.06623

Motivation: Dynamics of multiple M5-branes

To understand M-theory, an effective description of M5-branes would be very useful.

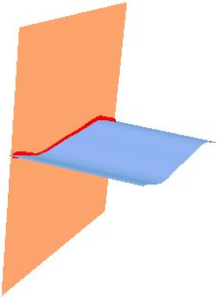


D-branes

- D-branes interact via strings.
- Effective description: theory of endpoints
- Parallel transport of these: Gauge theory
- Study string theory via gauge theory

M5-branes

- M5-branes interact via M2-branes.
- Eff. description: theory of self-dual strings
- Parallel transport: Higher gauge theory
- Long sought $(2,0)$-theory a HGT?



Outline

- The (2,0)-Theory: What we know and what we want
- Higher Gauge Theory: Lightening review
- Guidance from BPS self-dual strings
- The 6d superconformal field theory
- Open problems

The (2,0)-Theory: What we know and want

What we know about the (2,0)-theory

- 6d superconformal field theory, appears in type IIB on K3
- Also: parallel M5-branes — Witten, Strominger 1995/1996
  - self-dual strings: boundaries of M2- between M5-branes
  - become massless, if M5-branes approach each other
  - description of stacks of parallel M5-branes
- Field content: $\mathcal{N} = (2,0)$ tensor multiplet — Nahm 1978
  - a self-dual 3-form field strength
  - five (Goldstone) scalars
  - fermionic partners
- Observables: Wilson surfaces, i.e. parallel transport of strings
- Belief: No Lagrangian description
- Analogue of $\mathcal{N} = 4$ super Yang-Mills
- Huge interest in string theory: AGT, AdS_7-CFT_6, S-duality, ...
- Mathematics: Geom. Langlands, Khovanov Homology, ...

A successful M5-brane model should have the following properties:

- Contain an interacting, self-dual 2-form gauge potential
- Based on a sound mathematical foundation: higher bundles
- Field content of the $(2,0)$-theory, $\mathcal{N}=(1,0)$ supersymmetric
- Gauge structure natural, match some expectations (ADE, ...)
- Non-trivial coupling, interacting field theory
- Restriction to free $\mathcal{N}=(2,0)$ tensor multiplet possible
- contains the non-abelian self-dual string soliton as BPS state
- Reduction to 4d SYM theory with ADE gauge algebras
- and to 3d Chern–Simons-matter models with discrete coupling
- match expected moduli space of $(2,0)$-theory

- Non-abelian parallel transport of strings problematic?
  Non-abelian gerbes exist
  Remains: find relevant/interesting example
- No continuous coupling constant $\rightarrow$ no classical Lagrangian
  Same as for M2-branes
  Discrete coupling constant from geometry
- Reduction to 5d Yang–Mills theory seems impossible.
  Reduction to 4d Yang–Mills theory seems to work
- Action for self-dual 3-forms problematic
  PST formalism can be used
  Quantization (?) How useful (?)

Higher Gauge Theory: Lightening review

Guideline: Category Theory

"Category theory is the subject where
you can leave the definitions as exercises."

John Baez

N-manifolds, NQ-manifold

- $\mathbb{N}_0$-graded manifold with coordinates of degree $0, 1, 2, \ldots$

$$\begin{array}{c} M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots \end{array}$$

manifold — linear spaces

- NQ-manifold: vector field Q of degree 1, $Q^2 = 0$
- Physicists: think ghost numbers, BRST charge, SFT
- Functions on (M, Q) form differential graded algebra

Examples:

- Tangent algebroid $T[1]M$, $\mathcal{C}^\infty(T[1]M) \cong \Omega^\bullet(M)$, $Q = \mathrm{d}$
- Lie algebra $\mathfrak{g}[1]$, coordinates ξ^α of degree 1:

$$Q = -\tfrac{1}{2} f^\alpha_{\beta\gamma} \xi^\beta \xi^\gamma \frac{\partial}{\partial \xi^\alpha} \quad , \qquad \text{Jacobi identity} \Leftrightarrow \; Q^2 = 0$$

- Lie 2-algebra $\mathfrak{g}[1] \oplus \mathfrak{h}[2]$, Lie 3-algebra $\mathfrak{g}[1] \oplus \mathfrak{h}[2] \oplus \mathfrak{l}[3]$, …

Gauge Theory from dga-Morphisms

- Idea: Cartan, More: Strobl et al., Sati, Schreiber, Stasheff
- Local gauge theory: differential forms and Lie algebras
- Unify both in differential graded algebras from Weil algebra:

$$\mathrm{W}(\mathfrak{g}) := \mathcal{C}^\infty(T[1]\mathfrak{g}[1]) = \mathcal{C}^\infty(\mathfrak{g}[1] \oplus \mathfrak{g}[2]) \,, \quad \sigma : \mathfrak{g}^*[1] \xrightarrow{\cong} \mathfrak{g}^*[2]$$
$$Q|_{\mathcal{C}^\infty(\mathfrak{g}[1])} = Q_{\mathrm{CE}} + \sigma \,, \quad Q_{\mathrm{CE}}\sigma = -\sigma Q_{\mathrm{CE}}$$

- Potentials/curvatures/Bianchi identities from dga-morphisms

$$\begin{array}{rcl} (A,F) : \mathrm{W}(\mathfrak{g}) & \longrightarrow & W(M) = \Omega^\bullet(M) \\ \xi^\alpha & \longmapsto & A^\alpha \\ (\sigma\xi^\alpha) = Q\xi^\alpha + \tfrac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma & \longmapsto & F^\alpha = (\mathrm{d}A + \tfrac{1}{2}[A,A])^\alpha \\ Q(\sigma\xi^\alpha) = -f^\alpha_{\beta\gamma}(\sigma\xi^\beta)\xi^\gamma & \longmapsto & (\nabla F)^\alpha = 0 \end{array}$$

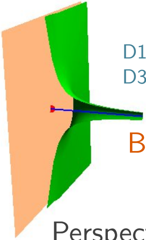
- Gauge transformations: homotopies between dga-morphisms
- Topological invariants: invariant polynomials on $\mathfrak{g}$ in $\mathrm{W}(\mathfrak{g})$

$\Rightarrow$ General notion of gauge theory from pairs of dgas

Which higher Lie algebra to take?

Guidance from BPS self-dual strings

Self-dual strings are the M-theory lift of monopoles.



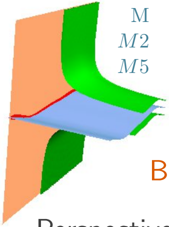
	0	1	2	3	4	5	6
D1	$\times$						$\times$
D3	$\times$	$\times$	$\times$	$\times$			

BPS configuration

Perspective of D3:

Bogomolny monopole eqn.

$$F = \nabla^2 = *\nabla\Phi \text{ on } \mathbb{R}^3$$



M	0	1	2	3	4	5	6
$M2$	$\times$					$\times$	$\times$
$M5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	

BPS configuration

Perspective of M5:

Abelian Self-dual string eqn.

$$H := \mathrm{d}B = *\mathrm{d}\Phi \text{ on } \mathbb{R}^4$$

Monopoles:

- Dirac monopole: Principal $\mathsf{U}(1)$-bundle over $\mathbb{R}^3\backslash\{0\}$ or S^2
  Hopf fibration: $\mathsf{U}(1)\hookrightarrow S^3 \to S^2$
- 't Hooft-Polyakov monopole: Principal $\mathsf{SU}(2)$-bundle over $\mathbb{R}^3$
- Non-abelianization and extension to $\mathbb{R}^3$:
  - want to preserve topology of Dirac monopole
  - want bundle over $\mathbb{R}^3$, which is necessarily trivial
  - $\to$ gauge group: total space $S^3 \cong \mathsf{SU}(2)$ of abelian bundle

Self-Dual String:

- Abelian: fundamental abelian gerbe $\mathscr{G}_F$ over S^3
- Non-abelian: principal 2-bundle over $\mathbb{R}^4$
- Analogy to above: Use total 2-space of $\mathscr{G}_F$ as gauge 2-group
- Indeed: $\mathscr{G}_F$ carries 2-group structure: string 2-group model

- String 2-group $\mathscr{G}_F$ and M-theory: many reasons, long story...
- $\mathscr{G}_F$ is analogue of $\mathsf{Spin}(3) \cong \mathsf{SU}(2)$ from many perspectives
- Lie differentiate (e.g. Demessie, CS (2016))
- Result:
  String Lie 2-algebra $\mathfrak{string}(3) = \left(\mathfrak{su}(2) \xleftarrow{\mu_1=0} \mathbb{R}[1]\right)$ with
  $$\mu_2(x_1, x_2) = [x_1, x_2]~, \quad \mu_3(x_1, x_2, x_3) = (x_1, [x_2, x_3])$$
  where $x_{1,2,3} \in \mathfrak{su}(2)$
- Equivalently: (quasi-isomorphic):
  $$P_0\mathfrak{su}(2) \hookleftarrow \hat{\Omega}\mathfrak{su}(2)$$
- Can be defined for any ADE Lie algebra $\mathfrak{g} \to \mathfrak{string}(\mathfrak{g})$

1. Kinematical data
   - Readily from dga-morphisms $W(\mathfrak{string}(3)) \to \Omega^\bullet(\mathbb{R}^4)$
   - Problem: fake curvatures need to vanish, many issues
   - Solution: twist Weil algebra   Sati, Schreiber, Stasheff (2009)
   - Get: string structures

$$A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{g} \;, \quad B \in \Omega^2(\mathbb{R}^4) \otimes \mathfrak{u}(1) \;,$$
$$F = \mathrm{d}A + \tfrac{1}{2}[A, A] \;, \quad H = \mathrm{d}B + \tfrac{1}{2}(A, \mathrm{d}A) + \tfrac{1}{3!}(A, [A, A]) \;,$$
$$\nabla F = 0 \;, \quad \mathrm{d}H = -(F, F)$$

   - Add by hand: Higgs field $\phi \in \Omega^0(\mathbb{R}^4) \otimes \mathfrak{u}(1)$

2. Dynamical principle   Schmidt, CS (2017)
   - Obvious: $H = *\mathrm{d}\phi$, implying $\mathrm{d}H = (F, F) = *\Box\phi$
   - Motivates: $F = \pm * F$
   - Full picture:
     $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, instanton + anti-instanton $c_2(F) = 0$

Conclusions so far

EOM matches story known from (1,0)-theories, so what?

- Higher analogue of $\mathsf{SU}(2) \cong \mathsf{Spin}(3)$ is $\mathsf{String}(3)$
- String structures allow for gauge invariant field equations
- Examples of truly non-abelian and non-trivial higher bundles
- Agnostic about quasi-isomorphs.: also for $P_0\mathfrak{su}(2) \leftarrow \hat{\Omega}\mathfrak{su}(2)$

The 6d superconformal field theory

Try to avoid hard SUSY computations and find in the literature:

6d (1,0)-model derived from tensor hierarchies
Samtleben, Sezgin, Wimmer (2011)

Open problems with this model:

- Issue 1: Choice of gauge structure unclear
- Issue 2: cubic interactions
- Issue 3: scalar fields with wrong sign kinetic term
- Issue 4: Self-duality of 3-form imposed by hand
- Issue 5: Unclear, how to fulfill "wishlist"

Previous observation:

- Gauge structure is Lie 3-algebra with "extra structure."
  Palmer, CS (2013), Samtleben et al. (2014)

New: Schmidt, CS (2017)

- Idea: use $\mathfrak{string}(\mathfrak{g})$ as gauge structure in this model
- Issue: need suitable notion of inner product for action
- Inner product/cyclic L_∞-algebras $\Leftrightarrow$ symplectic NQ-manifold
- Consequence: Extend $\mathfrak{string}(\mathfrak{g})$ from

$$(\mathfrak{g} \longleftarrow \mathbb{R} \xleftarrow{\text{id}} \mathbb{R}) \ \cong \mathfrak{g}$$

to symplectic graded vector space $T^*[2]\mathfrak{string}(\mathfrak{g})$:

$$\begin{array}{ccccc}
\mathbb{R}^* \xleftarrow{\quad\mu_1=\text{id}\quad} & \mathbb{R}^*[1] & & \mathfrak{g}^*[2] & \xleftarrow{\quad\mu_1=\text{id}\quad} \mathfrak{g}^*[3] \\
\oplus & \oplus & & \oplus & \\
\mathfrak{g} & \mathbb{R}[1] & \xleftarrow{\quad\mu_1=\text{id}\quad} & \mathbb{R}[2] &
\end{array}$$

- This carries natural inner product
- Has necessary extra structure for (1,0)-model

Field content:

- (1,0) tensor multiplet (ϕ, χ^i, B), values in $\mathbb{R}^2$, $\phi = \phi_s + \phi_r$, ...
- (1,0) vector multiplet (A, λ^i, Y^{ij}), values in $\mathfrak{g} \oplus \mathbb{R}$
- C-field, values in $\mathbb{R} \oplus \mathfrak{g}^*$

Action (schematically):

$$
\begin{aligned}
S = \int_{\mathbb{R}^{1,5}} \Big(& \mathcal{H}_r \wedge *\mathcal{H}_s + \mathrm{d}\phi_r \wedge *\mathrm{d}\phi_s - *\bar{\chi}_r \partial\!\!\!/ \chi_s + \mathcal{H}_s \wedge *(\bar{\lambda}, \gamma_{(3)}\lambda) + *(Y, \bar{\lambda})\chi_s \\
& + \phi_s\big((\mathcal{F}, *\mathcal{F}) - *(Y, Y) + *(\bar{\lambda}, \nabla\!\!\!\!/\,\lambda)\big) + (\bar{\lambda}, \mathcal{F}) \wedge *\gamma_{(2)}\chi_s \\
& + \mu_1(C) \wedge \mathcal{H}_s + B_s \wedge (\mathcal{F}, \mathcal{F}) + B_s \wedge ([A, A], [A, A]) \Big)
\end{aligned}
$$

This solves problems 1 and 2:

- Choice of gauge structure for ADE-(2,0)-theories clear(er).
- No cubic interaction term for scalar fields

Completing the theory

Adding Pasti-Sorokin-Tonin-type action:

- Recall: PST action has self-duality of H as equation of motion
- Bosonic part of (1,0)-theory was PST completed
  Bandos, Sorokin, Samtleben (2013)
- Full PST action announced, never appeared (not possible?)
- With string structure, construction possible and simplifies

Adding matter fields:

- Add hypermultiplet to get fields of (2,0)-tensor multiplet
- General construction and couplings discussed
  Samtleben, Sezgin, Wimmer (2012)
- Can make concrete choices with twisted string structures

$\Rightarrow$ A (1,0)-theory in 6d satisfying many of the "wishlist" items.

Wishlist

- ✓ Contain an interacting, self-dual 2-form gauge potential
- ✓ Based on a sound mathematical foundation: higher bundles
- ✓ Field content of the $(2,0)$-theory, $\mathcal{N}=(1,0)$ supersymmetric
- ✓ Gauge structure natural, match some expectations (ADE, ...)
- ✓ Non-trivial coupling, interacting field theory
- ✓ Restriction to free $\mathcal{N}=(2,0)$ tensor multiplet possible
- ✓ contains the non-abelian self-dual string soliton as BPS state
- $\rightarrow$ Reduction to 4d SYM theory with ADE gauge algebras
- $\rightarrow$ and to 3d Chern–Simons-matter models with discrete coupling
- ? match expected moduli space of $\mathcal{N}=(2,0)$-theory

Crucial consistency check: Reduction to D-branes/SYM theory

$$S = \int_{\mathbb{R}^{1,5}} \Big(\langle \mathcal{H}, *\mathcal{H} \rangle + \langle \mathrm{d}\phi, *\mathrm{d}\phi \rangle - *\langle \bar{\chi}, \partial\!\!\!/ \chi \rangle + \mathcal{H}_s \wedge *(\bar{\lambda}, \gamma_{(3)}\lambda) + *(Y, \bar{\lambda})\chi_s$$
$$+ \phi_s \big((\mathcal{F}, *\mathcal{F}) - *(Y, Y) + *(\bar{\lambda}, \nabla\!\!\!\!/ \lambda) \big) + (\bar{\lambda}, \mathcal{F}) \wedge *\gamma_{(2)}\chi_s$$
$$+ \mu_1(C) \wedge \mathcal{H}_s + B_s \wedge (\mathcal{F}, \mathcal{F}) + B_s \wedge ([A, A], [A, A]) \Big)$$

- Start from ADE-String Lie 3-algebra
- Anticipate 4d gauge couplings:

$$\tau = \tau_1 + \mathrm{i}\tau_2 = \frac{\theta}{2\pi} + \frac{\mathrm{i}}{g^2_{\mathrm{YM}}} \ ,$$

- VEVs from compactification on T^2 along x^9 and x^{10}

$$\langle \phi_s \rangle = -\frac{1}{32\pi^2} \frac{\tau_2}{R_9 R_{10}} \quad \text{and} \quad \langle B_s \rangle = \frac{1}{16\pi^2} \frac{\tau_1}{R_9 R_{10}}$$

- Strong coupling expansion around VEVs (cf. M2 $\rightarrow$ D2)
- $\Rightarrow$ 4d $\mathcal{N} = 4$ SYM with ADE-gauge group and θ-term

Additional consistency check: Reduction to M2-brane models

- Replace $\mathbb{R}^{1,5}$ by $\mathbb{R}^{1,2} \times S^3$.
- Assumptions:
  - String Lie 3-algebra of $\mathfrak{su}(n) \times \mathfrak{su}(n)$
  - A trivial on S^3, non-trivial on $\mathbb{R}^{1,2}$
  - B trivial on $\mathbb{R}^{1,2}$
  - B encodes abelian gerbe with DD class k on S^3.
- Recall: $\mathcal{H} = \mathrm{d}B + \mathrm{cs}(A)$
- Then we get the integer Chern–Simons coupling:

$$\mathcal{H} \wedge *\mathcal{H} \to k \mathrm{vol}_{S^3} \; \mathrm{cs}(A)$$

$$\int_{\mathbb{R}^{1,5}} \mathcal{H} \wedge *\mathcal{H} \to k \int_{\mathbb{R}^{1,2}} \mathrm{cs}(A)$$

- Altogether: Chern–Simons matter theory of ABJM type.
- Note: This theory has $\mathcal{N} = 4$, different potential from ABJM.

Open Problems

Our model is not the desired (2,0)-theory!

Try to improve this.

Mathematical issue: Model not agnostic about quasi-isomorphism:

$$\mathfrak{su}(2) \leftarrow \mathbb{R} \quad \cong \quad P_0\mathfrak{su}(2) \leftarrow \hat{\Omega}\mathfrak{su}(2)$$

Conclusion: the model of Samtleben et al. is too rigid:

$$(X_r)_s{}^t = f_{rs}^t + d_{rs}^t = f_{[rs]}^t + d_{(rs)}^t$$

Need to generalize the model, redo SUSY computations

(work in progress)

A free Yang–Mills multiplet contradicts $\mathcal{N} = (2, 0)$ supersymmetry.

Recall: M2-brane model also has additional gauge potential, but:

$$F = [X, *\nabla X] + \bar{\Psi}\Gamma^{(2)}\Psi$$

Need: Similar equation for F in our $(1, 0)$-model

3. Moduli space of vacua

Splitting stacks of Dp-branes into smaller stacks: Branching

$$\mathsf{U}(N) \quad \to \quad \mathsf{U}(n_1) \times \mathsf{U}(n_2)$$

Similar branching unclear for string Lie 2-algebra

Need extension:

$$\mathfrak{su}(2) \leftarrow \mathbb{R} \quad \longrightarrow \quad \mathfrak{su}(n) \leftarrow V$$

more freedom than in Lie 2-algebra case due to twist etc.

Ideas? (work in progress)

Two problems:

1. Scalar field with wrong sign kinetic term
2. PST mechanism requires $\phi_s > 0$

First issue may be solved in more general model.

5. Quantizing a PST action?

Potential quantization of PST action unclear.

Quantization of higher Chern–Simons theory: work in progress.

Summary:

- Higher gauge theory classically underlies M-theory
- Higher analogue of $\mathsf{SU}(2)$ is $\mathsf{String}(3)$
- There is non-abelian self-dual string
- There is classical action with many of desired features
- However: Clear differences to (2,0)-theory

Open problems:

- ▷ Compute more general model
- ▷ Construct more general string structures (branching)
- ▷ Study remaining open problems, e.g. quantization
- ▷ Explore model and implications further:
  categorified integrability, fuzzy S^3, etc. (future)

The Non-Abelian Self-Dual String and a 6d Superconformal Field Theory

Christian Sämann

School of Mathematical and Computer Sciences
Heriot-Watt University, Edinburgh

LMS/EPSRC Symposium, Durham, 15.8.2018