Symplectic dg manifolds: integration, differentiation, and boundary field theories

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Integration and differentiation in "higher Lie theory"

The integration/differentiation problem

Upgrade the correspondence

Lie algebra $\leftrightarrow$ Lie group
Lie algebroid $\leftrightarrow$ (local) Lie groupoid

to higher Lie theory

A part of the motivation:

- Poisson manifolds $\leftrightarrow$ (local) symplectic groupoids
- Courant algebroids $\leftrightarrow$ (local) symplectic 2-groupoids
  $+$ other symplectic manifolds coming from CAs (phase spaces of 2-dim sigma-models)

NQ-manifolds, or higher Lie algebroids

NQ-manifold = a $\mathbb{Z}_{\geq 0}$-graded manifold with a homological vector field Q ($Q^2 = 0$, $\deg Q = 1$)
($C^\infty(X)$ is a differential $\mathbb{Z}_{\geq 0}$-graded commutative algebra)
examples:

- $T[1]M$: $C^\infty(T[1]M) = \Omega(M)$, $Q = d$
- $\mathfrak{g}[1]$ for a Lie algebra $\mathfrak{g}$: $C^\infty(\mathfrak{g}[1]) = \bigwedge \mathfrak{g}^*$, $Q = d_{CE}$
- $A[1]$ for a Lie algebroid $A \to M$: $C^\infty(A[1]) = \Gamma(\bigwedge A^*)$, $Q = d_{CE}$

NQ-ideology: generalized manifolds

(and their homotopy groups/oids)

ideology (Sullivan): see NQ-maps$(T[1]M, X)$ as maps$(M, \hat{X})$ for a "generalized manifold" $\hat{X}$ (for $X = T[1]N$ we have $\hat{X} = N$)
in particular: a path in $\hat{X}$ is $T[1]I \to X$, a homotopy of paths is $T[1](I \times I) \to X$, etc.

Example

$X = \mathfrak{g}[1]$: $T[1]M \to \mathfrak{g}[1]$ = a flat $\mathfrak{g}$-connection on M,
$\pi_1(\widehat{\mathfrak{g}[1]}) = G$ the 1-connected Lie group
$X = A[1]$: $\Pi_1(\widehat{A[1]}) = \Gamma$ the source-1-connected (or local) Lie groupoid

$$\deg X := \text{the highest degree of a coordinate of } X$$

$\pi_n^{\text{local}}(\hat{X}) = 0$ for $n > \deg X$, i.e. $\hat{X}$ is a "local homotopy $(\deg X)$-type" (we should expect a local Lie $(\deg X)$-groupoid)

Solving the Maurer-Cartan PDE

Joint work with Michal Širaň

Is NQ-maps$(T[1]N, X)$ a manifold?

Describing NQ-maps: if $Q\xi^i = C^i(\xi)$ (ξ coordinates on X) then an NQ-map $T[1]N \to X$ is $A^i \in \Omega^{\deg \xi^i}(N)$ s.t. $dA^i = C^i(A)$ (generalized MC equation)

Theorem

Suppose N is contractible, h the de Rham homotopy operator. Then $dA = C(A)$ iff $dB = 0$, where $B = A - hC(A)$. $A \mapsto B$ is an open embedding (of Banach or Fréchet manifolds).

Corollary

maps$(\Delta^\bullet, \hat{X}) :=$ NQ-maps$(T[1]\Delta^\bullet, X)$ *is a Kan simplicial (Banach or Fréchet) manifold*

maps$(\Delta^\bullet, \hat{X})$ is the "big version" of the higher groupoid integrating X

Homotopies are easy

Joint work with Michal Širaň

Problem: find/describe all the NQ-maps $T[1](N \times I) \to X$ starting at a given NQ-map $T[1]N \to X$

Theorem

An NQ-map $A : T[1](N \times I) \to X$, $A^i = A^i_t + dt\, H^i_t$, *is uniquely specified by* $A_0 : T[1]N \to X$ *and by* $H^i_t \in \Omega(N)$. *Namely,* $A_t \in \Omega(N)$ *is the solution of the ODE*

$$\frac{d}{dt} A^i_t = dH^i_t + H^j_t \frac{\partial C^i}{\partial \xi^j}(A_t).$$

A_0 and H_t are arbitrary (such that the ODE has a solution).

Corollary

Local homotopy groups are manifolds, they vanish in dimensions higher than $\deg X$

Local Lie n-groupoid (following E. Getzler)

Joint work with Michal Širaň

Problem: replace the simplicial manifold ("big integration" of X) NQ-maps$(T[1]\Delta^\bullet, X)$ with an equivalent finite-dimensional one

Idea (Getzler): impose a gauge condition $sA = 0$, $dA = C(A)$ (and use only small A's)

Theorem

The gauge-fixed NQ-maps $T[1]\Delta^\bullet \to X$ form a finite-dimensional local $\deg X$-groupoid $\int X$, equivalent to the big integration. It is functorial up to coherent homotopies.

Differentiation

Main idea (Kontsevich): $T[1]M = \mathrm{maps}(\mathbb{R}^{0|1}, M)$,
NQ-structure = the action of $\mathrm{End}(\mathbb{R}^{0|1})^{op}$

Ideology

Any NQ-manifold should be of the form $\mathrm{maps}(\mathbb{R}^{0|1}, Z)$ for some "generalized manifold" (i.e. contravariant functor) Z

Example (tautological): if X is an NQ-manifold then
$X = \mathrm{maps}(\mathbb{R}^{0|1}, \hat{X})$
Any Lie n-groupoid K (a simplicial manifold) determines a generalized manifold: $\mathrm{maps}(M, \hat{K}) := \mathrm{maps}_{\mathrm{simpl}}(EM, K)$
Differentiation: $DK := \mathrm{maps}(\mathbb{R}^{0|1}, \hat{K})$ is an NQ-manifold

Differentiation is inverse to the integration

Joint work with Michal Širaň

Want to show $D \int X \cong X$:

$$X = \mathrm{maps}(\mathbb{R}^{0|1}, \hat{X}) = \text{NQ-maps}(T[1]\mathbb{R}^{0|1}, X) \to$$
$$\to \mathrm{maps}_{\mathrm{simpl}}(E\mathbb{R}^{0|1}, \textstyle\int X) = \mathrm{maps}(\mathbb{R}^{0|1}, \widehat{\textstyle\int X}) = D \textstyle\int X$$

One can show that $\to$ is bijective (Dold-Kan correspondence + deformation)

Symplectic structures

If X is an NQ-manifolds, $\omega \in \Omega^2(X)$ a symplectic form, $\deg \omega = n$, $L_Q\omega = 0$, and N a compact oriented n-dim manifold, then

$$\mathrm{maps}(N, \hat{X})/\text{homotopy rel } \partial N \tag{1}$$

is (formally) symplectic (a symplectic manifold if N is contractible and homotopies are small)
[$\int X$ has a symplectic and simplicially closed form on $(\int X)_n$ - a "symplectic n-groupoid"]

Example

$X = \mathfrak{g}[1]$, $n = 2$: moduli space of flat $\mathfrak{g}$-connections on N
$X = T^*[1]M$ (M Poisson), $n = 1$, $N = I$: the (local) symplectic groupoid integrating M

(1) is a great source of Hamiltonian systems (e.g. for T-duality)
(Hamiltonians are suitable functions of the boundary fields)

AKSZ model and its boundary

A space-time picture for the Hamiltonian systems

AKSZ: symplectic NQ manifold (X, ω) ($\deg \omega = n$) $\rightsquigarrow$ $n+1$-dim TFT (in BV formulation); classical solutions $=$ NQ-maps $T[1]K^{n+1} \to X$

Example

$X = \mathfrak{g}$, $\omega = \langle , \rangle$, $n = 2 \rightsquigarrow$ Chern-Simons
$X = T^*[1]M$, $n = 1$, $Q = [\pi, \cdot] \rightsquigarrow$ Poisson σ-model

Boundary condition $=$ an (exact) Lagrangian submanifold in the space of boundary fields
$\rightsquigarrow$ a boundary field theory (non-topological, n-dimensional; cf. CS/WZW)

Example: Chern-Simons and Poisson-Lie T-duality

$$S(A) = \int_K \Big(\frac{1}{2}\big\langle A, dA\big\rangle + \frac{1}{6}\big\langle [A,A], A\big\rangle\Big) \qquad A \in \Omega^1(K, \mathfrak{g})$$

$$\delta S = \int_K \langle \delta A, F\rangle + \frac{1}{2}\int_{\partial K} \langle \delta A, A\rangle$$

Boundary condition: (exact) Lagrangian submanifold in $\Omega^1(\partial K, \mathfrak{g})$

σ-model type boundary condition

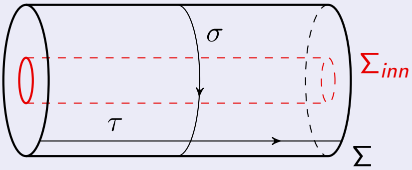
needs a pseudo-Riemannian metric on $\Sigma \subset \partial K$ and $V^+ \subset \mathfrak{g}$

$$*(A|_\Sigma) = \mathbf{V} A|_\Sigma$$

where $\mathbf{V} : \mathfrak{g} \to \mathfrak{g}$ is the reflection w.r.t. V_+ (generalized metric)

Example: Chern-Simons and Poisson-Lie T-duality

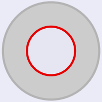
Hollow cylinder: The σ-model with the target G/H



Boundary condition: $*(A|_\Sigma) = \mathbf{V} A|_\Sigma$, $A|_{\Sigma_{inn}} \in \mathfrak{h}$

$$S(A) = \text{“}\int p\, dq - \mathcal{H} d\tau\text{”}, \quad \mathcal{H} = \frac{1}{2}\int_{S^1} \langle A_\sigma, \mathbf{V}(A_\sigma)\rangle\, d\sigma$$

Phase space: moduli space of flat $\mathfrak{g}$-connections on an annulus $\cong T^*(L(G/H))$



Full cylinder: The duality-invariant part (reduced phase space)

General picture and an open problem

Ingredients: symplectic NQ manifold X with $\deg \omega = n$

Phase space = NQ-maps$(T[1]D^n, X)$/htopy rel boundary
+ a Hamiltonian (a function of the boundary field)

Space-time picture: $n+1$-dim AKSZ model given by X, with a (non-topological) boundary condition

$n = 1$: $X = T^*[1]M$, Hamiltonian evolution on (the symplectic groupoid of) M.

$X = T^*[n]T[1]M$ - n-dim σ-model with the target M

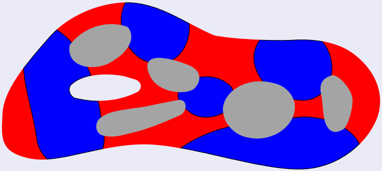
Lagrangian relations between X's give equivalencies/dualities

Problem for $n \geq 3$

Make it compatible with gauge symmetries, find non-trivial dualities of (higher) gauge theories

Open problem: quantization

Kramers-Wannier duality = Poincaré + Poisson



3-dim K

Σ = gray part of ∂K

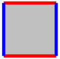
A finite Abelian group

$f : H^1(\Sigma, \partial\Sigma_{red}; A) \to \mathbb{C}$

(Boltzmann weight)

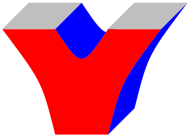
$$Z_{red}(f, A) := \sum_{\alpha \in H^1(Y, \partial Y_{red}; A)} f(i^* \alpha) \qquad Z_{red}(f, A) = Z_{blue}(\hat{f}, A^*)$$

Quantum: 3d TFT with colored boundary (RT TFT given by the double of H) [H semisimple: Turaev-Viro (Freed&Teleman)]

$H = Z($  $)$

Hopf algebra

$\mathfrak{h}, \mathfrak{h}^* \subset \mathfrak{g}$



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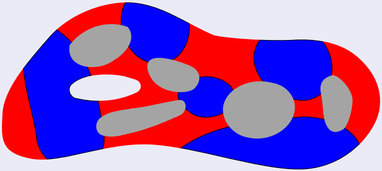

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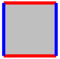
Thanks for your attention!

$H = Z($  $)$

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