# Optimal rate of convergence in periodic homogenization of Hamilton-Jacobi equations

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Assume $H(p, x) \in C(\mathbb{R}^n \times \mathbb{R}^n)$ is uniformly coercive in the $p$ variable and periodic in the $x$ variable.
For each $\epsilon > 0$, let $u^\epsilon \in C(\mathbb{R}^n \times [0, \infty))$ be the viscosity solution to the following Hamilton-Jacobi equation

$$\begin{cases} u_t^\epsilon + H\left(Du^\epsilon, \frac{x}{\epsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u^\epsilon(x, 0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1}$$

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It was known (**Lions-Papanicolaou-Varadhan, 1987**), that $u^\epsilon$, as $\epsilon \to 0$, converges locally uniformly to $u$, the solution of the effective equation,

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{2}$$

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$\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is called "effective Hamiltonian" or "$\alpha$ function", a nonlinear averaging of the original $H$.

Cell problem: for any $p \in \mathbb{R}^n$, there exists a **UNIQUE** number $\overline{H}(p)$ such that

$$H(p + Dv, y) = \overline{H}(p) \quad \text{in } \mathbb{T}^n.$$

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$$u^\epsilon(x, t) \approx u(x, t) + \epsilon v(\frac{x}{\epsilon}, Du).$$

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**However, there is NO way to justify this expansion rigorously!**

## Previous Results

Why does the expansion not hold generically?

(I) The solution of the effective equation $u(x,t)$ is in general not even $C^1$;
(2) There does not even exist a continuous selection of

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**Strategy:** (1) Using solutions to an auxiliary equation $v_\lambda$ to replace $v$.

$$\lambda v_\lambda + H(p + Dv_\lambda, x) = 0;$$

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• *Note: Armstrong, Cardaliaguet and Souganidis (2014) extended this to convex $H$ in the i.i.d setting and obtained* $O(\epsilon^{1/8})$

# Open Question

Whether the convergence rate $O(\epsilon^{1/3})$ can be improved?

In particular, when can we obtain the optimal one $O(\epsilon)$?

Note: It is basically impossible to modify or refine the **Capuzzo-Dolcetta–H. Ishii method** to achieve this goal. A completely new approach has to be introduced.

# Main Result 1: General Convex Case ($p \to H(p,x)$)

Theorem (Mitake, Tran, Y. 2018)

*Assume $H$ is onvex in $p$ and $g \in \mathrm{Lip}(\mathbb{R}^n)$.*
*(i)*

$$u^\epsilon(x,t) \geq u(x,t) - C\epsilon \qquad \textit{for all } (x,t) \in \mathbb{R}^n \times [0,\infty).$$

*The constant $C > 0$ in (i) and (ii) below depend only on $H$ and $\|Dg\|_{L^\infty(\mathbb{R}^n)}$.*

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*(ii) For fixed $(x,t) \in \mathbb{R}^n \times (0,\infty)$, if $u$ is differentiable at $(x,t)$ and $\overline{H}$ is twice differentiable at $p = Du(x,t)$, then*

$$u^\epsilon(x,t) \leq u(x,t) + \widetilde{C}_{x,t}\epsilon.$$

*if the initial data $g \in C^2(\mathbb{R}^n)$ with $\|g\|_{C^2(\mathbb{R}^n)} < \infty$. If $g$ is merely Lipschitz continuous, then*

$$u^\epsilon(x,t) \leq u(x,t) + C_{x,t}\sqrt{\epsilon}.$$

# Optimal Rate when $n = 2$

**Theorem (Mitake, Tran, Y. 2018)**

*Assume $n = 2$ and $g \in \mathrm{Lip}(\mathbb{R}^2)$. Assume further that $H$ is convex and positively homogeneous of degree $k$ in $p$ for some $k \geq 1$, that is, $H(\lambda p, x) = \lambda^k H(p, x)$ for all $(\lambda, x, p) \in [0, \infty) \times \mathbb{T}^2 \times \mathrm{R}^2$. Then,*

$$|u^\epsilon(x,t) - u(x,t)| \leq C\epsilon \qquad \text{for all } (x,t) \in \mathrm{R}^2 \times [0,\infty).$$

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Note that $k = 1$ corresponds to Hamiltonians associated with the front propagation, which is probably one of the most physically relevant situations in the homogenization theory. For example,

$$u_t + a(x)|Du| = 0 \quad \text{in crystal growth, etc}$$

and the well known **G-equation** in turbulent combustion

$$u_t + |Du| + V(x) \cdot Du = 0.$$

## Optimal Rate when $n = 1$

Theorem (Mitake, Tran, Y. 2018)

*Assume that* $n = 1$ *and* $H = H(p, x)$ *is convex in* $p$. *Assume further that* $g \in \mathrm{Lip}(\mathbb{R})$. *Then, for each* $T > 0$,

$$\|u^\epsilon - u\|_{L^\infty(\mathbb{R}\times[0,T])} \le C\epsilon.$$

*Here* $C$ *is a constant depending only on* $H$ *and* $\|g'\|_{L^\infty(\mathbb{R})}$.

- Son N.T. Tu extended to $H(u_x, x/\epsilon, x)$ when $n = 1$ for some $H$ (arxiv).

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• For the one dimension case, the remaining question is to find the optimal rate for general coercive $H$ (i.e. **Nonconvex** $H$). Recall that the **Capuzzo-Dolcetta–Ishii** result says that

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- We conjecture that the optimal rate is $O\left(\sqrt{\epsilon}\right)$.

# Sketch of Proof of the Lower Bound $u^\epsilon \geq u - C\epsilon$

$$u^\epsilon(0,1) = \inf_{\eta(0)=0} \left\{ g\left(\epsilon\eta\left(-\epsilon^{-1}\right)\right) + \epsilon \int_{-\epsilon^{-1}}^{0} L(\eta(t), \dot\eta(t))\, dt \right\}$$

Here $L(q,x) = \sup_{p\in\mathbb{R}^n}\{p\cdot q - H(p,x)\}$. Also,

$$u(0,1) = \inf_{y\in\mathbb{R}^n} \left\{ g(y) + \overline{L}(-y) \right\}.$$

For any $p \in \mathbb{R}^n$ and a "corrector" $v_p$:

$$H(p + Dv_p, y) = \overline{H}(p),$$

$$\int_{-\epsilon^{-1}}^{0} L(\eta(t), \dot\eta(t)) + \overline{H}(p)\, dt \geq p\cdot\eta(0) - p\cdot\eta\left(-\epsilon^{-1}\right) + v_p(\eta(0)) - v_p\left(\eta\left(-\epsilon^{-1}\right)\right)$$

Accordingly, since $\overline{L}(q) = \sup_{p\in\mathbb{R}^n}\{p\cdot q - \overline{H}(p)\}$,

$$\epsilon\int_{-\epsilon^{-1}}^{0} (L(\eta(t), \dot\eta(t))\, dt \geq \overline{L}\left(-\epsilon\eta\left(-\epsilon^{-1}\right)\right) - C\epsilon.$$

# The Upper Bound and the Hamiltonian System

For any $p \in \mathbb{R}^n$, if $\xi : \mathbb{R} \to \mathbb{R}^n$ is a **global charateristics** of a corrector $v_p$, i.e.,

$$p \cdot (\xi(t_2) - \xi(t_1)) + v_p(\xi(t_2)) - v_p(\xi(t_1)) = \int_{t_1}^{t_2} L(\dot{\xi}, \xi) + \overline{H}(p)\, ds.$$

for all $t_1 < t_2$. The collection of those $\xi$ is the so called **"Mané set"** in weak KAM theory. Such $\xi$ is an **absolute minimizer** of the action

$$A[\gamma] = \int L(\dot{\gamma}(t), \gamma(t)) + \overline{H}(p)\, dt.$$

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**Question:** Does the average slope

$$\frac{\xi(t)}{t}$$

converge as $t \to \infty$? More importantly, what is the **convergence rate**?

• It is known that in **weak KAM theory/Aubry-Mather theory** that if $\overline{H}$ is differentiable at $p$, then

$$\lim_{t\to\infty} \frac{\xi(t)}{t} = D\overline{H}(p). \tag{3}$$

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Connection with the convergence rate in homogenization:

$$(I). \quad \left|\frac{\xi(t)}{t} - D\overline{H}(p)\right| \leq \frac{C}{t} \Rightarrow |u^\epsilon - u| \leq O(\epsilon) \quad \text{for } g \in Lip(\mathbb{R}^n)$$

$$(II). \quad \left|\frac{\xi(t)}{t} - D\overline{H}(p)\right| \leq \frac{C}{\sqrt{t}} \Rightarrow \begin{cases} |u^\epsilon - u| \leq O(\sqrt{\epsilon}) & \text{for } g \in Lip(\mathbb{R}^n) \\ |u^\epsilon - u| \leq O(\epsilon) & \text{for } g \in C^2(\mathbb{R}^n). \end{cases}$$

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By modifying the argument of (3), it is easy to show that if $\overline{H}$ is twice differentiable at $p$, then (Gomes 2002)

$$\left|\frac{\xi(t)}{t}-D\overline{H}(p)\right|\leq\frac{C}{\sqrt{t}}.$$

## $n = 2$ and the Aubry-Mather Theory

**Key ingredient:** 2d topology + the fact that two absolute minimizers $\xi$ cannot intersect twice lead to good description of the structure of absolute minimizers (**Aubry-Mather sets** basically consist of recurrent ones).

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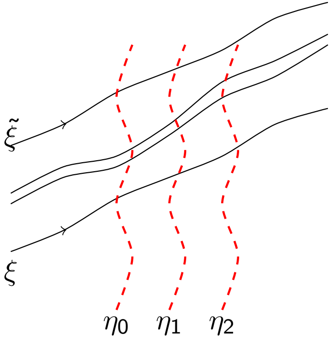
**Key ingredient:** 2d topology + the fact that two absolute minimizers $\xi$ cannot intersect twice lead to good description of the structure of absolute minimizers (**Aubry-Mather sets** basically consist of recurrent ones).

• In particular, each absolute minimizer can be identifed with a **circle map**: $f : \mathbb{R} \to \mathbb{R}$, continuous, increasing and $f(x+1) = f(x) + 1$.




There exists a rotation number $\alpha$ such that $|f^i(x) - x - \alpha i| \leq 1$ for all $i$.

## Connection with the Convergence Rate

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• Combining with the circle map identification and some weak KAM type calcuations, we can deduce that for any global charateristics $\xi : \mathbb{R} \to \mathbb{R}$:

$$\left| \frac{\xi(t)}{t} - D\overline{H}(p) \right| \le \frac{C}{t}.$$

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**Conjecture:** For a general convex and coercive $H(p, x)$ when $n = 2$, we are working on to show that

$$|u^{\epsilon}(x, t) - u(x, t)| \leq C_{x,t}\, \epsilon \quad \text{for a.e. } (x, t) \in \mathbb{R}^2 \times (0, +\infty).$$

# Some Remarks about the Higher Dimension Case $n \geq 3$

• When $n \geq 3$, there is **NO** topological obstructions for absolutely minimizing curves. The generalized **Aubry-Mather theory** has very limited applicability in obtaining properties of $\overline{H}$.

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For example, consider a simple metric Hamiltonian with smooth, positive and periodic $a(x)$

$$H(p,x) = a(x)|p|.$$

and the associated effective Hamiltonian $\overline{H}(p)$:

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Does there exist a non-constant smooth $a(x)$ such that $\overline{H} \equiv |p|$?

When $n = 2$, the answer is "No" (**Bangert, 1994**) based on Aubry-Mather

# Lack of Examples with Fractional Convergence Rate

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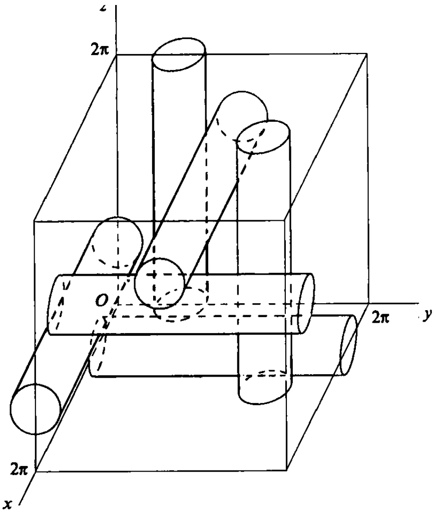
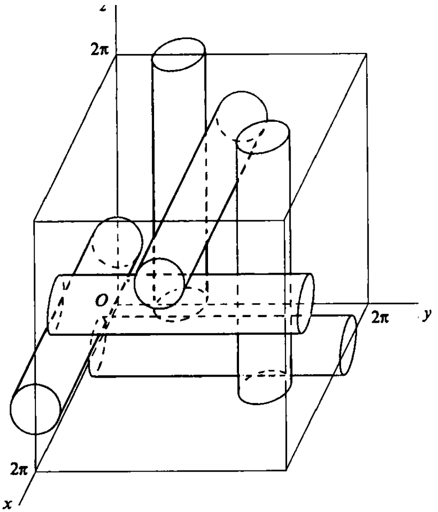
When $n \geq 3$, the only well-understood interesting example is the classical **Hedlund example**: The metric function $a(x)$ is a smooth periodic singular pertubation of 1 such that any minimizing geodesics is basically confined in a small neighbourhood of one of three disjoint parallel lines.

So the **Aubry-Mather set** is very small and

$$\overline{H}(p) = C \max\{|p_1|,\ |p_2|,\ |p_3|\}.$$

• The level surface is a cube, in particular not $C^1$, which is **different** from $n = 2$.


z
2π
O
2π
y
2π
x



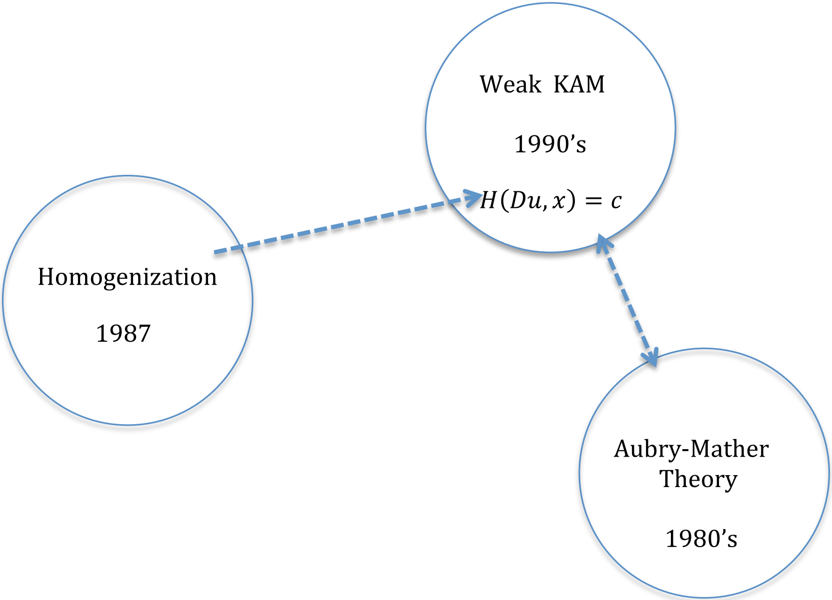

However, for this sort of "bad" example, the convergence rate is $O(\epsilon)$.

Homogenization

1987

Aubry-Mather Theory

1980's


Weak KAM
1990's
$H(Du,x) = c$
Homogenization
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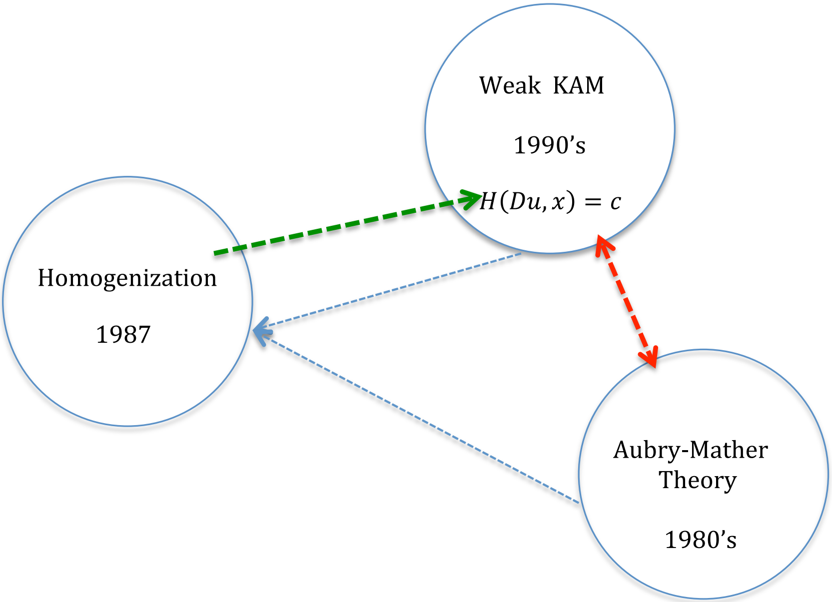
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