Quantization of Magnetic Poisson Structures

Richard Szabo

HERIOT WATT UNIVERSITY

MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES

cost Action MP 1405

Quantum Structure of Spacetime

HORIZON 2020

Higher Structures in M-Theory

LMS/EPSRC Durham Symposium

August 17, 2018

Outline

- Magnetic Poisson structures: Description & motivation
- Deformation quantization
- Symplectic realization
- Higher geometric quantization

Magnetic Poisson Structures

Magnetic Poisson Structures

- $M = \mathbb{R}^d$ 'configuration space' x, M^* 'momentum space' p, $\mathcal{M} = T^*M = M \times M^*$ 'phase space' $X = (x, p)$, with canonical symplectic form $\sigma_0(X, X') = p \cdot x' - p' \cdot x$

Magnetic Poisson Structures

- $M = \mathbb{R}^d$ 'configuration space' x, M^* 'momentum space' p, $\mathcal{M} = T^*M = M \times M^*$ 'phase space' $X = (x, p)$, with canonical symplectic form $\sigma_0(X, X') = p \cdot x' - p' \cdot x$
- $\rho \in \Omega^2(M)$ 'magnetic field' deforms σ_0 to almost symplectic form:

$$\sigma_\rho = \sigma_0 - \rho$$

Magnetic Poisson Structures

- $M = \mathbb{R}^d$ 'configuration space' x, M^* 'momentum space' p, $\mathcal{M} = T^*M = M \times M^*$ 'phase space' $X = (x, p)$, with canonical symplectic form $\sigma_0(X, X') = p \cdot x' - p' \cdot x$
- $\rho \in \Omega^2(M)$ 'magnetic field' deforms σ_0 to almost symplectic form:

$$\sigma_\rho = \sigma_0 - \rho$$

- $\theta_\rho = \sigma_\rho^{-1}$ defines magnetic Poisson algebra $\{f, g\}_\rho = \theta_\rho(\mathrm{d}f \wedge \mathrm{d}g)$ on $C^\infty(\mathcal{M})$:

$$\{x^i, x^j\}_\rho = 0 \quad , \quad \{x^i, p_j\}_\rho = \delta^i{}_j \quad , \quad \{p_i, p_j\}_\rho = -\rho_{ij}(x)$$

Magnetic Poisson Structures

- $M = \mathbb{R}^d$ 'configuration space' x, M^* 'momentum space' p, $\mathcal{M} = T^*M = M \times M^*$ 'phase space' $X = (x, p)$, with canonical symplectic form $\sigma_0(X, X') = p \cdot x' - p' \cdot x$

- $\rho \in \Omega^2(M)$ 'magnetic field' deforms σ_0 to almost symplectic form:

$$\sigma_\rho = \sigma_0 - \rho$$

- $\theta_\rho = \sigma_\rho^{-1}$ defines magnetic Poisson algebra $\{f, g\}_\rho = \theta_\rho(\mathrm{d}f \wedge \mathrm{d}g)$ on $C^\infty(\mathcal{M})$:

$$\{x^i, x^j\}_\rho = 0 \quad , \quad \{x^i, p_j\}_\rho = \delta^i{}_j \quad , \quad \{p_i, p_j\}_\rho = -\rho_{ij}(x)$$

- H-twisted Poisson structure on $\mathcal{M}$ with $H = \mathrm{d}\rho$ 'magnetic charge' $[\theta_\rho, \theta_\rho]_{\mathrm{S}} = \bigwedge^3 \theta_\rho^\sharp(\mathrm{d}\sigma_\rho)$ gives nonassociative algebra with Jacobiators $\{f, g, h\}_\rho = [\theta_\rho, \theta_\rho]_{\mathrm{S}}(\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h)$:

$$\{p_i, p_j, p_k\}_\rho = -H_{ijk}(x)$$

(Günaydin & Zumino '84)

Magnetic Monopoles

Magnetic Monopoles

- $d = 3$ and $\rho_{ij} = e\,\varepsilon_{ijk}\,B^k$ governs motion of electric charge e in a magnetic field $\vec{B}$ on $\mathbb{R}^3$

Magnetic Monopoles

- $d = 3$ and $\rho_{ij} = e\, \varepsilon_{ijk}\, B^k$ governs motion of electric charge e in a magnetic field $\vec{B}$ on $\mathbb{R}^3$
- $\mathrm{d}\rho = 0$ gives classical Maxwell theory $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{B} = \vec{\nabla} \times \vec{A}$

Magnetic Monopoles

- $d = 3$ and $\rho_{ij} = e\,\varepsilon_{ijk}\,B^k$ governs motion of electric charge e in a magnetic field $\vec{B}$ on $\mathbb{R}^3$
- $\mathrm{d}\rho = 0$ gives classical Maxwell theory $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{B} = \vec{\nabla} \times \vec{A}$
- Dirac monopole field on $\mathbb{R}^3 \setminus \{0\}$:

$$\vec{B}_{\mathrm{D}} \ = \ g\,\frac{\vec{x}}{|\vec{x}|^3} \ = \ \vec{\nabla} \times \vec{A}_{\mathrm{D}} \qquad , \qquad \vec{A}_{\mathrm{D}} \ = \ \frac{g}{|\vec{x}|}\,\frac{\vec{x} \times \vec{n}}{|\vec{x}| - \vec{x} \cdot \vec{n}}$$

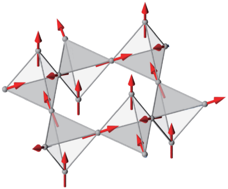
Magnetic Monopoles

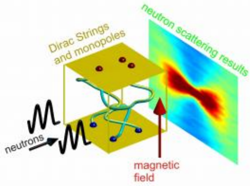
- $d = 3$ and $\rho_{ij} = e\,\varepsilon_{ijk}\,B^k$ governs motion of electric charge e in a magnetic field $\vec{B}$ on $\mathbb{R}^3$
- $\mathrm{d}\rho = 0$ gives classical Maxwell theory $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{B} = \vec{\nabla}\times\vec{A}$
- Dirac monopole field on $\mathbb{R}^3 \setminus \{0\}$:

$$\vec{B}_{\rm D} = g\,\frac{\vec{x}}{|\vec{x}|^3} = \vec{\nabla}\times\vec{A}_{\rm D} \quad , \quad \vec{A}_{\rm D} = \frac{g}{|\vec{x}|}\,\frac{\vec{x}\times\vec{n}}{|\vec{x}| - \vec{x}\cdot\vec{n}}$$

- In the lab: Neutron scattering off spin ice pyrochlore lattices

(Castelnovo, Moessner & Sondhi '08; Morris *et al.* '09; ...)





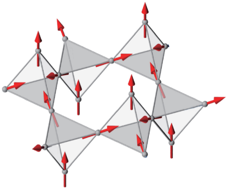
Magnetic Monopoles

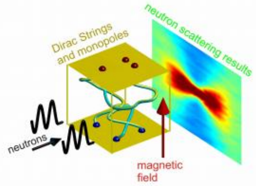
- $d = 3$ and $\rho_{ij} = e\,\varepsilon_{ijk}\,B^k$ governs motion of electric charge e in a magnetic field $\vec{B}$ on $\mathbb{R}^3$
- $\mathrm{d}\rho = 0$ gives classical Maxwell theory $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{B} = \vec{\nabla}\times\vec{A}$
- Dirac monopole field on $\mathbb{R}^3 \setminus \{0\}$:

$$\vec{B}_{\rm D} = g\,\frac{\vec{x}}{|\vec{x}|^3} = \vec{\nabla}\times\vec{A}_{\rm D} \qquad , \qquad \vec{A}_{\rm D} = \frac{g}{|\vec{x}|}\,\frac{\vec{x}\times\vec{n}}{|\vec{x}| - \vec{x}\cdot\vec{n}}$$

- In the lab: Neutron scattering off spin ice pyrochlore lattices

(Castelnovo, Moessner & Sondhi '08; Morris *et al.* '09; . . .)





- Smooth $H = \mathrm{d}\rho \neq 0$ gives smooth distributions of magnetic charge

Locally Non-Geometric Fluxes

Locally Non-Geometric Fluxes

- Magnetic duality $(x,p) \longmapsto (p,-x)$ preserves σ_0, maps $\rho \in \Omega^2(M) \longmapsto \beta \in \Omega^2(M^*)$ with twisted Poisson brackets:

$$\{x^i, x^j\}_\beta = -\beta^{ij}(p) \quad , \quad \{x^i, p_j\}_\beta = \delta^i{}_j \quad , \quad \{p_i, p_j\}_\beta = 0$$

Locally Non-Geometric Fluxes

- Magnetic duality $(x,p) \longmapsto (p,-x)$ preserves σ_0, maps $\rho \in \Omega^2(M) \longmapsto \beta \in \Omega^2(M^*)$ with twisted Poisson brackets:

$$\{x^i, x^j\}_\beta = -\beta^{ij}(p) \quad , \quad \{x^i, p_j\}_\beta = \delta^i{}_j \quad , \quad \{p_i, p_j\}_\beta = 0$$

- Twisting by 'R-flux' $R = \mathrm{d}\beta \in \Omega^3(M^*)$ gives nonassociative configuration space:

$$\{x^i, x^j, x^k\}_\beta = -R^{ijk}(p)$$

Locally Non-Geometric Fluxes

- Magnetic duality $(x,p) \longmapsto (p,-x)$ preserves σ_0, maps $\rho \in \Omega^2(M) \longmapsto \beta \in \Omega^2(M^*)$ with twisted Poisson brackets:

$$\{x^i, x^j\}_\beta = -\beta^{ij}(p) \quad , \quad \{x^i, p_j\}_\beta = \delta^i{}_j \quad , \quad \{p_i, p_j\}_\beta = 0$$

- Twisting by 'R-flux' $R = \mathrm{d}\beta \in \Omega^3(M^*)$ gives nonassociative configuration space:

$$\{x^i, x^j, x^k\}_\beta = -R^{ijk}(p)$$

- *R-flux model:* Phase space of closed strings propagating in 'locally non-geometric' R-flux backgrounds (Blumenhagen & Plauschinn '10; Lüst '10; Blumenhagen, Deser, Lüst, Plauschinn & Rennecke '11; Mylonas, Schupp & Sz '12; Freidel, Leigh & Minic '17; ...)

Locally Non-Geometric Fluxes

- Magnetic duality $(x,p)\longmapsto(p,-x)$ preserves σ_0, maps $\rho\in\Omega^2(M)\ \longmapsto\ \beta\in\Omega^2(M^*)$ with twisted Poisson brackets:

$$\{x^i,x^j\}_\beta=-\beta^{ij}(p)\quad,\quad\{x^i,p_j\}_\beta=\delta^i{}_j\quad,\quad\{p_i,p_j\}_\beta=0$$

- Twisting by 'R-flux' $R=\mathrm{d}\beta\in\Omega^3(M^*)$ gives nonassociative configuration space:

$$\{x^i,x^j,x^k\}_\beta=-R^{ijk}(p)$$

- R-flux model: Phase space of closed strings propagating in 'locally non-geometric' R-flux backgrounds (Blumenhagen & Plauschinn '10; Lüst '10; Blumenhagen, Deser, Lüst, Plauschinn & Rennecke '11; Mylonas, Schupp & Sz '12; Freidel, Leigh & Minic '17; . . .)

- **Questions:**

Locally Non-Geometric Fluxes

- Magnetic duality $(x,p) \longmapsto (p,-x)$ preserves σ_0, maps $\rho \in \Omega^2(M) \ \longmapsto \ \beta \in \Omega^2(M^*)$ with twisted Poisson brackets:

$$\{x^i, x^j\}_\beta = -\beta^{ij}(p) \quad , \quad \{x^i, p_j\}_\beta = \delta^i{}_j \quad , \quad \{p_i, p_j\}_\beta = 0$$

- Twisting by 'R-flux' $R = \mathrm{d}\beta \in \Omega^3(M^*)$ gives nonassociative configuration space:

$$\{x^i, x^j, x^k\}_\beta = -R^{ijk}(p)$$

- R-flux model: Phase space of closed strings propagating in 'locally non-geometric' R-flux backgrounds (Blumenhagen & Plauschinn '10; Lüst '10; Blumenhagen, Deser, Lüst, Plauschinn & Rennecke '11; Mylonas, Schupp & Sz '12; Freidel, Leigh & Minic '17; ...)

- **Questions:**
    - What substitutes for canonical quantization of locally non-geometric closed strings?

Locally Non-Geometric Fluxes

- Magnetic duality $(x,p)\longmapsto(p,-x)$ preserves σ_0, maps $\rho\in\Omega^2(M) \longmapsto \beta\in\Omega^2(M^*)$ with twisted Poisson brackets:

$$\{x^i,x^j\}_\beta=-\beta^{ij}(p) \quad , \quad \{x^i,p_j\}_\beta=\delta^i{}_j \quad , \quad \{p_i,p_j\}_\beta=0$$

- Twisting by 'R-flux' $R=\mathrm{d}\beta\in\Omega^3(M^*)$ gives nonassociative configuration space:

$$\{x^i,x^j,x^k\}_\beta=-R^{ijk}(p)$$

- *R*-flux model: Phase space of closed strings propagating in 'locally non-geometric' R-flux backgrounds (Blumenhagen & Plauschinn '10; Lüst '10; Blumenhagen, Deser, Lüst, Plauschinn & Rennecke '11; Mylonas, Schupp & Sz '12; Freidel, Leigh & Minic '17; ...)

- **Questions:**
  - What substitutes for canonical quantization of locally non-geometric closed strings?
  - What is a sensible nonassociative quantum mechanics?

Quantization of Magnetic Poisson Structures

Quantization of Magnetic Poisson Structures

- Quantization $f \longmapsto \mathcal{O}_f$ for $f \in C^\infty(\mathcal{M})$:

$$[\mathcal{O}_f, \mathcal{O}_g] = \mathrm{i}\,\hbar\, \mathcal{O}_{\{f,g\}_\rho} + O(\hbar^2)$$

Quantization of Magnetic Poisson Structures

- Quantization $f \longmapsto \mathcal{O}_f$ for $f \in C^\infty(\mathcal{M})$:

$$[\mathcal{O}_f, \mathcal{O}_g] = \mathrm{i}\,\hbar\,\mathcal{O}_{\{f,g\}_\rho} + O(\hbar^2)$$

$$[\mathcal{O}_{x^i}, \mathcal{O}_{x^j}] = 0 \quad , \quad [\mathcal{O}_{x^i}, \mathcal{O}_{p_j}] = \mathrm{i}\,\hbar\,\delta^i{}_j\,\mathbb{1} \quad , \quad [\mathcal{O}_{p_i}, \mathcal{O}_{p_j}] = -\,\mathrm{i}\,\hbar\,\rho_{ij}(\mathcal{O}_x)$$

Quantization of Magnetic Poisson Structures

- Quantization $f \longmapsto \mathcal{O}_f$ for $f \in C^\infty(\mathcal{M})$:

$$[\mathcal{O}_f, \mathcal{O}_g] = \mathrm{i}\,\hbar\,\mathcal{O}_{\{f,g\}_\rho} + O(\hbar^2)$$

$$[\mathcal{O}_{x^i}, \mathcal{O}_{x^j}] = 0 \quad , \quad [\mathcal{O}_{x^i}, \mathcal{O}_{p_j}] = \mathrm{i}\,\hbar\,\delta^i{}_j\,\mathbb{1} \quad , \quad [\mathcal{O}_{p_i}, \mathcal{O}_{p_j}] = -\mathrm{i}\,\hbar\,\rho_{ij}(\mathcal{O}_x)$$

- Magnetic translation operators $\mathcal{P}_v = \exp\big(\frac{\mathrm{i}}{\hbar}\,\mathcal{O}_{p\cdot v}\big)$:

$$\mathcal{P}_v^{-1}\,\mathcal{O}_x\,\mathcal{P}_v = \mathcal{O}_{x+v}$$

Quantization of Magnetic Poisson Structures

- Quantization $f \longmapsto \mathcal{O}_f$ for $f \in C^\infty(\mathcal{M})$:

$$[\mathcal{O}_f, \mathcal{O}_g] = \mathrm{i}\,\hbar\,\mathcal{O}_{\{f,g\}_\rho} + O(\hbar^2)$$

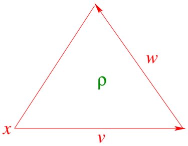
$$[\mathcal{O}_{x^i}, \mathcal{O}_{x^j}] = 0 \quad , \quad [\mathcal{O}_{x^i}, \mathcal{O}_{p_j}] = \mathrm{i}\,\hbar\,\delta^i{}_j\,\mathbb{1} \quad , \quad [\mathcal{O}_{p_i}, \mathcal{O}_{p_j}] = -\,\mathrm{i}\,\hbar\,\rho_{ij}(\mathcal{O}_x)$$

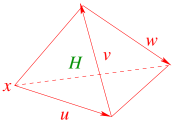
- Magnetic translation operators $\mathcal{P}_v = \exp\big(\frac{\mathrm{i}}{\hbar}\,\mathcal{O}_{p\cdot v}\big)$:

$$\mathcal{P}_v^{-1}\,\mathcal{O}_x\,\mathcal{P}_v = \mathcal{O}_{x+v}$$

- Representation of translation group $\mathbb{R}^d$? (Jackiw '85)

$$\mathcal{P}_w\,\mathcal{P}_v = \mathrm{e}^{\,\mathrm{i}\,\Phi_2(x;v,w)}\,\mathcal{P}_{v+w} \quad , \quad \mathcal{P}_w\,(\mathcal{P}_v\,\mathcal{P}_u) = \mathrm{e}^{\,\mathrm{i}\,\Phi_3(x;u,v,w)}\,(\mathcal{P}_w\,\mathcal{P}_v)\,\mathcal{P}_u$$





Quantization with $\mathrm{d}\rho = 0$

Quantization with $\mathrm{d}\rho = 0$

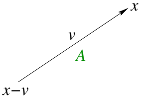
- $\rho = \mathrm{d}A = F_{\nabla^L}$ is curvature of a (trivial) line bundle $L \longrightarrow M = \mathbb{R}^d$

Quantization with $\mathrm{d}\rho = 0$

- $\rho = \mathrm{d}A = F_{\nabla^L}$ is curvature of a (trivial) line bundle $L \longrightarrow M = \mathbb{R}^d$
- $\mathcal{O}_x = x$, $\mathcal{O}_p = -\mathrm{i}\,\hbar\,\nabla^L = -\mathrm{i}\,\hbar\,\mathrm{d} + A$
  act on quantum Hilbert space $\mathcal{H} = \mathrm{L}^2(M, L)$

Quantization with $\mathrm{d}\rho = 0$

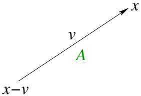
- $\rho = \mathrm{d}A = F_{\nabla^L}$ is curvature of a (trivial) line bundle $L \longrightarrow M = \mathbb{R}^d$
- $\mathcal{O}_x = x \ , \ \mathcal{O}_p = -\mathrm{i}\,\hbar\,\nabla^L = -\mathrm{i}\,\hbar\,\mathrm{d} + A$
  act on quantum Hilbert space $\mathcal{H} = \mathrm{L}^2(M, L)$
- Magnetic translations given by parallel transport in L:



$$(\mathcal{P}_v\psi)(x) = \exp\Big(-\frac{\mathrm{i}}{\hbar}\int_{\triangle^1(x;v)} A\Big)\,\psi(x-v)$$

Quantization with $\mathrm{d}\rho = 0$

- $\rho = \mathrm{d}A = F_{\nabla^L}$ is curvature of a (trivial) line bundle $L \longrightarrow M = \mathbb{R}^d$
- $\mathcal{O}_x = x \ , \ \mathcal{O}_p = -\mathrm{i}\,\hbar\,\nabla^L = -\mathrm{i}\,\hbar\,\mathrm{d} + A$
  act on quantum Hilbert space $\mathcal{H} = \mathrm{L}^2(M, L)$
- Magnetic translations given by parallel transport in L:



$$(\mathcal{P}_v\psi)(x) = \exp\Big(-\frac{\mathrm{i}}{\hbar}\int_{\triangle^1(x;v)} A\Big)\,\psi(x-v)$$

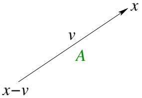
- Defines weak projective representation of translation group $\mathbb{R}^d$ on $\mathcal{H}$:

$$(\mathcal{P}_w\,\mathcal{P}_v\psi)(x) = \omega_{v,w}(x)\ (\mathcal{P}_{v+w}\psi)(x)$$

$$\omega_{v,w}(x) = \exp\Big(-\frac{\mathrm{i}}{\hbar}\int_{\triangle^2(x;w,v)} \rho\Big)$$

Quantization with $\mathrm{d}\rho = 0$

- $\rho = \mathrm{d}A = F_{\nabla^L}$ is curvature of a (trivial) line bundle $L \longrightarrow M = \mathbb{R}^d$
- $\mathcal{O}_x = x \ , \ \mathcal{O}_p = -\mathrm{i}\,\hbar\,\nabla^L = -\mathrm{i}\,\hbar\,\mathrm{d} + A$
  act on quantum Hilbert space $\mathcal{H} = \mathrm{L}^2(M, L)$
- Magnetic translations given by parallel transport in L:



$$(\mathcal{P}_v\psi)(x) = \exp\Big(-\frac{\mathrm{i}}{\hbar}\int_{\triangle^1(x;v)} A\Big)\,\psi(x-v)$$

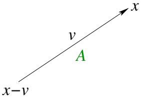
- Defines weak projective representation of translation group $\mathbb{R}^d$ on $\mathcal{H}$:

$$(\mathcal{P}_w\,\mathcal{P}_v\psi)(x) = \omega_{v,w}(x)\ (\mathcal{P}_{v+w}\psi)(x)$$

$$\omega_{v,w}(x) = \exp\Big(-\frac{\mathrm{i}}{\hbar}\int_{\triangle^2(x;w,v)} \rho\Big) \qquad \Big(=\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}\,\rho(v,w)} \text{ for } \rho \text{ constant}\Big)$$

Quantization with $\mathrm{d}\rho = 0$

- $\rho = \mathrm{d}A = F_{\nabla^L}$ is curvature of a (trivial) line bundle $L \longrightarrow M = \mathbb{R}^d$
- $\mathcal{O}_x = x$, $\mathcal{O}_p = -\mathrm{i}\,\hbar\,\nabla^L = -\mathrm{i}\,\hbar\,\mathrm{d} + A$
  act on quantum Hilbert space $\mathcal{H} = \mathrm{L}^2(M, L)$
- Magnetic translations given by parallel transport in L:



$$(\mathcal{P}_v\psi)(x) = \exp\Big(-\frac{\mathrm{i}}{\hbar}\int_{\triangle^1(x;v)} A\Big)\,\psi(x-v)$$

- Defines weak projective representation of translation group $\mathbb{R}^d$ on $\mathcal{H}$:

$$(\mathcal{P}_w\,\mathcal{P}_v\psi)(x) = \omega_{v,w}(x)\ (\mathcal{P}_{v+w}\psi)(x)$$

$$\omega_{v,w}(x) = \exp\Big(-\frac{\mathrm{i}}{\hbar}\int_{\triangle^2(x;w,v)}\rho\Big) \qquad \Big(= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}\,\rho(v,w)} \text{ for } \rho \text{ constant}\Big)$$

- $\omega_{v,w}(x-u)\,\omega^{-1}_{u+v,w}(x)\,\omega_{u,v+w}(x)\,\omega^{-1}_{v,w}(x) = 1$
  2-cocycle on $\mathbb{R}^d$ with values in $C^\infty(M, U(1))$

Quantization with $\mathrm{d}\rho = 0$

- Magnetic Weyl transform $f \in C^\infty(\mathcal{M}) \longmapsto \mathcal{O}_f \in \mathrm{End}(\mathcal{H})$:

$$W(x,p) : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad \big(W(x,p)\psi\big)(y) = \mathrm{e}^{\,\frac{\mathrm{i}\,\hbar}{2}\,p\cdot x}\,\mathrm{e}^{\,-\,\mathrm{i}\,p\cdot y}\,(\mathcal{P}_x\psi)(y)$$

Quantization with $\mathrm{d}\rho = 0$

- Magnetic Weyl transform $f \in C^\infty(\mathcal{M}) \longmapsto \mathcal{O}_f \in \mathrm{End}(\mathcal{H})$:

$$W(x,p) : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad \big(W(x,p)\psi\big)(y) = \mathrm{e}^{\,\frac{\mathrm{i}\,\hbar}{2}\,p\cdot x}\,\mathrm{e}^{\,-\,\mathrm{i}\,p\cdot y}\,(\mathcal{P}_x\psi)(y)$$

$$\mathcal{O}_f = \int_{\mathcal{M}} \bigg(\int_{\mathcal{M}} \mathrm{e}^{\,\mathrm{i}\,\sigma_0(X,Y)}\,f(Y)\,\frac{\mathrm{d}Y}{(2\pi)^d} \bigg)\,W(X)\,\frac{\mathrm{d}X}{(2\pi)^d}$$

Quantization with $\mathrm{d}\rho = 0$

- Magnetic Weyl transform $f \in C^\infty(\mathcal{M}) \longmapsto \mathcal{O}_f \in \mathrm{End}(\mathcal{H})$:

$$W(x,p) : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad \big(W(x,p)\psi\big)(y) = \mathrm{e}^{\,\frac{\mathrm{i}\,\hbar}{2}\,p\cdot x}\,\mathrm{e}^{\,-\,\mathrm{i}\,p\cdot y}\,(\mathcal{P}_x\psi)(y)$$

$$\mathcal{O}_f = \int_{\mathcal{M}} \bigg(\int_{\mathcal{M}} \mathrm{e}^{\,\mathrm{i}\,\sigma_0(X,Y)}\, f(Y)\, \frac{\mathrm{d}Y}{(2\pi)^d} \bigg)\, W(X)\, \frac{\mathrm{d}X}{(2\pi)^d}$$

- Magnetic Moyal–Weyl star product $\mathcal{O}_{f \star_\rho g} = \mathcal{O}_f\, \mathcal{O}_g$:

$$(f \star_\rho g)(X) = \frac{1}{(\pi\,\hbar)^d} \int_{\mathcal{M}} \int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_0(Y,Z)}\, \omega_{x+y-z,x-y+z}(x-y-z)\, f(X-Y)\, g(X-Z)\, \mathrm{d}Y\, \mathrm{d}Z$$

Quantization with $\mathrm{d}\rho = 0$

- Magnetic Weyl transform $f \in C^\infty(\mathcal{M}) \longmapsto \mathcal{O}_f \in \mathrm{End}(\mathcal{H})$:

$$W(x,p) : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad \big(W(x,p)\psi\big)(y) = \mathrm{e}^{\,\frac{\mathrm{i}\,\hbar}{2}\,p\cdot x}\ \mathrm{e}^{\,-\,\mathrm{i}\,p\cdot y}\,(\mathcal{P}_x\psi)(y)$$

$$\mathcal{O}_f = \int_{\mathcal{M}} \Big(\int_{\mathcal{M}} \mathrm{e}^{\,\mathrm{i}\,\sigma_0(X,Y)}\, f(Y)\, \frac{\mathrm{d}Y}{(2\pi)^d} \Big)\, W(X)\, \frac{\mathrm{d}X}{(2\pi)^d}$$

- Magnetic Moyal–Weyl star product $\mathcal{O}_{f\star_\rho g} = \mathcal{O}_f\,\mathcal{O}_g$:

$$(f\star_\rho g)(X) = \frac{1}{(\pi\,\hbar)^d} \int_{\mathcal{M}} \int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_0(Y,Z)}\,\omega_{x+y-z,x-y+z}(x-y-z)\,f(X-Y)\,g(X-Z)\,\mathrm{d}Y\,\mathrm{d}Z$$

$$\Big(= \frac{1}{(\pi\,\hbar)^d} \int_{\mathcal{M}} \int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_\rho(Y,Z)}\, f(X-Y)\,g(X-Z)\,\mathrm{d}Y\,\mathrm{d}Z \quad \text{for } \rho \text{ constant} \Big)$$

Quantization with $\mathrm{d}\rho = 0$

- Magnetic Weyl transform $f \in C^\infty(\mathcal{M}) \longmapsto \mathcal{O}_f \in \mathrm{End}(\mathcal{H})$:

$$W(x,p) : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad \big(W(x,p)\psi\big)(y) = \mathrm{e}^{\,\frac{\mathrm{i}\,\hbar}{2}\,p\cdot x}\,\mathrm{e}^{\,-\,\mathrm{i}\,p\cdot y}\,(\mathcal{P}_x\psi)(y)$$

$$\mathcal{O}_f = \int_{\mathcal{M}} \bigg(\int_{\mathcal{M}} \mathrm{e}^{\,\mathrm{i}\,\sigma_0(X,Y)}\, f(Y)\, \frac{\mathrm{d}Y}{(2\pi)^d} \bigg)\, W(X)\, \frac{\mathrm{d}X}{(2\pi)^d}$$

- Magnetic Moyal–Weyl star product $\mathcal{O}_{f\star_\rho g} = \mathcal{O}_f\,\mathcal{O}_g$:

$$(f\star_\rho g)(X) = \frac{1}{(\pi\,\hbar)^d} \int_{\mathcal{M}}\int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_0(Y,Z)}\,\omega_{x+y-z,x-y+z}(x-y-z)\, f(X-Y)\, g(X-Z)\,\mathrm{d}Y\,\mathrm{d}Z$$

$$\bigg(= \frac{1}{(\pi\,\hbar)^d} \int_{\mathcal{M}}\int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_\rho(Y,Z)}\, f(X-Y)\, g(X-Z)\,\mathrm{d}Y\,\mathrm{d}Z \quad \text{for } \rho \text{ constant} \bigg)$$

- Magnetic translation operators bridge geometric quantization (canonical quantum mechanics) with deformation quantization (phase space quantum mechanics):

Quantization with $\mathrm{d}\rho = 0$

- Magnetic Weyl transform $f \in C^\infty(\mathcal{M}) \longmapsto \mathcal{O}_f \in \mathrm{End}(\mathcal{H})$:

$$W(x,p) : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad \big(W(x,p)\psi\big)(y) = \mathrm{e}^{\,\frac{\mathrm{i}\,\hbar}{2}\,p\cdot x}\ \mathrm{e}^{\,-\,\mathrm{i}\,p\cdot y}\,(\mathcal{P}_x\psi)(y)$$

$$\mathcal{O}_f = \int_{\mathcal{M}} \Big(\int_{\mathcal{M}} \mathrm{e}^{\,\mathrm{i}\,\sigma_0(X,Y)}\, f(Y)\ \frac{\mathrm{d}Y}{(2\pi)^d} \Big)\, W(X)\ \frac{\mathrm{d}X}{(2\pi)^d}$$

- Magnetic Moyal–Weyl star product $\mathcal{O}_{f\star_\rho g} = \mathcal{O}_f\,\mathcal{O}_g$:

$$(f\star_\rho g)(X) = \frac{1}{(\pi\,\hbar)^d}\int_{\mathcal{M}}\int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_0(Y,Z)}\ \omega_{x+y-z,x-y+z}(x-y-z)\, f(X-Y)\, g(X-Z)\ \mathrm{d}Y\ \mathrm{d}Z$$

$$\Big(= \frac{1}{(\pi\,\hbar)^d}\int_{\mathcal{M}}\int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_\rho(Y,Z)}\ f(X-Y)\, g(X-Z)\ \mathrm{d}Y\ \mathrm{d}Z \quad \text{for } \rho \text{ constant}\Big)$$

- Magnetic translation operators bridge geometric quantization (canonical quantum mechanics) with deformation quantization (phase space quantum mechanics):
  - Observables/states: (real) functions on phase space

Quantization with $\mathrm{d}\rho = 0$

- Magnetic Weyl transform $f \in C^\infty(\mathcal{M}) \longmapsto \mathcal{O}_f \in \mathrm{End}(\mathcal{H})$:

$$W(x,p) : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad \big(W(x,p)\psi\big)(y) = \mathrm{e}^{\,\frac{\mathrm{i}\,\hbar}{2}\,p\cdot x}\,\mathrm{e}^{\,-\,\mathrm{i}\,p\cdot y}\,(\mathcal{P}_x\psi)(y)$$

$$\mathcal{O}_f = \int_{\mathcal{M}} \bigg(\int_{\mathcal{M}} \mathrm{e}^{\,\mathrm{i}\,\sigma_0(X,Y)}\, f(Y)\,\frac{\mathrm{d}Y}{(2\pi)^d} \bigg)\, W(X)\,\frac{\mathrm{d}X}{(2\pi)^d}$$

- Magnetic Moyal–Weyl star product $\mathcal{O}_{f\star_\rho g} = \mathcal{O}_f\,\mathcal{O}_g$:

$$(f\star_\rho g)(X) = \frac{1}{(\pi\,\hbar)^d} \int_{\mathcal{M}}\int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_0(Y,Z)}\,\omega_{x+y-z,x-y+z}(x-y-z)\,f(X-Y)\,g(X-Z)\,\mathrm{d}Y\,\mathrm{d}Z$$

$$\bigg(= \frac{1}{(\pi\,\hbar)^d} \int_{\mathcal{M}}\int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_\rho(Y,Z)}\, f(X-Y)\,g(X-Z)\,\mathrm{d}Y\,\mathrm{d}Z \quad \text{for } \rho \text{ constant} \bigg)$$

- Magnetic translation operators bridge geometric quantization (canonical quantum mechanics) with deformation quantization (phase space quantum mechanics):
  - Observables/states: (real) functions on phase space
  - Operator product: star product , traces: integration

Quantization with $\mathrm{d}\rho = 0$

- Magnetic Weyl transform $f \in C^\infty(\mathcal{M}) \longmapsto \mathcal{O}_f \in \mathrm{End}(\mathcal{H})$:

$$W(x,p) : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad \big(W(x,p)\psi\big)(y) = \mathrm{e}^{\,\frac{\mathrm{i}\,\hbar}{2}\,p\cdot x}\,\mathrm{e}^{\,-\,\mathrm{i}\,p\cdot y}\,(\mathcal{P}_x\psi)(y)$$

$$\mathcal{O}_f = \int_{\mathcal{M}} \bigg(\int_{\mathcal{M}} \mathrm{e}^{\,\mathrm{i}\,\sigma_0(X,Y)}\, f(Y)\, \frac{\mathrm{d}Y}{(2\pi)^d} \bigg)\, W(X)\, \frac{\mathrm{d}X}{(2\pi)^d}$$

- Magnetic Moyal–Weyl star product $\mathcal{O}_{f \star_\rho g} = \mathcal{O}_f\, \mathcal{O}_g$:

$$(f \star_\rho g)(X) = \frac{1}{(\pi\,\hbar)^d} \int_{\mathcal{M}} \int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_0(Y,Z)}\, \omega_{x+y-z,x-y+z}(x-y-z)\, f(X-Y)\, g(X-Z)\, \mathrm{d}Y\, \mathrm{d}Z$$

$$\bigg(= \frac{1}{(\pi\,\hbar)^d} \int_{\mathcal{M}} \int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_\rho(Y,Z)}\, f(X-Y)\, g(X-Z)\, \mathrm{d}Y\, \mathrm{d}Z \quad \text{for } \rho \text{ constant} \bigg)$$

- Magnetic translation operators bridge geometric quantization (canonical quantum mechanics) with deformation quantization (phase space quantum mechanics):
  - Observables/states: (real) functions on phase space
  - Operator product: star product , traces: integration
  - State function (density matrix): $S \geqslant 0$, $\int_{\mathcal{M}} S = 1$

Quantization with $\mathrm{d}\rho = 0$

- Magnetic Weyl transform $f \in C^\infty(\mathcal{M}) \longmapsto \mathcal{O}_f \in \mathrm{End}(\mathcal{H})$:

$$W(x,p) : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad \big(W(x,p)\psi\big)(y) = \mathrm{e}^{\,\frac{\mathrm{i}\,\hbar}{2}\,p\cdot x}\,\mathrm{e}^{\,-\,\mathrm{i}\,p\cdot y}\,(\mathcal{P}_x\psi)(y)$$

$$\mathcal{O}_f = \int_{\mathcal{M}} \Big(\int_{\mathcal{M}} \mathrm{e}^{\,\mathrm{i}\,\sigma_0(X,Y)}\, f(Y)\, \frac{\mathrm{d}Y}{(2\pi)^d} \Big)\, W(X)\, \frac{\mathrm{d}X}{(2\pi)^d}$$

- Magnetic Moyal–Weyl star product $\mathcal{O}_{f \star_\rho g} = \mathcal{O}_f\, \mathcal{O}_g$:

$$(f \star_\rho g)(X) = \frac{1}{(\pi\,\hbar)^d} \int_{\mathcal{M}} \int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_0(Y,Z)}\, \omega_{x+y-z,x-y+z}(x-y-z)\, f(X-Y)\, g(X-Z)\, \mathrm{d}Y\, \mathrm{d}Z$$

$$\Big(= \frac{1}{(\pi\,\hbar)^d} \int_{\mathcal{M}} \int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_\rho(Y,Z)}\, f(X-Y)\, g(X-Z)\, \mathrm{d}Y\, \mathrm{d}Z \quad \text{for } \rho \text{ constant} \Big)$$

- Magnetic translation operators bridge geometric quantization (canonical quantum mechanics) with deformation quantization (phase space quantum mechanics):
  - Observables/states: (real) functions on phase space
  - Operator product: star product , traces: integration
  - State function (density matrix): $S \geqslant 0$, $\int_{\mathcal{M}} S = 1$
  - Expectation values: $\langle \mathcal{O} \rangle = \int_{\mathcal{M}} \mathcal{O} \star_\rho S$ ...

Quantization with $H = \mathrm{d}\rho \neq 0$

Quantization with $H = \mathrm{d}\rho \neq 0$

- Operator/state formulation of canonical quantization (geometric quantization) cannot handle nonassociative magnetic Poisson algebras

Quantization with $H = \mathrm{d}\rho \neq 0$

- Operator/state formulation of canonical quantization (geometric quantization) cannot handle nonassociative magnetic Poisson algebras

- For Dirac monopole $\vec{B}_{\mathrm{D}} = g\,\vec{x}/|\vec{x}|^3$:

Quantization with $H = \mathrm{d}\rho \neq 0$

- Operator/state formulation of canonical quantization (geometric quantization) cannot handle nonassociative magnetic Poisson algebras
- For Dirac monopole $\vec{B}_{\rm D} = g\,\vec{x}/|\vec{x}|^3$:
  - Magnetic Poisson algebra is associative on $M^\circ = \mathbb{R}^3 \setminus \{0\}$, $\rho = \mathrm{d}A_{\rm D}$ locally

Quantization with $H = \mathrm{d}\rho \neq 0$

- Operator/state formulation of canonical quantization (geometric quantization) cannot handle nonassociative magnetic Poisson algebras

- For Dirac monopole $\vec{B}_{\mathrm{D}} = g\,\vec{x}/|\vec{x}|^3$:
  - Magnetic Poisson algebra is associative on $M^\circ = \mathbb{R}^3 \setminus \{0\}$, $\rho = \mathrm{d}A_{\mathrm{D}}$ locally
  - Quantum Hilbert space is $\mathcal{H} = \mathrm{L}^2(M^\circ, L)$ for a non-trivial line bundle $L \longrightarrow M^\circ$ iff Dirac charge quantization: $\dfrac{2\,e\,g}{\hbar} \in \mathbb{Z}$

    (Wu & Yang '76)

Quantization with $H = \mathrm{d}\rho \neq 0$

- Operator/state formulation of canonical quantization (geometric quantization) cannot handle nonassociative magnetic Poisson algebras

- For Dirac monopole $\vec{B}_{\rm D} = g\,\vec{x}/|\vec{x}|^3$:
  - Magnetic Poisson algebra is associative on $M^\circ = \mathbb{R}^3 \setminus \{0\}$, $\rho = \mathrm{d}A_{\rm D}$ locally
  - Quantum Hilbert space is $\mathcal{H} = \mathrm{L}^2(M^\circ, L)$ for a non-trivial line bundle $L \longrightarrow M^\circ$ iff Dirac charge quantization: $\frac{2\,e\,g}{\hbar} \in \mathbb{Z}$ (Wu & Yang '76)
  - Magnetic Weyl transform on $\mathcal{M}^\circ$ induces associative phase space star product (Soloviev '17)

Quantization with $H = \mathrm{d}\rho \neq 0$

- Operator/state formulation of canonical quantization (geometric quantization) cannot handle nonassociative magnetic Poisson algebras

- For Dirac monopole $\vec{B}_{\mathrm{D}} = g\,\vec{x}/|\vec{x}|^3$:
  - Magnetic Poisson algebra is associative on $M^\circ = \mathbb{R}^3 \setminus \{0\}$, $\rho = \mathrm{d}A_{\mathrm{D}}$ locally
  - Quantum Hilbert space is $\mathcal{H} = \mathrm{L}^2(M^\circ, L)$ for a non-trivial line bundle $L \longrightarrow M^\circ$ iff Dirac charge quantization: $\frac{2\,e\,g}{\hbar} \in \mathbb{Z}$
    (Wu & Yang '76)
  - Magnetic Weyl transform on $\mathcal{M}^\circ$ induces associative phase space star product
    (Soloviev '17)

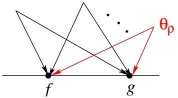
- For generic smooth distributions $H \in \Omega^3(M)$, standard geometric quantization breaks down

Deformation Quantization

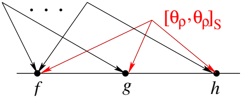
Deformation Quantization

- For any $H = \mathrm{d}\rho \in \Omega^3(M)$, Kontsevich formality provides noncommutative and nonassociative star product on $C^\infty(\mathcal{M})[[\hbar]]$:

$$f \star_H g = f\, g + \frac{\mathrm{i}\,\hbar}{2}\, \{f,g\}_\rho + \sum_{n\geqslant 2} \frac{(\mathrm{i}\,\hbar)^n}{n!}\ \mathtt{b}_n(f,g)$$



$$[f,g,h]_{\star_H} = -\hbar^2\, \{f,g,h\}_\rho + \sum_{n\geqslant 3} \frac{(\mathrm{i}\,\hbar)^n}{n!}\ \mathtt{t}_n(f,g,h)$$

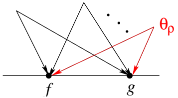


where $\mathtt{b}_n = U_n(\theta_\rho,\dots,\theta_\rho)$ and $\mathtt{t}_n = U_{n+1}([\theta_\rho,\theta_\rho]_{\mathrm{S}},\theta_\rho,\dots,\theta_\rho)$ are bi-/tri-differential operators (Mylonas, Schupp & Sz '12)

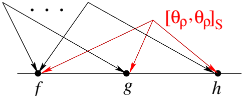
Deformation Quantization

- For any $H = \mathrm{d}\rho \in \Omega^3(M)$, Kontsevich formality provides noncommutative and nonassociative star product on $C^\infty(\mathcal{M})[[\hbar]]$:

$$f \star_H g = f\,g + \frac{\mathrm{i}\,\hbar}{2}\,\{f,g\}_\rho + \sum_{n\geqslant 2} \frac{(\mathrm{i}\,\hbar)^n}{n!}\ \mathtt{b}_n(f,g)$$



$$[f,g,h]_{\star_H} = -\hbar^2\,\{f,g,h\}_\rho + \sum_{n\geqslant 3} \frac{(\mathrm{i}\,\hbar)^n}{n!}\ \mathtt{t}_n(f,g,h)$$



where $\mathtt{b}_n = U_n(\theta_\rho,\dots,\theta_\rho)$ and $\mathtt{t}_n = U_{n+1}([\theta_\rho,\theta_\rho]_{\mathrm{S}},\theta_\rho,\dots,\theta_\rho)$ are bi-/tri-differential operators (Mylonas, Schupp & Sz '12)

- For H constant, $\rho_{ij}(x) = \frac{1}{3}\,H_{ijk}\,x^k$:

$$(f\star_H g)(X) = \frac{1}{(\pi\,\hbar)^d}\int_{\mathcal{M}}\int_{\mathcal{M}} \mathrm{e}^{-\frac{2\,\mathrm{i}}{\hbar}\,\sigma_\rho(Y,Z)}\,f(X-Y)\,g(X-Z)\,\mathrm{d}Y\,\mathrm{d}Z$$

Deformation Quantization

- Nonassociative magnetic translations $\mathcal{P}_v := \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\, p\cdot v}$ give 3-cocycle:

$$\mathcal{P}_v \star_H \mathcal{P}_w \ = \ \Pi_{v,w}(x)\ \mathcal{P}_{v+w}$$

$$\big(\mathcal{P}_u \star_H \mathcal{P}_v\big) \star_H \mathcal{P}_w \ = \ \omega_{u,v,w}(x)\ \mathcal{P}_u \star_H \big(\mathcal{P}_v \star_H \mathcal{P}_w\big)$$

where $\Pi_{v,w}(x) \ = \ \mathrm{e}^{-\frac{\mathrm{i}}{6\hbar}\, H(x,v,w)}$ and $\omega_{u,v,w}(x) \ = \ \mathrm{e}^{\frac{\mathrm{i}}{6\hbar}\, H(u,v,w)}$

Deformation Quantization

- Nonassociative magnetic translations $\mathcal{P}_v := \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\, p\cdot v}$ give 3-cocycle:

$$\mathcal{P}_v \star_H \mathcal{P}_w \;=\; \Pi_{v,w}(x)\; \mathcal{P}_{v+w}$$

$$\big(\mathcal{P}_u \star_H \mathcal{P}_v\big) \star_H \mathcal{P}_w \;=\; \omega_{u,v,w}(x)\; \mathcal{P}_u \star_H \big(\mathcal{P}_v \star_H \mathcal{P}_w\big)$$

where $\Pi_{v,w}(x) \;=\; \mathrm{e}^{-\frac{\mathrm{i}}{6\hbar}\, H(x,v,w)}$ and $\omega_{u,v,w}(x) \;=\; \mathrm{e}^{\frac{\mathrm{i}}{6\hbar}\, H(u,v,w)}$

- Phase space formulation of nonassociative quantum mechanics is physically sensible and gives novel quantitative predictions

(Mylonas, Schupp & Sz '13)

Deformation Quantization

- Nonassociative magnetic translations $\mathcal{P}_v := \mathrm{e}^{\,\frac{\mathrm{i}}{\hbar}\, p\cdot v}$ give 3-cocycle:

$$\mathcal{P}_v \star_H \mathcal{P}_w \;=\; \Pi_{v,w}(x)\, \mathcal{P}_{v+w}$$

$$\big(\mathcal{P}_u \star_H \mathcal{P}_v\big) \star_H \mathcal{P}_w \;=\; \omega_{u,v,w}(x)\, \mathcal{P}_u \star_H \big(\mathcal{P}_v \star_H \mathcal{P}_w\big)$$

where $\Pi_{v,w}(x) \;=\; \mathrm{e}^{\,-\frac{\mathrm{i}}{6\hbar}\, H(x,v,w)}$ and $\omega_{u,v,w}(x) \;=\; \mathrm{e}^{\,\frac{\mathrm{i}}{6\hbar}\, H(u,v,w)}$

- Phase space formulation of nonassociative quantum mechanics is physically sensible and gives novel quantitative predictions

(Mylonas, Schupp & Sz '13)

- R-flux model: Expectation values of oriented volume uncertainty operators $V^{ijk} \;=\; \big\langle \frac{1}{2}\, [\Delta x^i, \Delta x^j, \Delta x^k]_{\star_H} \big\rangle$ give quantum of volume

$$V^{ijk} \;=\; \tfrac{1}{2}\, \hbar^2\, R^{ijk}$$

Deformation Quantization

- Nonassociative magnetic translations $\mathcal{P}_v := \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\, p\cdot v}$ give 3-cocycle:

$$\mathcal{P}_v \star_H \mathcal{P}_w \;=\; \Pi_{v,w}(x)\, \mathcal{P}_{v+w}$$

$$\big(\mathcal{P}_u \star_H \mathcal{P}_v\big) \star_H \mathcal{P}_w \;=\; \omega_{u,v,w}(x)\, \mathcal{P}_u \star_H \big(\mathcal{P}_v \star_H \mathcal{P}_w\big)$$

where $\Pi_{v,w}(x) \;=\; \mathrm{e}^{-\frac{\mathrm{i}}{6\hbar}\, H(x,v,w)}$ and $\omega_{u,v,w}(x) \;=\; \mathrm{e}^{\frac{\mathrm{i}}{6\hbar}\, H(u,v,w)}$

- Phase space formulation of nonassociative quantum mechanics is physically sensible and gives novel quantitative predictions

(Mylonas, Schupp & Sz '13)

- R-flux model: Expectation values of oriented volume uncertainty operators $V^{ijk} \;=\; \big\langle \frac{1}{2}\,[\Delta x^i, \Delta x^j, \Delta x^k]_{\star_H} \big\rangle$ give quantum of volume

$$V^{ijk} \;=\; \tfrac{1}{2}\,\hbar^2\, R^{ijk}$$

For $d = 3$, no D0-branes in locally non-geometric string backgrounds

Deformation Quantization

- Nonassociative magnetic translations $\mathcal{P}_v := \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\, p\cdot v}$ give 3-cocycle:

$$\mathcal{P}_v \star_H \mathcal{P}_w \;=\; \Pi_{v,w}(x)\; \mathcal{P}_{v+w}$$

$$\big(\mathcal{P}_u \star_H \mathcal{P}_v\big) \star_H \mathcal{P}_w \;=\; \omega_{u,v,w}(x)\; \mathcal{P}_u \star_H \big(\mathcal{P}_v \star_H \mathcal{P}_w\big)$$

  where $\Pi_{v,w}(x) \;=\; \mathrm{e}^{-\frac{\mathrm{i}}{6\hbar}\, H(x,v,w)}$ and $\omega_{u,v,w}(x) \;=\; \mathrm{e}^{\frac{\mathrm{i}}{6\hbar}\, H(u,v,w)}$

- Phase space formulation of nonassociative quantum mechanics is physically sensible and gives novel quantitative predictions

  (Mylonas, Schupp & Sz '13)

- R-flux model: Expectation values of oriented volume uncertainty operators $V^{ijk} \;=\; \big\langle \frac{1}{2}\,[\Delta x^i, \Delta x^j, \Delta x^k]_{\star_H}\big\rangle$ give quantum of volume

$$V^{ijk} \;=\; \tfrac{1}{2}\,\hbar^2\, R^{ijk}$$

  For $d=3$, no D0-branes in locally non-geometric string backgrounds

- **Problems:** Quantization formal in $\hbar$ for non-constant H, usual issues with phase space quantum mechanics, ...

Symplectic Realization

Symplectic Realization

- Symplectic realization of a Poisson structure θ on M:
  Symplectic manifold (S, Ω) with surjective submersion $S \longrightarrow M$ which is a Poisson map (Weinstein '83; Karasev '87; Coste, Dazord & Weinstein '87; Cattaneo & Xu '04)

Symplectic Realization

- Symplectic realization of a Poisson structure θ on M:
  Symplectic manifold (S, Ω) with surjective submersion $S \longrightarrow M$ which is a Poisson map (Weinstein '83; Karasev '87; Coste, Dazord & Weinstein '87; Cattaneo & Xu '04)

- Local symplectic realization of magnetic Poisson algebra "doubles" $\mathcal{M}$ to extended phase space $(x^i, \tilde{x}^i, p_i, \tilde{p}_i)$ using local Darboux coordinates (x^i, π_i) and $(\tilde{x}^i, \tilde{\pi}_i)$ with generalized Bopp shifts $p_i = \pi_i - \frac{1}{2}\, \rho_{ij}(x)\, \tilde{x}^j \;,\; \tilde{p}_i = \tilde{\pi}_i$: (Kupriyanov & Sz '18)

Symplectic Realization

- Symplectic realization of a Poisson structure θ on M: Symplectic manifold (S,Ω) with surjective submersion $S \longrightarrow M$ which is a Poisson map (Weinstein '83; Karasev '87; Coste, Dazord & Weinstein '87; Cattaneo & Xu '04)

- Local symplectic realization of magnetic Poisson algebra "doubles" $\mathcal{M}$ to extended phase space $(x^i, \tilde{x}^i, p_i, \tilde{p}_i)$ using local Darboux coordinates (x^i, π_i) and $(\tilde{x}^i, \tilde{\pi}_i)$ with generalized Bopp shifts $p_i = \pi_i - \frac{1}{2}\,\rho_{ij}(x)\,\tilde{x}^j$, $\tilde{p}_i = \tilde{\pi}_i$: (Kupriyanov & Sz '18)

$$\{x^i, p_j\} = \{\tilde{x}^i, p_j\} = \{x^i, \tilde{p}_j\} = \delta^i{}_j$$

$$\{p_i, p_j\} = \rho_{ij}(x) + \tfrac{1}{2}\,\tilde{x}^k\,\big(\partial_k\rho_{ij}(x) - H_{ijk}(x)\big)$$

$$\{p_i, \tilde{p}_j\} = \{\tilde{p}_i, p_j\} = \tfrac{1}{2}\,\rho_{ij}(x)$$

Symplectic Realization

- Symplectic realization of a Poisson structure θ on M: Symplectic manifold (S,Ω) with surjective submersion $S \longrightarrow M$ which is a Poisson map (Weinstein '83; Karasev '87; Coste, Dazord & Weinstein '87; Cattaneo & Xu '04)

- Local symplectic realization of magnetic Poisson algebra "doubles" $\mathcal{M}$ to extended phase space $(x^i,\tilde{x}^i,p_i,\tilde{p}_i)$ using local Darboux coordinates (x^i,π_i) and $(\tilde{x}^i,\tilde{\pi}_i)$ with generalized Bopp shifts $p_i = \pi_i - \frac{1}{2}\,\rho_{ij}(x)\,\tilde{x}^j \;,\; \tilde{p}_i = \tilde{\pi}_i$: (Kupriyanov & Sz '18)

$$\{x^i,p_j\} = \{\tilde{x}^i,p_j\} = \{x^i,\tilde{p}_j\} = \delta^i{}_j$$

$$\{p_i,p_j\} = \rho_{ij}(x) + \tfrac{1}{2}\,\tilde{x}^k\,\big(\partial_k\rho_{ij}(x) - H_{ijk}(x)\big)$$

$$\{p_i,\tilde{p}_j\} = \{\tilde{p}_i,p_j\} = \tfrac{1}{2}\,\rho_{ij}(x)$$

- Quantization on $C^\infty(\mathcal{M})$: $\widehat{\tilde{p}}_i = \mathrm{i}\,\hbar\,\frac{\partial}{\partial x^i} \;,\; \widehat{\tilde{x}}^i = -\,\mathrm{i}\,\hbar\,\frac{\partial}{\partial p_i}$ coincide with associative composition algebra $(\mathrm{Diff}(\mathcal{M}),\circ_H)$ of observables in nonassociative quantum mechanics $(f\circ_H g)\star_H\varphi := f\star_H(g\star_H\varphi)$

Symplectic Realization

- $O(d,d) \times O(d,d)$-invariant Hamiltonian:

$$\mathcal{H} \;=\; \frac{1}{m}\, p_I\, \eta^{IJ}\, p_J \quad , \quad p_I \;=\; (p_i, \tilde{p}_i) \quad , \quad \eta \;=\; \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix}$$

For $d = 3$ reproduces Lorentz force $\dot{\vec{p}} \;=\; \frac{e}{m}\, \vec{p} \times \vec{B} \;\; , \;\; \vec{p} \;=\; m\, \dot{\vec{x}}$

Symplectic Realization

- $O(d,d) \times O(d,d)$-invariant Hamiltonian:

$$\mathcal{H} \;=\; \frac{1}{m}\, p_I\, \eta^{IJ}\, p_J \quad , \quad p_I \;=\; (p_i, \tilde{p}_i) \quad , \quad \eta \;=\; \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix}$$

  For $d = 3$ reproduces Lorentz force $\dot{\vec{p}} \;=\; \frac{e}{m}\, \vec{p} \times \vec{B}$, $\vec{p} \;=\; m\, \dot{\vec{x}}$

- Consistent Hamiltonian reduction eliminates auxiliary coordinates iff $H \;=\; 0$: **No** polarisation of extended symplectic algebra consistent with Lorentz force and nonassociative magnetic Poisson algebra

Symplectic Realization

- $O(d,d)\times O(d,d)$-invariant Hamiltonian:

$$\mathcal{H} \ = \ \frac{1}{m}\, p_I\, \eta^{IJ}\, p_J \quad , \quad p_I \ = \ (p_i, \tilde{p}_i) \quad , \quad \eta \ = \ \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix}$$

  For $d=3$ reproduces Lorentz force $\dot{\vec{p}} \ = \ \frac{e}{m}\,\vec{p}\times\vec{B}$, $\vec{p} \ = \ m\,\dot{\vec{x}}$

- Consistent Hamiltonian reduction eliminates auxiliary coordinates iff $H \ = \ 0$: **No** polarisation of extended symplectic algebra consistent with Lorentz force and nonassociative magnetic Poisson algebra

- Dynamics: For H constant, equivalent to motion in Dirac monopole field $\vec{B}_{\rm D}$ with additional frictional forces (Bakas & Lüst '13)
  Extra degrees of freedom represent reservoir?

Symplectic Realization

- $O(d,d) \times O(d,d)$-invariant Hamiltonian:

$$\mathcal{H} \ = \ \frac{1}{m}\, p_I\, \eta^{IJ}\, p_J \quad , \quad p_I \ = \ (p_i, \tilde{p}_i) \quad , \quad \eta \ = \ \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix}$$

  For $d=3$ reproduces Lorentz force $\dot{\vec{p}} \ = \ \frac{e}{m}\, \vec{p} \times \vec{B}$, $\vec{p} \ = \ m\, \dot{\vec{x}}$

- Consistent Hamiltonian reduction eliminates auxiliary coordinates iff $H \ = \ 0$: **No** polarisation of extended symplectic algebra consistent with Lorentz force and nonassociative magnetic Poisson algebra

- Dynamics: For H constant, equivalent to motion in Dirac monopole field $\vec{B}_{\rm D}$ with additional frictional forces (Bakas & Lüst '13)
  Extra degrees of freedom represent reservoir?

- **Problems:** Physical meaning of spurious degrees of freedom, 3-cocycles for magnetic translations "hidden" in extra variables, ...

Symplectic Realization

- $O(d,d) \times O(d,d)$-invariant Hamiltonian:

$$\mathcal{H} \;=\; \frac{1}{m}\, p_I\, \eta^{IJ}\, p_J \quad , \quad p_I \;=\; (p_i, \tilde{p}_i) \quad , \quad \eta \;=\; \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix}$$

  For $d=3$ reproduces Lorentz force $\dot{\vec{p}} \;=\; \frac{e}{m}\, \vec{p} \times \vec{B}$, $\vec{p} \;=\; m\, \dot{\vec{x}}$

- Consistent Hamiltonian reduction eliminates auxiliary coordinates iff $H \;=\; 0$: **No** polarisation of extended symplectic algebra consistent with Lorentz force and nonassociative magnetic Poisson algebra

- Dynamics: For H constant, equivalent to motion in Dirac monopole field $\vec{B}_{\rm D}$ with additional frictional forces (Bakas & Lüst '13)
  Extra degrees of freedom represent reservoir?

- **Problems:** Physical meaning of spurious degrees of freedom, 3-cocycles for magnetic translations "hidden" in extra variables, ...

- **Higher structures:** Replace Hilbert spaces with 2-Hilbert spaces of sections of a suitable geometric object which encodes $H = {\rm d}\rho \neq 0$

Bundle Gerbes

Bundle Gerbes

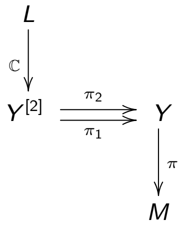
- For $\pi : Y \longrightarrow M$ surjective submersion: $Y^{[p]} := Y \times_M \cdots \times_M Y$ forms a simplicial space with face maps $\pi_i : Y^{[p]} \longrightarrow Y^{[p-1]}$

Bundle Gerbes

- For $\pi : Y \longrightarrow M$ surjective submersion: $Y^{[p]} := Y \times_M \cdots \times_M Y$ forms a simplicial space with face maps $\pi_i : Y^{[p]} \longrightarrow Y^{[p-1]}$
- $\big(Y^{[2]} \rightrightarrows Y\big)$ = pair groupoid with source/target maps π_2/π_1, orbit space M

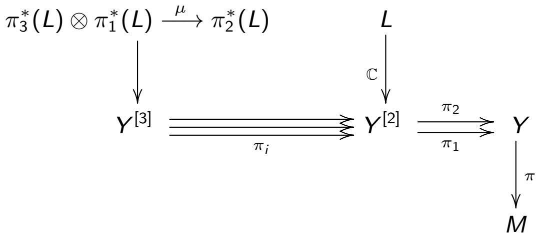
Bundle Gerbes

- For $\pi : Y \longrightarrow M$ surjective submersion: $Y^{[p]} := Y \times_M \cdots \times_M Y$ forms a simplicial space with face maps $\pi_i : Y^{[p]} \longrightarrow Y^{[p-1]}$
- $\big(Y^{[2]} \rightrightarrows Y\big) =$ pair groupoid with source/target maps π_2/π_1, orbit space M
- Bundle gerbe $(L, Y) =$ groupoid central extension: (Murray '96)



Bundle Gerbes

- For $\pi : Y \longrightarrow M$ surjective submersion: $Y^{[p]} := Y \times_M \cdots \times_M Y$ forms a simplicial space with face maps $\pi_i : Y^{[p]} \longrightarrow Y^{[p-1]}$
- $\big(Y^{[2]} \rightrightarrows Y\big)$ = pair groupoid with source/target maps π_2/π_1, orbit space M
- Bundle gerbe (L, Y) = groupoid central extension: (Murray '96)



- Groupoid multiplication gives bundle gerbe multiplication $\mu : \pi_3^*(L) \otimes \pi_1^*(L) \xrightarrow{\simeq} \pi_2^*(L)$ over $Y^{[3]}$, associative over $Y^{[4]}$

Bundle Gerbes

- For $\pi : Y \longrightarrow M$ surjective submersion: $Y^{[p]} := Y \times_M \cdots \times_M Y$ forms a simplicial space with face maps $\pi_i : Y^{[p]} \longrightarrow Y^{[p-1]}$
- $\big(Y^{[2]} \rightrightarrows Y\big) =$ pair groupoid with source/target maps π_2/π_1, orbit space M
- Bundle gerbe (L, Y) = groupoid central extension: (Murray '96)

$$
\begin{array}{ccccc}
\pi_3^*(L) \otimes \pi_1^*(L) \xrightarrow{\ \mu\ } \pi_2^*(L) & & L & & \\
\downarrow & & \downarrow{\scriptstyle \mathbb{C}} & & \\
Y^{[3]} & \overset{\longrightarrow}{\underset{\pi_i}{\substack{\longrightarrow \\ \longrightarrow}}} & Y^{[2]} & \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} & Y \\
 & & & & \downarrow{\scriptstyle \pi} \\
 & & & & M
\end{array}
$$

- Groupoid multiplication gives bundle gerbe multiplication $\mu : \pi_3^*(L) \otimes \pi_1^*(L) \xrightarrow{\ \simeq\ } \pi_2^*(L)$ over $Y^{[3]}$, associative over $Y^{[4]}$
- Connection: $\rho \in \Omega^2(Y)$ satisfying $\pi_2^*(\rho) - \pi_1^*(\rho) = F_{\nabla^L}$, curvature is $\pi^* H = \mathrm{d}\rho$, $H \in \Omega^3(M)$

Sections of Bundle Gerbes

Sections of Bundle Gerbes

- Bundle gerbes $\mathcal{G} = (L, Y)$ are objects in a symmetric monoidal 2-category (Waldorf '07; Bunk, Sämann & Sz '16)

Sections of Bundle Gerbes

- Bundle gerbes $\mathcal{G} = (L, Y)$ are objects in a symmetric monoidal 2-category (Waldorf '07; Bunk, Sämann & Sz '16)
- 2-Hilbert space of sections $\Gamma(M, \mathcal{G})$: Hilb-module category of morphisms $\mathcal{I}_0 \longrightarrow \mathcal{G}$ from trivial bundle gerbe $\mathcal{I}_0$ with:

Sections of Bundle Gerbes

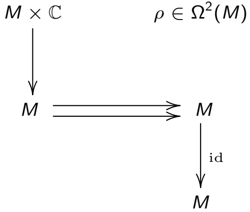
- Bundle gerbes $\mathcal{G} = (L, Y)$ are objects in a symmetric monoidal 2-category (Waldorf '07; Bunk, Sämann & Sz '16)
- 2-Hilbert space of sections $\Gamma(M, \mathcal{G})$: Hilb-module category of morphisms $\mathcal{I}_0 \longrightarrow \mathcal{G}$ from trivial bundle gerbe $\mathcal{I}_0$ with:
  - Rig module category structure over rig category HVbdl(M)

Sections of Bundle Gerbes

- Bundle gerbes $\mathcal{G} = (L, Y)$ are objects in a symmetric monoidal 2-category (Waldorf '07; Bunk, Sämann & Sz '16)
- 2-Hilbert space of sections $\Gamma(M, \mathcal{G})$: Hilb-module category of morphisms $\mathcal{I}_0 \longrightarrow \mathcal{G}$ from trivial bundle gerbe $\mathcal{I}_0$ with:
  - Rig module category structure over rig category HVbdl(M)
  - Inner product bifunctor $\langle\ ,\ \rangle : \Gamma(M, \mathcal{G})^{\rm op} \times \Gamma(M, \mathcal{G}) \longrightarrow$ Hilb

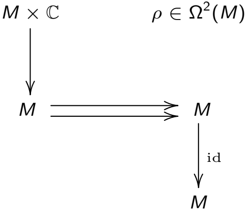
Sections of Bundle Gerbes

- Bundle gerbes $\mathcal{G} = (L, Y)$ are objects in a symmetric monoidal 2-category (Waldorf '07; Bunk, Sämann & Sz '16)
- 2-Hilbert space of sections $\Gamma(M, \mathcal{G})$: Hilb-module category of morphisms $\mathcal{I}_0 \longrightarrow \mathcal{G}$ from trivial bundle gerbe $\mathcal{I}_0$ with:
  - Rig module category structure over rig category HVbdl(M)
  - Inner product bifunctor $\langle \ , \ \rangle : \Gamma(M, \mathcal{G})^{\mathrm{op}} \times \Gamma(M, \mathcal{G}) \longrightarrow \mathrm{Hilb}$
- Simple description on $M = \mathbb{R}^d$ for trivial bundle gerbes $\mathcal{G} = \mathcal{I}_\rho$:



Sections of Bundle Gerbes

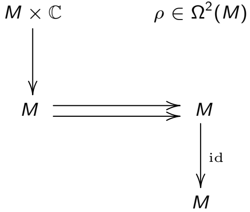
- Bundle gerbes $\mathcal{G} = (L, Y)$ are objects in a symmetric monoidal 2-category (Waldorf '07; Bunk, Sämann & Sz '16)
- 2-Hilbert space of sections $\Gamma(M, \mathcal{G})$: Hilb-module category of morphisms $\mathcal{I}_0 \longrightarrow \mathcal{G}$ from trivial bundle gerbe $\mathcal{I}_0$ with:
  - Rig module category structure over rig category $\mathrm{HVbdl}(M)$
  - Inner product bifunctor $\langle\ ,\ \rangle : \Gamma(M, \mathcal{G})^{\mathrm{op}} \times \Gamma(M, \mathcal{G}) \longrightarrow \mathrm{Hilb}$
- Simple description on $M = \mathbb{R}^d$ for trivial bundle gerbes $\mathcal{G} = \mathcal{I}_\rho$:



  - Objects are vector bundles with connection: $\eta \in \Omega^1(M, u(n))$

Sections of Bundle Gerbes

- Bundle gerbes $\mathcal{G} = (L, Y)$ are objects in a symmetric monoidal 2-category (Waldorf '07; Bunk, Sämann & Sz '16)
- 2-Hilbert space of sections $\Gamma(M, \mathcal{G})$: Hilb-module category of morphisms $\mathcal{I}_0 \longrightarrow \mathcal{G}$ from trivial bundle gerbe $\mathcal{I}_0$ with:
  - Rig module category structure over rig category HVbdl(M)
  - Inner product bifunctor $\langle\ ,\ \rangle : \Gamma(M, \mathcal{G})^{\mathrm{op}} \times \Gamma(M, \mathcal{G}) \longrightarrow$ Hilb
- Simple description on $M = \mathbb{R}^d$ for trivial bundle gerbes $\mathcal{G} = \mathcal{I}_\rho$:



  - Objects are vector bundles with connection: $\eta \in \Omega^1(M, u(n))$
  - Morphisms are parallel morphisms: $f : \eta \longrightarrow \eta'$ is a function $f : M \longrightarrow \mathrm{Mat}(n \times n')$ satisfying $\mathrm{i}\,\eta' f = \mathrm{i} f \eta - \mathrm{d} f$

Nonassociative Magnetic Translations

Nonassociative Magnetic Translations

- Parallel transport functor $\mathcal{P}_v : \Gamma(M,\mathcal{I}_\rho) \longrightarrow \Gamma(M,\mathcal{I}_\rho)$:

(Bunk, Müller & Sz '18)

$$\mathcal{P}_v(\eta)|_x(a) \;=\; \eta|_{x-v}(a) + \frac{1}{\hbar}\int_{\triangle^1(x;v)} \iota_a\rho \quad , \quad \mathcal{P}_v(f)(x) \;=\; f(x-v)$$

Weak module functor: $\mathcal{P}_v(\xi\otimes\eta) = v^*(\xi)\otimes\mathcal{P}_v(\eta)$, $\;\xi\in\Omega^1(M,u(k))$

Nonassociative Magnetic Translations

- Parallel transport functor $\mathcal{P}_v : \Gamma(M,\mathcal{I}_\rho) \longrightarrow \Gamma(M,\mathcal{I}_\rho)$:

(Bunk, Müller & Sz '18)

$$\mathcal{P}_v(\eta)|_x(a) \ = \ \eta|_{x-v}(a) + \frac{1}{\hbar}\int_{\triangle^1(x;v)} \iota_a\rho \quad , \quad \mathcal{P}_v(f)(x) \ = \ f(x-v)$$

Weak module functor: $\mathcal{P}_v(\xi\otimes\eta) = v^*(\xi)\otimes\mathcal{P}_v(\eta)$, $\ \xi\in\Omega^1(M,u(k))$

- Coherence isomorphisms $\Pi_{v,w} : \mathcal{P}_v\circ\mathcal{P}_w \Longrightarrow \chi_{v,w}\otimes\mathcal{P}_{v+w}$:

$\chi_{v,w}|_x(a) \ = \ \frac{1}{\hbar}\int_{\triangle^2(x;w,v)} \iota_a H$ (connection 1-form of trivial line bundle on M)

$$\Pi_{v,w|\eta}(x) := \exp\Big(-\frac{\mathrm{i}}{\hbar}\int_{\triangle^2(x;w,v)} \rho\Big)$$

Nonassociative Magnetic Translations

- Parallel transport functor $\mathcal{P}_v : \Gamma(M, \mathcal{I}_\rho) \longrightarrow \Gamma(M, \mathcal{I}_\rho)$:

(Bunk, Müller & Sz '18)

$$\mathcal{P}_v(\eta)|_x(a) \ = \ \eta|_{x-v}(a) + \frac{1}{\hbar} \int_{\triangle^1(x;v)} \iota_a \rho \quad , \quad \mathcal{P}_v(f)(x) \ = \ f(x-v)$$

Weak module functor: $\mathcal{P}_v(\xi \otimes \eta) = v^*(\xi) \otimes \mathcal{P}_v(\eta)$, $\xi \in \Omega^1(M, u(k))$

- Coherence isomorphisms $\Pi_{v,w} : \mathcal{P}_v \circ \mathcal{P}_w \Longrightarrow \chi_{v,w} \otimes \mathcal{P}_{v+w}$:

$$\chi_{v,w}|_x(a) \ = \ \frac{1}{\hbar} \int_{\triangle^2(x;w,v)} \iota_a H \quad \text{(connection 1-form of trivial line bundle on } M\text{)}$$

$$\Pi_{v,w|\eta}(x) := \exp\Big(-\frac{\mathrm{i}}{\hbar} \int_{\triangle^2(x;w,v)} \rho \Big) \quad \Big(= \mathrm{e}^{-\frac{\mathrm{i}}{6\hbar} H(x,v,w)} \text{ for } H \text{ constant} \Big)$$

Nonassociative Magnetic Translations

- Parallel transport functor $\mathcal{P}_v : \Gamma(M,\mathcal{I}_\rho) \longrightarrow \Gamma(M,\mathcal{I}_\rho)$:

(Bunk, Müller & Sz '18)

$$\mathcal{P}_v(\eta)|_x(a) \;=\; \eta|_{x-v}(a) + \frac{1}{\hbar}\int_{\triangle^1(x;v)} \iota_a\rho \quad , \quad \mathcal{P}_v(f)(x) \;=\; f(x-v)$$

Weak module functor: $\mathcal{P}_v(\xi\otimes\eta) = v^*(\xi)\otimes\mathcal{P}_v(\eta)$, $\xi\in\Omega^1(M,u(k))$

- Coherence isomorphisms $\Pi_{v,w} : \mathcal{P}_v\circ\mathcal{P}_w \Longrightarrow \chi_{v,w}\otimes\mathcal{P}_{v+w}$:

$$\chi_{v,w}|_x(a) \;=\; \frac{1}{\hbar}\int_{\triangle^2(x;w,v)} \iota_a H$$ (connection 1-form of trivial line bundle on M)

$$\Pi_{v,w|\eta}(x) := \exp\Big(-\frac{\mathrm{i}}{\hbar}\int_{\triangle^2(x;w,v)}\rho\Big)$$ $\Big(=\mathrm{e}^{-\frac{\mathrm{i}}{6\hbar}H(x,v,w)}$ for H constant$\Big)$

- "Nonassociativity" of $\mathcal{P}_u\circ\mathcal{P}_v\circ\mathcal{P}_w$:

$$\Pi_{u+v,w}\circ\Pi_{u,v}(x) \;=\; \omega_{u,v,w}(x)\;\Pi_{u,v+w}\circ\mathcal{P}_u(\Pi_{v,w})(x)$$

$$\omega_{u,v,w} : \chi_{u+v,w}\otimes\chi_{u,v} \longrightarrow \chi_{u,v+w}\otimes u^*(\chi_{v,w})$$

$$\omega_{u,v,w}(x) := \exp\Big(\frac{\mathrm{i}}{\hbar}\int_{\triangle^3(x;w,v,u)} H\Big)$$

Nonassociative Magnetic Translations

- Parallel transport functor $\mathcal{P}_v : \Gamma(M, \mathcal{I}_\rho) \longrightarrow \Gamma(M, \mathcal{I}_\rho)$:

(Bunk, Müller & Sz '18)

$$\mathcal{P}_v(\eta)|_x(a) \;=\; \eta|_{x-v}(a) + \frac{1}{\hbar} \int_{\triangle^1(x;v)} \iota_a \rho \quad , \quad \mathcal{P}_v(f)(x) \;=\; f(x-v)$$

Weak module functor: $\mathcal{P}_v(\xi \otimes \eta) = v^*(\xi) \otimes \mathcal{P}_v(\eta)$, $\xi \in \Omega^1(M, u(k))$

- Coherence isomorphisms $\Pi_{v,w} : \mathcal{P}_v \circ \mathcal{P}_w \Longrightarrow \chi_{v,w} \otimes \mathcal{P}_{v+w}$:

$$\chi_{v,w}|_x(a) \;=\; \frac{1}{\hbar} \int_{\triangle^2(x;w,v)} \iota_a H \quad \text{(connection 1-form of trivial line bundle on } M\text{)}$$

$$\Pi_{v,w|\eta}(x) := \exp\Big(-\frac{\mathrm{i}}{\hbar} \int_{\triangle^2(x;w,v)} \rho \Big) \quad \Big(= \mathrm{e}^{\,-\frac{\mathrm{i}}{6\hbar}\, H(x,v,w)} \text{ for } H \text{ constant} \Big)$$

- "Nonassociativity" of $\mathcal{P}_u \circ \mathcal{P}_v \circ \mathcal{P}_w$:

$$\Pi_{u+v,w} \circ \Pi_{u,v}(x) \;=\; \omega_{u,v,w}(x)\; \Pi_{u,v+w} \circ \mathcal{P}_u(\Pi_{v,w})(x)$$

$$\omega_{u,v,w} : \chi_{u+v,w} \otimes \chi_{u,v} \longrightarrow \chi_{u,v+w} \otimes u^*(\chi_{v,w})$$

$$\omega_{u,v,w}(x) := \exp\Big(\frac{\mathrm{i}}{\hbar} \int_{\triangle^3(x;w,v,u)} H \Big) \quad \Big(= \mathrm{e}^{\,\frac{\mathrm{i}}{6\hbar}\, H(u,v,w)} \text{ for } H \text{ constant} \Big)$$

Higher Projective Representations

Higher Projective Representations

- $\omega_{u,v,w}$ define a 3-cocycle on $\mathbb{R}^d$ with values in $C^\infty(M, U(1))$

Higher Projective Representations

- $\omega_{u,v,w}$ define a 3-cocycle on $\mathbb{R}^d$ with values in $C^\infty(M, U(1))$
- $(\chi_{v,w}, \omega_{u,v,w})$ define a higher weak 2-cocycle on $\mathbb{R}^d$ with values in the Hilb-algebra category $\mathsf{HVbdl}(M)$

Higher Projective Representations

- $\omega_{u,v,w}$ define a 3-cocycle on $\mathbb{R}^d$ with values in $C^\infty(M, U(1))$
- $(\chi_{v,w}, \omega_{u,v,w})$ define a higher weak 2-cocycle on $\mathbb{R}^d$ with values in the Hilb-algebra category HVbdl(M)
- **Theorem:** (Bunk, Müller & Sz '18)
  $(\mathcal{P}_v, \Pi_{v,w})$ define a weak projective 2-representation of the translation group $\mathbb{R}^d$ on the HVbdl(M)-module category $\Gamma(M, \mathcal{I}_\rho)$ (2-Hilbert space of sections of the bundle gerbe $\mathcal{I}_\rho$)

Higher Projective Representations

- $\omega_{u,v,w}$ define a 3-cocycle on $\mathbb{R}^d$ with values in $C^\infty(M, U(1))$
- $(\chi_{v,w}, \omega_{u,v,w})$ define a higher weak 2-cocycle on $\mathbb{R}^d$ with values in the Hilb-algebra category $\mathsf{HVbdl}(M)$
- **Theorem:** (Bunk, Müller & Sz '18)
  $(\mathcal{P}_v, \Pi_{v,w})$ define a weak projective 2-representation of the translation group $\mathbb{R}^d$ on the $\mathsf{HVbdl}(M)$-module category $\Gamma(M, \mathcal{I}_\rho)$ (2-Hilbert space of sections of the bundle gerbe $\mathcal{I}_\rho$)
- **Open issues:**

Higher Projective Representations

- $\omega_{u,v,w}$ define a 3-cocycle on $\mathbb{R}^d$ with values in $C^\infty(M, U(1))$
- $(\chi_{v,w}, \omega_{u,v,w})$ define a higher weak 2-cocycle on $\mathbb{R}^d$ with values in the Hilb-algebra category HVbdl(M)
- **Theorem:** (Bunk, Müller & Sz '18)
  $(\mathcal{P}_v, \Pi_{v,w})$ define a weak projective 2-representation of the translation group $\mathbb{R}^d$ on the HVbdl(M)-module category $\Gamma(M, \mathcal{I}_\rho)$ (2-Hilbert space of sections of the bundle gerbe $\mathcal{I}_\rho$)
- **Open issues:**
  - Understand physical significance of 2-Hilbert space $\Gamma(M, \mathcal{I}_\rho)$: states, observables, . . .

Higher Projective Representations

- $\omega_{u,v,w}$ define a 3-cocycle on $\mathbb{R}^d$ with values in $C^\infty(M, U(1))$
- $(\chi_{v,w}, \omega_{u,v,w})$ define a higher weak 2-cocycle on $\mathbb{R}^d$ with values in the Hilb-algebra category HVbdl(M)
- **Theorem:** (Bunk, Müller & Sz '18)
  $(\mathcal{P}_v, \Pi_{v,w})$ define a weak projective 2-representation of the translation group $\mathbb{R}^d$ on the HVbdl(M)-module category $\Gamma(M, \mathcal{I}_\rho)$ (2-Hilbert space of sections of the bundle gerbe $\mathcal{I}_\rho$)
- **Open issues:**
  - Understand physical significance of 2-Hilbert space $\Gamma(M, \mathcal{I}_\rho)$: states, observables, ...
  - Develop "Higher magnetic Weyl transform" to bridge higher geometric quantization with deformation quantization

Many Thanks to . . .

Many Thanks to . . .

Brano, Christian, Urs & Martin

for a great week!