Supergravity fluxes and generalised geometry

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General motivation

- $\exists$ "good" understanding of sugra/strings on "nice" geometries

(Special holonomy manifolds e.g. CY3 [Candelas, Horowitz, Strominger & Witten '85])

- Want: to understand general SUSY backgrounds (with fluxes)

  - for phenomenology (moduli stabilisation, SUSY breaking)
  - for AdS/CFT (CFT data $\leftrightarrow$ geometry)
  - for theoretical understanding

- Much work on this! [Too many people to list!]

- At first: Mathematical picture much worse than for e.g. CY3

- Recently: New mathematical picture... Generalised geometry!

[Hitchin '02] [Gualtieri '04]

Talk of two parts

- Overview of generalised geometry and flux backgrounds

  (and some open questions / wish list)

- Recent work on $\mathcal{N} = 1$ heterotic geometry and moduli

  (features L_∞ algebras & holomorphic Chern-Simons)

Generalised geometry and supergravity [Coimbra, CS-C & Waldram '11,'12]

Practical viewpoint: Natural geometry for supergravity

- Geometry of extended tangent bundle $E \simeq TM \oplus$ ("forms")
- Combines : $\begin{cases} \text{diffeo} + \text{gauge symmetry} \rightarrow \text{"Generalised diffeos"} \\ \text{metric} + \text{gauge fields} \rightarrow \text{Generalised metric} \end{cases}$
- Enhanced sym $\begin{cases} GL(d,\mathbb{R}) \rightarrow O(d,d) \times \mathbb{R}^+, \ E_{d(d)} \times \mathbb{R}^+, \text{etc.} \\ SO(d) \rightarrow SO(d) \times SO(d), \ H_d, \text{etc.} \end{cases}$
- $\exists$ Analogues of Levi-Civita connection $D : Q \rightarrow E^* \otimes Q$
- Natural operators and curvatures built from $D \rightarrow$ Supergravity

$$\delta\psi = D \times \epsilon \qquad\qquad R_{AB} = 0$$

- Universal formulae! (& E.g. $E_{d(d)} \times \mathbb{R}^+$ unifies IIA, IIB and 11d)

E.g. Dorfman derivative $\quad L_V = \partial_V - (\partial \times_{\mathrm{ad}} V)\cdot$

$O(d,d)$ Generalised geometry [Hitchin '02] [Gualtieri '04]

E.g. $E \simeq TM \oplus T^*M$ NS-NS fields (g, B_2) (& ϕ via $\mathbb{R}^+$)

- Natural $O(d,d)$ metric $\langle v+\lambda, v+\lambda\rangle = i_v\lambda$
- Dorfman derivative $L_V = (\mathcal{L}_v - \mathrm{d}\lambda\cdot)$
- Generalised metric $G = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \in \frac{O(6,6)}{O(6)\times O(6)}$
- Generalised Levi-Civita D (torsion-free: $L_V^{(D)} = L_V$)

$$\to \text{e.g. SUSY} \begin{cases} \delta\psi_a^- = D_a\epsilon^- = (\nabla_a - \frac{1}{8}H_{abc}\gamma^{bc})\epsilon^- \\ \delta\rho^- = \gamma^{\bar{a}}D_{\bar{a}}\epsilon^- = (\nabla\!\!\!/ - \frac{1}{4}H\!\!\!/ - \partial\!\!\!/\phi)\epsilon^- \end{cases}$$

$$\to \text{Generalised Ricci} \quad \mathcal{R}_{a\bar{b}} = R_{a\bar{b}} - \frac{1}{4}H_a{}^{cd}H_{\bar{b}cd} + \nabla_a\nabla_{\bar{b}}\phi$$

Ordinary G-structures

G-structure $\equiv$ cover of M with local frames for TM related by G

- E.g. Orthonormal frames $g(\hat{e}_a, \hat{e}_b) = \delta_{ab}$

$$\hat{e}'_a = \Lambda_a{}^b \hat{e}_b \text{ with } \Lambda \in O(d) \quad \to \quad O(d) \text{ structure}$$

- E.g. real spinor ϵ in 7d $\quad \to \quad$ Stabiliser $G_2 \subset spin(7)$

$$\to \text{frames } \hat{e}_a \text{ with } \epsilon \propto (1, 0, \ldots, 0) \quad \to \quad G_2 \text{ structure}$$

- Or define via 3-form $\phi_{mnp} \sim \epsilon^T \gamma_{mnp} \epsilon$

- E.g. Almost $\mathbb{C}$-structure $J^2 = -1$ in $2n$ dim

$$\to \quad GL(n, \mathbb{C}) \text{ structure}, \quad \text{frames } \{\hat{e}_a, \hat{e}_{\bar{a}}\}$$

Integrability:

$\boxed{\text{Integrable} \equiv \text{Intrinsic torsion vanishes}}$ E.g. $\begin{cases} \nabla^{\mathsf{LC}} \epsilon = 0 \\ \mathrm{d}\phi = \mathrm{d} * \phi = 0 \end{cases}$

Generalised G-structures and SUSY backgrounds

Gen G-str $\equiv$ cover of M with local frames for E related by G

E.g. $SU(3) \times SU(3) \subset O(6) \times O(6) \subset O(6,6) \times \mathbb{R}^+$

[Grana, Minasian, Petrini & Tomasiello '04]

- Defined by $\epsilon = (\epsilon_1, \epsilon_2)$ Or equiv $\Phi^\pm \in \Lambda^\pm T^*$
- $\mathcal{N} = 2$ SUSY. (Unifies $SU(2)$ and $SU(3)$ structures on TM)
- $\Phi^+ = \mathrm{e}^{\mathrm{i}\omega}$ and $\Phi^- = \Omega$ for plain $SU(3)$ structure
- Integrable $\Leftrightarrow$ $D^{\mathsf{LC}}\epsilon = 0$ $\Leftrightarrow$ $\mathrm{d}\Phi^\pm = 0$

Generalised G-structures and SUSY backgrounds

General picture: [Coimbra, CS-C & Waldram '14] [Coimbra & CS-C '16]

- Unique generalised structure group $\mathcal{G}_{\mathcal{N}}$ for each $\mathcal{N}$

$$\boxed{\mathcal{N}\text{-SUSY Minkowski background} \Leftrightarrow \text{Integrable } \mathcal{G}_{\mathcal{N}} \text{ structure}}$$

E.g. For Type IIA/B and 11d sugra :

d	$\tilde{H}$	$\mathcal{G}_{\mathcal{N}}$ (∼Generalised holonomy)
4	$SU(8)$	$SU(8-\mathcal{N})$
5	$Sp(8)$	$Sp(8-2\mathcal{N})$
6	$Sp(4)\times Sp(4)$	$Sp(4-2\mathcal{N}_+)\times Sp(4-2\mathcal{N}_-)$
7	$Sp(4)$	$Sp(4-2\mathcal{N})$

Consistent truncations [Lee, CS-C & Waldram '14] [Hohm & Samtleben '14]

Formalism: (Maximal SUSY)

- Gen i.d.-str $\equiv$ global frame $\hat{E}_A$ for E
- Get consistent truncation if closes $L_{\hat{E}_A}\hat{E}_B = X_{AB}{}^C \hat{E}_C$

  w/ gauge algebra for embedding tensor $X_{AB}{}^C$
- Gen. Scherk-Schwarz ansatz e.g. $G^{AB} = \delta^{CD} U_C{}^A(x) U_D{}^B(x)$

E.g. Spheres

- S^d Parallelisable in $GL(d+1,\mathbb{R})$ gen geom $E \simeq T \oplus \Lambda^{d-2}T^*$
- Extends to $E_{d(d)} \times \mathbb{R}^+$ for famous S^4, S^5 & S^7
- Structure group of $E \subset$ Structure group of TM

Also, half-maximal via gen G-structures : [Malek '16, '17]

Wish list:

- Full understanding of moduli spaces of (fully back-reacted) SUSY flux backgrounds [Ashmore & Waldram][Garcia-Fernandez, Rubio & Tipler]
- Eff theory including all light modes and gaugings (existence?)
  c.f. [Guerri, Louis, Micu & Waldram '02] [Kashani-Poor & Minasian '06] [Katmadas & Tomasiello '17]
- Duality with flux? (Generalised mirror sym?)
  [Guerri, Louis, Micu & Waldram '02] [Fidanza, Minasian & Tomasiello '03]
- "Parallel transport description" of generalised holonomy?
- New concrete e.g.s / classification of consistent truncations
  [Anderson '17] [d'Inverso '17]
- Extension to non-geometric backgrounds?
- Description of higher-derivative corrections (in e.g. M theory)
  [Coimbra & Minasian '17]
- Mathematical picture of doubled/extended spaces?
  [Hohm, Zwiebach, Berman, Cederwall, Papadopoulos, . . .] [Deser & Sämann]

Generalised geometry for heterotic?

Heterotic supergravity : [Garcia-Fernandez '13] [Coimbra, Minasian, Triendl & Waldram '14]

$$E \simeq T \oplus \mathrm{End}(V) \oplus T^*$$

Find much structure the same:

$\exists$ generalised connection $\rightarrow$ SUSY and eqns of motion etc.

But : Connection is not torsion-free (Torsion $\sim F$)

SUSY $\neq$ Integrability

Holomorphic structures and $\mathcal{N}=1$ heterotic moduli

- Consider $\mathcal{N}=1$ Minkowski backgrounds of heterotic sugra (always $\mathbb{C}$-manifolds!)
- Infinitesimal moduli given by $H^{0,1}_{\bar{D}}(Q)$ where:

$$Q \simeq T^{1,0} \oplus \mathrm{End}(V) \oplus T^{*(1,0)}$$

has holomorphic structure [de la Ossa & Svanes '14][Anderson, Gray & Sharpe '14]

$$\bar{D} : \Omega^{0,p}(Q) \longrightarrow \Omega^{0,p+1}(Q) \quad \text{s.t.} \quad \bar{D}^2 = 0$$

$$\bar{D} \sim \bar{\partial} + F + H \qquad (\Leftrightarrow \text{Bianchi})$$

- How to extend this to (small but) finite deformations?
- Does some generalised geometry based on Q play a role?

Outline of 2nd part of talk

Based on 1806.08367 with Anthony Ashmore, Xenia de la Ossa, Ruben Minasian & Eirik Eik Svanes

- Review: Heterotic supergravity and Hull-Strominger system

  $\rightarrow$ Derive F-terms from superpotential

- Space of off-shell $\mathcal{N} = 1$ geometries

  $\rightarrow$ Expand superpotential in $\mathbb{C}$-coords

- Holomorphic Courant algebroids

  $\rightarrow$ Superpotential as holomorphic Chern-Simons

- F-term conditions from holomorphic L_3 algebra

- Questions for the future

Heterotic supergravity

Field content : $$(g_{MN}, B_{MN}, \phi, A^\alpha; \quad \psi_M, \lambda, \chi^\alpha)$$

$$H_3 := \mathrm{d}B + \frac{\alpha'}{4}(\omega_{\mathsf{CS}}(A) - \omega_{\mathsf{CS}}(\Theta))$$

$$\Rightarrow \quad \text{Bianchi} \quad \mathrm{d}H_3 = \frac{\alpha'}{4}(\mathrm{tr}\, F \wedge F - \mathrm{tr}\, R \wedge R)$$

Trick :

Think of Θ as extra gauge connection!

$$H_3 := \mathrm{d}B + \frac{\alpha'}{4}\omega_{\mathsf{CS}}(A)$$

$$\Rightarrow \quad \mathrm{d}H_3 = \frac{\alpha'}{4}\,\mathrm{tr}\, F \wedge F$$

[Bergshoeff & de Roo '89] [de la Ossa & Svanes '14] [Coimbra, Minasian, Triendl & Waldram '14]

Hull-Strominger system

[Hull '86, Strominger '86]

Minkowski backgrounds : $M^{3,1} \times_{\mathsf{w}} X_6$

(Off-shell) $\mathcal{N} = 1$ SUSY :

$\Rightarrow \exists$ spinor field $\eta \neq 0$ on $X_6 \;\rightarrow\; SU(3)$ structure

$\Rightarrow \exists$ forms : $\omega_{mn} = -\mathrm{i}\,\eta^\dagger \gamma_{mn} \eta, \qquad \Omega_{mnp} = \mathrm{e}^{-2\phi} \eta^{\mathsf{T}} \gamma_{mnp} \eta,$

$\mathcal{N} = 1$ SUSY :

$$\left.\begin{aligned} \mathrm{d}\Omega &= 0 \\ \mathrm{i}(\partial - \bar{\partial})\omega &= H \\ \Omega \wedge F &= 0 \end{aligned}\right\} \text{F-terms}$$

$$\left.\begin{aligned} \omega \lrcorner F &= 0 \\ \mathrm{d}(\mathrm{e}^{-2\phi} \omega \wedge \omega) &= 0 \end{aligned}\right\} \text{D-terms}$$

$$\left[\text{And Bianchi! } : \mathrm{d}H_3 = -2\mathrm{i}\partial\bar{\partial}\omega = \frac{\alpha'}{4} \operatorname{tr} F \wedge F\right]$$

The Superpotential

Superpotential : [Gurrieri, Lukas & Micu '04]

$$W = \int (H + \mathrm{i}\mathrm{d}\omega) \wedge \Omega$$

Stationary zero of W

$$W = \delta W = 0$$

$\Leftrightarrow$ geometry satisfies F-term equations [see: de la Ossa & Svanes '15]

$$\begin{array}{rcll} \mathrm{d}\Omega & = & 0 & \Rightarrow \text{ Complex manifold} \\ \mathrm{i}(\partial - \bar{\partial})\omega & = & H & \\ \Omega \wedge F & = & 0 & \Rightarrow \text{ Holomorphic bundle} \end{array}$$

But: What is the space of geometries?

Space of $\mathcal{N} = 1$ geometries

Degrees of freedom

- For now, ignore gauge field A^α!
- $\mathcal{N} = 1 \;\Rightarrow\; \exists\; SU(3)$ structure on TX $\qquad (\omega, \Omega) \sim \frac{GL(6,\mathbb{R})}{SU(3)}$
- Total d.o.f: $SU(3)$ structure (ω, Ω), 2-form B and ϕ

Geometry

- $\mathcal{N} = 1 \;\Rightarrow\;$ complex structure on space of fields? (Kähler)

How can we understand this?

- Useful tool: "holomorphic variation" operator

$$\Delta = \sum_{n=1}^{\infty} \frac{1}{n!} \epsilon^n \frac{\partial^n}{\partial t^n} \qquad \begin{array}{l} t \to t + \epsilon \\ \bar{t} \to \bar{t} \end{array}$$

Space of almost complex structures

Almost $\mathbb{C}$-structure $(GL(3,\mathbb{C}) \subset GL(6,\mathbb{R}))$

- $J \in \mathrm{End}(TX)$ s.t. $J^2 = -1$
- $TX_{\mathbb{C}} \to T^{1,0} \oplus T^{0,1}$ J specified by choice of $T^{0,1}$

Deforming $J \to J'$

- New $T^{0,1}$ graph of map $\mu : T^{0,1} \to T^{1,0}$
- Think $\mu \in \Omega^{0,1}(T^{1,0})$ local $\mathbb{C}$-coord
- $J' =$ ∞-series in μ and $\bar{\mu}$
- But $\Delta J = -2\mathrm{i}\mu$ (Only $\mu \neq 0$ by definition)

Integrability

$\mathbb{C}$-structure

- Integrable $\Leftrightarrow \bar\partial^2 = 0$
- Deformed $(\bar\partial + \mu^a \partial_a)^2 = 0 \quad \to \quad$ Maurer-Cartan equation

$$\bar\partial\mu + \tfrac{1}{2}[\mu,\mu] = 0$$

Kodaira-Spencer DLGA

- On $\Omega^{0,\bullet}(T^{1,0})$ have $\begin{cases} \bar\partial : \Omega^{0,p}(T^{1,0}) \to \Omega^{0,p+1}(T^{1,0}) \\ [\cdot,\cdot] : \Omega^{0,p}(T^{1,0}) \times \Omega^{0,q}(T^{1,0}) \to \Omega^{0,p+q}(T^{1,0}) \end{cases}$

$$\bar\partial_{[\bar a_1}\mu_{\bar a_2\ldots\bar a_{p+1}]}{}^c \qquad v_{[\bar a_1\ldots\bar a_p}{}^c\partial_c w_{\bar b_1\ldots\bar b_q]}{}^a \pm w_{[\bar b_1\ldots\bar b_q}{}^c\partial_c v_{\bar a_1\ldots\bar a_p]}{}^a$$

- $\bar\partial[v,w] = [\bar\partial v, w] \pm [v, \bar\partial w]$ and (graded) Jacobi

Parameterisation

Key point: Elegant expressions due to nice parameterisation μ

How to pick nice variable μ?

- Use group theory
- $\mathfrak{gl}(6,\mathbb{R}) \to \mathfrak{gl}(3,\mathbb{C}) \oplus \{\mu\} \oplus \{\bar{\mu}\}$
- Complex coordinate on homogenous space $\frac{GL(6,\mathbb{R})}{GL(3,\mathbb{C})}$

Space of $\mathcal{N}=1$ geometries

Think generalised geometry!

Generalised tangent bundle $E \simeq T \oplus T^*$

$$\to \text{Generalised metric} \sim \frac{SO(6,6)\times\mathbb{R}^+}{SO(6)\times SO(6)} \sim (g, B, \phi)$$

$\mathcal{N}=1$ SUSY $\to$ generalised $SU(3)\times SO(6)$ structure

$$\tilde{\chi} \sim \frac{SO(6,6)\times\mathbb{R}^+}{SU(3)\times SO(6)} \sim (\omega, \Omega, B, \phi)$$

Lie algebra decomposition provides analogue of μ

$\to$ corresponds to $(\mu \propto \Delta J, x := \Delta(B+\mathrm{i}\omega)^{1,1}, b := \Delta(B+\mathrm{i}\omega)^{0,2})$

$\to$ $\mathbb{C}$-coords on space of $\mathcal{N}=1$ structures

The superpotential

$\mathcal{N} = 1 \quad \Rightarrow \quad$ Superpotential holomorphic: $W' = W + \Delta W$

Expand superpotential around SUSY vacuum ($W = \delta W = 0$)

$$\begin{aligned}\Delta W &= \int_X \Big(H + \mathrm{i}\mathrm{d}\omega + \mathrm{d}(\Delta B + \mathrm{i}\Delta\omega)\Big) \wedge \Big(\Omega + \Delta\Omega\Big) \\ &= 2\int_X \Big(\mu^d \wedge \bar{\partial} x_d + \tfrac{1}{2}\mu^d \wedge \mu^e \wedge H_{de\bar{c}} e^{\bar{c}} \\ &\qquad + \mu^d \wedge \mu^e \wedge \partial_d x_e - \tfrac{1}{2}\mu^d \wedge \partial_d \tilde{b}\Big) \wedge \Omega\end{aligned}$$

Recall:

$$\mu \in \Omega^{0,1}(T^{1,0}) \qquad x \in \Omega^{1,1}_{\mathbb{C}} \sim \Omega^{0,1}(T^{*(1,0)}) \qquad b \in \Omega^{0,2}$$

Holomorphic Courant algebroids [Gualtieri '10]

Exact sequence of bundles

$$0 \longrightarrow T^{*(1,0)} \longrightarrow Q \longrightarrow T^{(1,0)} \longrightarrow 0$$

Other formulae as for Courant algebroids but with $\frac{\partial}{\partial x^m} \to \frac{\partial}{\partial z^a}$

Pairing: $\langle v+\lambda, v+\lambda\rangle = i_v\lambda$

Courant bracket:

$$[v+\lambda, v'+\lambda'] = [v,v'] + i_v\partial\lambda' - i_{v'}\partial\lambda + \tfrac{1}{2}\partial(i_v\lambda' - i_{v'}\lambda) - i_v i_{v'} H^{3,0}$$

Holomorphic Courant algebroids [Gualtieri '10]

Holomorphic structure

$$\begin{aligned}\bar{D} : \Omega^{0,p}(Q) &\longrightarrow \Omega^{0,p+1}(Q) \\ \bar{D}(v+\lambda) &= \bar{\partial} v + (\bar{\partial}\lambda - i_v \wedge H^{2,1})\end{aligned}$$

$$\bar{D}^2 = 0 \quad \Leftrightarrow \quad \bar{\partial} H^{2,1} = 0$$

More on $\Omega^{0,\bullet}(Q)$

Courant bracket extends to bracket on $\Omega^{0,\bullet}(Q)$

$$\bar{D}[y, y'] = [\bar{D}y, y'] \pm [y, \bar{D}y']$$

$\therefore$ $[\ ,\ ]$ well-defined on $\bar{D}$-holomorphic sections of Q

The superpotential revisited

Package variables: $\mu + x \in \Omega^{0,1}(Q) \qquad b \in \Omega^{0,2}$

Expression becomes

$$\Delta W = \int_X \langle y, \bar{D}y - \tfrac{1}{3}[y,y] - \partial b\rangle \wedge \Omega$$

(c.f. Holomorphic Chern-Simons ...)

$\Rightarrow$ F-term conditions

$$\begin{aligned} \bar{D}y - \tfrac{1}{2}[y,y] - \tfrac{1}{2}\partial b = 0 \quad &\rightarrow \quad \text{Maurer-Cartan? DGLA?} \\ \bar{\partial} b - \tfrac{1}{2}\langle y, \partial b\rangle + \tfrac{1}{3!}\langle y, [y,y]\rangle = 0 \quad &\rightarrow \quad \text{???} \\ \partial \imath_\mu \Omega = 0 \quad &\rightarrow \quad \text{volume preservation} \end{aligned}$$

L_∞ algebras [see Hohm & Zwiebach '17 for a nice discussion]

Graded vector space :

$$\mathcal{Y} = \bigoplus_n \mathcal{Y}_n, \qquad n \in \mathbb{Z}$$

Multilinear brackets : $\ell_1(\cdot) \;\; \ell_2(\cdot,\cdot) \;\; \ell_3(\cdot,\cdot,\cdot) \;\; \ldots$ degree $2-k$

Field equation / Maurer-Cartan equation : $Y \in \mathcal{Y}_1$

$$\mathcal{F}(Y) = \ell_1(Y) - \tfrac{1}{2}\ell_2(Y,Y) - \tfrac{1}{3!}\ell_3(Y,Y,Y) + \ldots$$

Gauge transformation / Symmetry : $\Lambda \in \mathcal{Y}_0$

$$\delta_\Lambda Y = \ell_1(\Lambda) + \ell_2(\Lambda, Y) - \tfrac{1}{2}\ell_3(\Lambda,Y,Y) - \tfrac{1}{3!}\ell_4(\Lambda,Y,Y,Y) + \ldots$$

Courant algebroid as L_3 algebra [Roytenberg & Weinstein '98]

Graded vector space :

$$\mathcal{Y}_0 = \mathcal{Q} \qquad \mathcal{Y}_{-1} = \mathcal{O}_X$$

Multilinear brackets : $V, V' \in \mathcal{Q} \qquad \varphi \in \mathcal{O}_X$

$$\begin{array}{ccc} \ell_1(V) = 0 & \ell_2(V,V') = [V,V'] & \ell_3(V,V',V'') = \langle V, [V',V'']\rangle \\ \ell_1(\varphi) = 0 + \partial\varphi & \ell_2(V,\varphi) = \langle V, \partial\varphi\rangle & +\text{cycles} \end{array}$$

Crucial point : Jacobiator of Courant bracket is ∂-exact

$$\mathrm{Jac}(V,V',V'') = -\tfrac{1}{3}\partial\langle V, [V',V'']\rangle + \text{cycles}$$

Another L_3 algebra

Graded vector space :

$$\mathcal{Y} = \bigoplus_{n \geq -1} \mathcal{Y}_n = \bigoplus_{n \geq -1} \Omega^{0,n}(Q) \oplus \Omega^{0,n+1}$$

Multilinear brackets : $\qquad Y = (y, b)$

$$\begin{aligned}
\ell_1(Y) &= (\bar{D}y + \tfrac{1}{2}(-1)^{Y} \partial b, \bar{\partial} b), \\
\ell_2(Y, Y') &= ([y, y'], \tfrac{1}{2}(\langle y, \partial b' \rangle + (-1)^{1+YY'} \langle y', \partial b \rangle)), \\
\ell_3(Y, Y', Y'') &= \tfrac{1}{3}(-1)^{Y+Y'+Y''}(0, \langle y, [y', y''] \rangle \\
&\qquad + (-1)^{Y(Y'+Y'')} \langle y', [y'', y] \rangle, \\
&\qquad + (-1)^{Y''(Y+Y')} \langle y'', [y, y'] \rangle), \\
\ell_{k \geq 4} &= 0
\end{aligned}$$

F-terms as L_3 Maurer-Cartan

Package variables: $Y = (y, b) \;\in\; \Omega^{0,1}(Q) \oplus \Omega^{0,2} = \mathcal{Y}_1$

Result: F-term conditions

$$\left.\begin{array}{rcl} \bar{D}y - \frac{1}{2}[y,y] - \frac{1}{2}\partial b & = & 0 \\ \bar{\partial} b - \frac{1}{2}\langle y, \partial b\rangle + \frac{1}{3!}\langle y, [y,y]\rangle & = & 0 \end{array}\right\} \leftrightarrow \mathcal{F}(Y) = 0$$

$$\partial \imath_\mu \Omega \;=\; 0 \;\} \rightarrow \text{volume preservation}$$

Including the gauge fields

New $\mathbb{C}$-coordinate: $\alpha \in \Omega^{0,1}(\mathrm{End}(V))$ joins (μ, x, b) from before

New Courant algebroid [c.f. Garcia-Fernandez '13] [Coimbra, Minasian, Triendl & Waldram '14]

$$y = (\mu, \alpha, x) \in \Omega^{0,1}(Q) \quad \text{with} \quad Q \simeq T^{1,0} \oplus \mathrm{End}(V) \oplus T^{*(1,0)}$$

$\exists$ natural modifications of $[\ ,\ ]$, $\bar{D}$, etc. including this part

Superpotential

$$\Delta W = \int_X \langle y, \bar{D}y - \tfrac{1}{3}[y, y] - \partial b \rangle \wedge \Omega$$

(α terms are holomorphic Chern-Simons [Witten '86])

Including the gauge fields

L_3 algebra :

$$\mathcal{Y} = \bigoplus_{n \geq -1} \Omega^{0,n}(Q) \oplus \Omega^{0,n+1}$$

F-term conditions

$$\left.\begin{array}{rcl} \bar{D}y - \frac{1}{2}[y,y] - \frac{1}{2}\partial b & = & 0 \\ \bar{\partial} b - \frac{1}{2}\langle y, \partial b\rangle + \frac{1}{3!}\langle y, [y,y]\rangle & = & 0 \end{array}\right\} \leftrightarrow \mathcal{F}(Y) = 0$$

$$\left.\partial \imath_\mu \Omega \;=\; 0 \right\} \rightarrow \text{volume preservation}$$

Matching infinitesimal story

At first order:

$$\bar{D}y = \tfrac{1}{2}\partial b \qquad \bar{\partial} b = 0 \qquad \partial \imath_\mu \Omega = 0$$

$\partial\bar{\partial}$-lemma $\quad\Rightarrow\quad$ ∂b & $\partial \imath_\mu \Omega$ $\quad$ are $\bar{\partial}$-exact

$$\Rightarrow \begin{cases} \text{can shift } y \text{ to get } \bar{D}y = 0 \\ \text{up to diffeo } \partial \imath_\mu \Omega = 0 \end{cases}$$

So matches earlier result:

$$\text{Infinitesimal moduli space} = H^{0,1}_{\bar{D}}(Q)$$

[de la Ossa & Svanes '14] [Anderson, Gray & Sharpe '14]

Relation to holomorphic L_3 algebra

Dolbeault resolutions: of $\mathcal{O}_X$ and $\mathcal{Q}$ (holomorphic sections of Q)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\iota"] \arrow[d, "\partial"'] & \mathcal{C}^\infty(\mathbb{C}) \arrow[r, "\bar\partial"] \arrow[d, "\partial"'] & \Omega^{(0,1)} \arrow[r, "\bar\partial"] \arrow[d, "\partial"'] & \Omega^{(0,2)} \arrow[r, "\bar\partial"] \arrow[d, "\partial"'] & {} \\
0 \arrow[r] & \mathcal{Q} \arrow[r, "\iota"] & Q \arrow[r, "\bar D"] & \Omega^{(0,1)}(Q) \arrow[r, "\bar D"] & \Omega^{(0,2)}(Q) \arrow[r, "\bar D"] & {}
\end{tikzcd}$$

Total complex $\cong$ original complex

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\partial"] \arrow[d, "\iota"'] & \mathcal{Q} \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & \mathcal{C}^\infty(\mathbb{C}) \arrow[r, "\ell_1"] & \Gamma(Q) \oplus \Omega^{(0,1)} \arrow[r, "\ell_1"] & \Omega^{(0,1)}(Q) \oplus \Omega^{(0,2)} \arrow[r, "\ell_1"] & {}
\end{tikzcd}$$

Relation to holomorphic L_3 algebra

L_∞ quasi-isomorphism :

$$(\mathcal{O}_X \xrightarrow{\partial} \mathcal{Q}) \quad \overset{\text{quasi}}{\cong} \quad \left(\mathcal{Y} = \bigoplus_{n \geq -1} \Omega^{0,n}(Q) \oplus \Omega^{0,n+1}\right)$$

Our L_3 algebra is rewriting of LHS using $\mathcal{C}^\infty$ objects

Conclusions

Holomorphic Courant algebroid $\to L_3$ algebra:

$$Q \simeq T^{1,0} \oplus \mathrm{End}(V) \oplus T^{*(1,0)} \qquad \mathcal{Y} = \bigoplus_{n \geq -1} \Omega^{0,n}(Q) \oplus \Omega^{0,n+1}$$

Superpotential

$$\Delta W = \int_X \langle y, \bar{D}y - \tfrac{1}{3}[y,y] - \partial b \rangle \wedge \Omega$$

F-term conditions

$$\left.\begin{array}{rcl} \bar{D}y - \frac{1}{2}[y,y] - \frac{1}{2}\partial b & = & 0 \\ \bar{\partial} b - \frac{1}{2}\langle y, \partial b\rangle + \frac{1}{3!}\langle y, [y,y]\rangle & = & 0 \end{array}\right\} \leftrightarrow \mathcal{F}(Y) = 0$$

$$\partial \imath_\mu \Omega \;=\; 0 \;\} \to \text{volume preservation}$$

Further points (see paper!)

- Sol$^{\mathsf{n}}$ of F-terms $\Rightarrow \exists$ Sol$^{\mathsf{n}}$ of D-terms

  $\rightarrow$ D-terms are "gauge-fixing conditions"

  $\rightarrow$ Our Maurer-Cartan set is full moduli space

- Effective field theory

Decompose into massless and massive parts: $y = y_0 + y_h$

$$y = y_0 + y_h \qquad \bar{D} y_0 = 0 \quad \rightarrow \quad \text{modes } [y] \in H^{0,1}_{\bar{D}}(Q)$$

Integrate out $b^{0,2}$ and argue that low energy Yukawa couplings are

$$\Delta W_{\mathsf{Yuk}} = \int \langle y_0, [y_0, y_0] \rangle \wedge \Omega$$

c.f. [Witten '86]

Questions

- Obstructions : analyse by formal power series expansion ?

  [c.f. Tian-Todorov construction?]

- Superpotential $\sim$ holomorphic Chern-Simons functional ?
  - Anti-holomorphic coords $\sim$ spacetime?
  - Holomorphic generalised diffeos $\sim$ gauge group ?
  - Couples $\mathbb{C}$-structure and "Kähler" moduli to hCS
  - Quantisation? TQFT? Donaldson-Thomas invariants?
  - SFT for some topological string?

    (c.f. Witten '92, BCOV '93)

Questions

- Worldsheet picture ?
  - How to see L_3 algebra? (Extending Melnikov & Sharpe '11)
  - Holomorphic β-γ system? [see Nekrasov '05]
- G_2 or $Spin(7)$ manifolds? [de la Ossa, Larfors & Svanes]
- How to impose Θ is connection $\nabla + \frac{1}{2}H$ on TM?
- Kähler potential/metric and more on effective theory?

  (See Candelas, de la Ossa & McOrist '16; McOrist '16)

The End

Thanks for your attention!