A Liouville theorem for stationary and ergodic ensembles of parabolic systems

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I. Introduction

We consider random ensembles of parabolic systems

$$\partial_t u = \nabla \cdot a \nabla u \;\text{ in }\; \mathbb{R}^d \times (0, \infty),$$

with space-time dependent coefficients $a := a(x,t)$.

Qualitative theory (essentially) due to Papanicolaou and Vardhan [15] and Kozlov [14].

- intrinsic (extended) corrector
- large-scale regularity estimate
- Liouville principles
- Simon [16] (deterministic), Avellaneda and Lin [4, 5, 6] (periodic)

I. Introduction

The coefficients are indexed by a probability space $a \in (\Omega, \mathcal{F}, \mathbb{P})$.

- *Stationarity*: The environment is statistically homogenous.
- *Ergodicity*: The environment is mixing: stationary random variables exhibit deterministic averaging effects on large scales.
- *Uniform Ellipticity*: There exists $\lambda \in (0,1]$ such that, for $\langle\cdot\rangle$-a.e a,

$$|a\xi| \leq |\xi| \;\text{ and }\; \lambda|\xi|^2 \leq \xi \cdot a\xi \;\text{ for each }\; \xi \in \mathbb{R}^d.$$

II. Deterministic elliptic setting

The space of strictly sub-quadratic harmonic functions

$$\limsup_{|x|\to\infty} \frac{|u(x)|}{|x|^2} = 0 \text{ with } -\Delta u = 0 \text{ in } \mathbb{R}^d,$$

are spanned by the constant and coordinate functions

$$x \in \mathbb{R}^d \mapsto x_i \text{ for } i \in \{1, \ldots, d\}.$$

Higher order statements hold: for $n \geq 1$, the space

$$\limsup_{|x|\to\infty} \frac{|u(x)|}{|x|^{1+n}} = 0 \text{ with } -\Delta u = 0 \text{ in } \mathbb{R}^d,$$

is the space of harmonic polynomials of degree $\leq n$.

II. Deterministic elliptic setting

For an α-Hölder continuous function,

$$\begin{aligned} R^{-2\alpha} \fint_{B_R} |u(x)-u(0)|^2 \,\mathrm{d}x &\leq [u]_{C^{0,\alpha}(B_R)} \fint_{B_R} |x|^{2\alpha} \,\mathrm{d}x \\ &\lesssim [u]_{C^{0,\alpha}(B_R)} R^{-(2\alpha+d)} \int_0^R r^{2\alpha+d-1} \,\mathrm{d}r \\ &\lesssim [u]_{C^{0,\alpha}(B_R)}. \end{aligned}$$

Indeed, from Campanato [10],

$$[u]_{C^{0,\alpha}(B_R)} \simeq \sup_{z\in B_R} \sup_{r\in(0,d(z,\partial B_R))} \inf_{c\in\mathbb{R}} r^{-2\alpha} \fint_{B_r(z)} |u(x)-c|^2 \,\mathrm{d}x.$$

III. Random elliptic setting

The homogenization corrector ϕ_i solves

$$-\nabla \cdot a\left(\nabla \phi_i + e_i\right) = 0 \ \text{ in } \ \mathbb{R}^d \ \text{ for } \ i \in \{1, \ldots, d\},$$

with stationary, finite-energy gradient. It is sub-linear (hence, quadratic) in the sense that

$$\limsup_{R \to \infty} \frac{1}{R} \left(\fint_{B_R} |\phi_i|^2 \right)^{\frac{1}{2}} = 0.$$

The correctors define the a-harmonic coordinates:

$$x \in \mathbb{R}^d \mapsto x_i + \phi_i(x) \ \text{ for } \ i \in \{1, \ldots, d\}.$$

III. Random elliptic setting

The correctors define the coordinate functions of the flux:

$$q_i := a(\nabla \phi_i + e_i) \text{ for } i \in \{1, \ldots, d\}.$$

The homogenized coefficient field is the expectation of the flux:

$$a_{\text{hom}} e_i := \langle a(\nabla \phi_i + e_i) \rangle .$$

On large-scales, a-harmonic functions inherit the properties of a_{hom}-harmonic functions

$$-\nabla \cdot a_{\text{hom}} \nabla v = 0 \text{ in } \mathbb{R}^d,$$

measured with respect to the intrinsic geometry of the coefficient field.

III. Random elliptic setting

Versions of excess were introduced by

- Avellaneda and Lin (periodic)
- Armstrong and Smart [3] (random)
- Gloria, Neukamm, and Otto [13] (random)

For each $R > 0$, the excess on scale R is

$$\mathrm{Exc}(R; u) := \inf_{\xi \in \mathbb{R}^d} \fint_{B_R} (\nabla u - \xi - \nabla \phi_\xi) \cdot a \left(\nabla u - \xi - \nabla \phi_\xi\right),$$

for the corrector

$$\phi_\xi := \xi_i \phi_i.$$

III. Random elliptic setting

The excess defines an intrinsic large-scale α-Hölder norm:

$$R^{-2\alpha}\mathrm{Exc}(R;u) \simeq R^{-2\alpha} \fint_{B_R} (\nabla u - \xi - \nabla\phi_\xi) \cdot a\,(\nabla u - \xi - \nabla\phi_\xi)\,.$$

Theorem [13] (Gloria, Neukamm, Otto)

Assume stationarity, ergodicity, and uniform ellipticity. For $\langle\cdot\rangle$-a.e. a, there exists $r_*(a) \in (0,\infty)$ such that, for every $r_* < R_1 < R_2 < \infty$,

$$R_1^{-2\alpha}\mathrm{Exc}(R_1;u) \lesssim R_2^{-2\alpha}\mathrm{Exc}(R_2;u).$$

Or,

$$\mathrm{Exc}(R_1;u) \leq \left(\frac{R_1}{R_2}\right)^{2\alpha} \mathrm{Exc}(R_2;u).$$

III. Random elliptic setting

The energy of the homogenization error

$$w := u - (1 + \eta\phi_i\partial_i)v,$$

provides a good proxy for the excess, where

$$\begin{cases} -\nabla \cdot a_{\text{hom}} \nabla v = 0 & \text{in } B_R, \\ v = u & \text{on } \partial B_R, \end{cases}$$

and where η is a cutoff ensuring vanishing boundary conditions.

It suffices to show that, for all $R > 0$ sufficiently large,

$$\fint_{B_{\frac{R}{2}}} \nabla w \cdot a \nabla w \leq \left(\frac{1}{2}\right)^{2\alpha} \fint_{B_R} \nabla u \cdot a \nabla u.$$

III. Random elliptic setting

The flux q is divergence-free:

$$-\nabla \cdot q_i = -\nabla \cdot a\left(\nabla \phi_i + e_i\right) = 0 \;\text{ for }\; i \in \{1, \ldots, d\}.$$

As a closed $(d-1)$ form, the flux correction solves the exterior equation

$$\nabla \cdot \sigma_i = q_i - \langle q_i \rangle \;\text{ for }\; i \in \{1, \ldots, d\},$$

with the choice of gauge

$$\Delta \sigma_{ijk} = \partial_k q_j - \partial_j q_k \;\text{ for }\; i, j, k \in \{1, \ldots, d\}.$$

Here, the σ_i are skew-symmetric matrices ($(d-2)$-forms) and

$$\left(\nabla \cdot \sigma_i\right)_j := \partial_k \sigma_{ijk}.$$

III. Random elliptic setting

The homogenization error solves, in B_R,

$$\begin{aligned} -\nabla \cdot a\nabla w =& \nabla \cdot ((1-\eta)(a-a_{\mathrm{hom}})\nabla v + (\phi_i a - \sigma_i)\nabla(\eta\partial_i v)) \\ =& \nabla \cdot ((\phi_i a - \sigma_i)\nabla(\eta\partial_i v)) + \text{ boundary terms,} \end{aligned}$$

After testing this equation with w, and using the interior/boundary regularity of v:

$$\int_{B_{\frac{R}{2}}} \nabla w \cdot a\nabla w \lesssim \left(\frac{1}{R^2} \fint_{B_R} |\phi|^2 + |\sigma|^2 \,\mathrm{d}x\right) \fint_{B_R} \nabla u \cdot a\nabla u.$$

III. Random elliptic setting

The sub-linearity of (ϕ, σ) is essentially classical:

$$\lim_{R\to\infty} \frac{1}{R} \left(\fint_{B_R} |\phi|^2 + |\sigma|^2 \right)^{\frac{1}{2}} = 0.$$

Hence, for any $\alpha \in (0,1)$, for all $R > 0$ sufficiently large:

$$\int_{B_{\frac{R}{2}}} \nabla w \cdot a \nabla w \lesssim \left(\frac{1}{2}\right)^{2\alpha} \fint_{B_R} \nabla u \cdot a \nabla u.$$

To recover the excess, the argument is invariant after replacing u with

$$u_\xi(x) := u + \xi \cdot x + \phi_\xi(x).$$

III. Random elliptic setting

Theorem [13] (Gloria, Neukamm, Otto)

Assume stationarity, ergodicity, and uniform ellipticity. Suppose that u is an a-harmonic function that is strictly subquadratic in the sense that, for some $\alpha \in (0,1)$,

$$\lim_{R\to\infty} \frac{1}{R^{1+\alpha}} \left(\fint_{\mathcal{C}_R} |u|^2 \right)^{\frac{1}{2}} = 0.$$

Then, there exists $c \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$ such that

$$u(x) = c + \xi \cdot x + \phi_\xi(x) \ \text{ for } \ x \in \mathbb{R}^d.$$

- Armstrong, Kuusi, and Mourrat [3] (higher order)
- Fischer and Otto [12] (higher order)

IV. Random parabolic setting

The parabolic excess is defined by

$$\mathrm{Exc}(R) := \inf_{\xi \in \mathbb{R}^d} \fint_{\mathcal{C}_R} (\nabla u - \xi - \nabla \phi_\xi) \cdot a(\nabla u - \xi - \nabla \phi_\xi),$$

for the parabolic correctors solving

$$\partial_t \phi_i = \nabla \cdot a\left(\nabla \phi_i + e_i\right) \;\; \text{in } \; \mathbb{R}^d \times (-\infty, \infty),$$

and for the parabolic cylinder

$$\mathcal{C}_R := B_R \times (-R^2, 0].$$

The aim is to prove that, for all $R_1 < R_2$ sufficiently large,

$$R_1^{-2\alpha} \mathrm{Exc}(R_1; u) \lesssim R_2^{-2\alpha} \mathrm{Exc}(R_2; u).$$

IV. Random parabolic setting

The flux q is defined by

$$q_i := a\left(\nabla \phi_i + e_i\right) \quad \text{for} \quad i \in \{1, \ldots, d\},$$

which defines the homogenized coefficients

$$a_{\mathrm{hom}} e_i := \left\langle a\left(\nabla \phi_i + e_i\right)\right\rangle = \left\langle q_i \right\rangle.$$

However, the flux is no longer divergence-free: the equation

$$\nabla \cdot \sigma_i = q_i - \left\langle q_i \right\rangle \quad \text{for} \quad i \in \{1, \ldots, d\},$$

is not solvable. To overcome this, we introduce a four part extended corrector $(\phi, \psi, \sigma, \zeta)$.

IV. Random parabolic setting

We essentially perform a Weyl decomposition of the flux:

$$q_i := q_{\text{pot}} + q_{\text{sol}} + c,$$

where ψ is constructed to correct the potential part. That is,

$$\Delta\psi_i = \nabla \cdot q_i \;\text{ for }\; i \in \{1, \ldots, d\}.$$

It is then immediate that

$$\nabla \cdot (q_i - \nabla\psi_i) = 0 \;\text{ for each }\; i \in \{1, \ldots, d\}.$$

IV. Random parabolic setting

The solenoidal part of the flux is corrected similarly to the elliptic case: σ solves

$$\nabla \cdot \sigma_i = (q_i - \nabla \psi_i) - \langle q_i \mid \mathcal{F}_{\mathbb{R}^d} \rangle \,,$$

where $\langle \cdot \mid \mathcal{F}_{\mathbb{R}^d} \rangle$ denotes the conditional expectation with respect to the sub-sigma-algebra $\mathcal{F}_{\mathbb{R}^d}$ of subsets left invariant by spatial translations of the coefficient field.

- According to the choice of gauge

$$\Delta \sigma_{ijk} = \partial_k (q_i - \nabla \psi_i)_j - \partial_j (q_i - \nabla \psi_i)_k.$$

- We use implicitly that

$$\langle \nabla \psi_i \mid \mathcal{F}_{\mathbb{R}^d} \rangle = \langle \nabla \cdot \sigma_i \mid \mathcal{F}_{\mathbb{R}^d} \rangle = 0.$$

IV. Random parabolic setting

The following informal calculation motivated by [13] proves that the difference

$$\nabla \cdot \sigma_i - (q_i - \nabla \psi_i),$$

is invariant with respect to spatial shifts:

$$\begin{aligned}\Delta \left(\nabla \cdot \sigma_i\right)_j =& \Delta \left(\partial_k \sigma_{ijk}\right) = \partial_k \left(\Delta \sigma_{ijk}\right) \\ =& \partial_k \partial_k (q_i - \nabla \psi_i)_j - \partial_k \partial_j (q_i - \nabla \psi_i)_k \\ =& \Delta (q_i - \nabla \psi_i)_j - \partial_j \nabla \cdot (q_i - \nabla \psi_i) \\ =& \Delta (q_i - \nabla \psi_i)_j.\end{aligned}$$

Hence,

$$\nabla \cdot \sigma_i - (q_i - \nabla \psi_i) = \left\langle \nabla \cdot \sigma_i - (q_i - \nabla \psi_i) \mid \mathcal{F}_{\mathbb{R}^d} \right\rangle = \left\langle q_i \mid \mathcal{F}_{\mathbb{R}^d} \right\rangle.$$

The extended corrector: ζ

It remains to correct the oscillations of the conditional expectation about the mean: ζ solves

$$\partial_t \zeta_i = \langle q_i \mid \mathcal{F}_{\mathbb{R}^d} \rangle - \langle q_i \rangle = \langle q_i \mid \mathcal{F}_{\mathbb{R}^d} \rangle - a_{\text{hom}} e_i.$$

It is essential to the analysis that ζ is constant in space.

In total, the extended corrector $(\phi, \psi, \sigma, \zeta)$ solves:

Parabolic

i. $\partial_t \phi_i = \nabla \cdot a \left(\nabla \phi_i + e_i \right)$

ii. $\Delta \psi_i = \nabla \cdot q_i$

iii. $\nabla \cdot \sigma_i = (q_i - \nabla \psi_i) - \langle q_i \mid \mathcal{F}_{\mathbb{R}^d} \rangle$

iv. $\partial_t \zeta_i = \langle q_i \mid \mathcal{F}_{\mathbb{R}^d} \rangle - a_{\text{hom}} e_i$

Elliptic

i. $-\nabla a \left(\nabla \phi_i + e_i \right) = 0$

ii. $\psi_i = 0$

iii. $\nabla \cdot \sigma_i = q_i - \langle q_i \rangle$

iv. $\zeta_i = 0$

IV. Random parabolic setting

Proposition [7] (Bella, Chiarini, F.)

Assume stationarity, ergodicity, and uniform ellipticity. For $\langle\cdot\rangle$-a.e a,

$$\lim_{R\to\infty}\frac{1}{R}\left(\fint_{C_R}|\phi|^2+|\psi|^2+|\sigma|^2\right)^{\frac{1}{2}}=0$$

and

$$\limsup_{R\to\infty}\frac{1}{R^2}\left(\fint_{\mathcal{C}_R}|\zeta|^2\right)^{\frac{1}{2}}=0.$$

- new proof of sub-linearity for ϕ
- (ψ,σ) are constructed to be stationary in time
- sub-linearity of ζ respects the scaling in time

IV. Random parabolic setting

Informally, for an a_{hom}-caloric function v, the homogenization error

$$w := u - (1 + \phi_i \partial_i) v,$$

solves the parabolic equation

$$\begin{aligned} \partial_t w - \nabla \cdot a \nabla w =& \nabla \cdot ((\phi_i a + \psi_i - \sigma_i) \nabla (\partial_i v)) \\ & + \partial_t \zeta_i \cdot \nabla (\partial_i v) - \phi_i (\partial_i v)_t - \psi_i \Delta (\partial_i v) \\ & (+\text{boundary terms}). \end{aligned}$$

- v is a_{hom}-caloric in a parabolic cylinder with boundary data u
- parabolic estimates require a smoothing boundary data
- boundary terms handled using interior/boundary regularity of a_{hom}-caloric functions

IV. Random parabolic setting

Ignoring the boundary terms:

$$\begin{aligned}\partial_t w - \nabla \cdot a\nabla w =& \nabla \cdot ((\phi_i a + \psi_i - \sigma_i)\nabla(\partial_i v)) \\ &+ \partial_t \zeta_i \cdot \nabla(\partial_i v) - \phi_i (\partial_i v)_t - \psi_i \Delta(\partial_i v).\end{aligned}$$

Testing this equation with w yields the energy estimate

$$\begin{aligned}\fint_{B_{\frac{R}{2}}} \nabla w \cdot a\nabla w \lesssim & \frac{1}{R^2}\left(\fint_{B_R} |\phi|^2 + |\psi|^2 + |\sigma|^2\right)\fint_{B_R} \nabla u \cdot a\nabla u \\ &+ \left|\fint_{B_R} (\partial_t \zeta_i \cdot \nabla(\partial_i v))\, w\right|.\end{aligned}$$

- integrate by parts in time
- expand the homogenization error, use the equations satisfied by (u, v, ϕ) and the spatial homogeneity of ζ

Proposition [7] (Bella, Chiarini, F.)

Assume stationarity, ergodicity, and uniform ellipticity. For every a-caloric function u on $\mathcal{C}_R$ and $\epsilon \in (0, \frac{R}{4})$, for each $\rho \in (0, \frac{1}{8})$,

$$\begin{aligned}
\fint_{\mathcal{C}_{\frac{R}{4}}} \nabla w \cdot a \nabla w \leq & C\epsilon \fint_{\mathcal{C}_R} |\nabla u|^2 + C\frac{\rho^{\frac{2}{d}}}{\epsilon^2} \fint_{\mathcal{C}_R} |\nabla u|^2 \\
& + \frac{C}{R^2 \rho^{d+4}} \fint_{\mathcal{C}_R} \left(|\phi|^2 + |\psi|^2 + |\sigma|^2\right) \fint_{\mathcal{C}_R} |\nabla u|^2 \\
& + \left(\frac{C}{R^2 \rho^{\frac{d}{2}+3}} \left(\int_{\mathcal{C}_R} |\zeta|^2 \right)^{\frac{1}{2}} + \frac{C}{R^4 \rho^{d+6}} \int_{\mathcal{C}_R} |\zeta|^2 \right) \int_{\mathcal{C}_R} |\nabla u|^2 \\
& + \frac{C}{R^2 \rho^{d+4}} \left(\fint_{\mathcal{C}_R} |\zeta|^2 \right)^{\frac{1}{2}} \left(\fint_{\mathcal{C}_R} |q|^2 \right)^{\frac{1}{2}} \fint_{\mathcal{C}_R} |\nabla u|^2
\end{aligned}$$

IV. Random parabolic setting

Theorem [7] (Bella, Chiarini, F.)

Assume stationarity, ergodicity, and uniform ellipticity. Fix a Hölder exponent $\alpha \in (0,1)$. Then, there exist constants $C_0 = C_0(\alpha, d, \lambda) > 0$ and $C_1(\alpha, d, \lambda) > 0$ with the following property:
If $R_1 < R_2$ are two radii such that, for each $R \in [R_1, R_2]$,

$$\frac{1}{R}\left(\fint_{\mathcal{C}_R} |\phi|^2 + |\psi|^2 + |\sigma|^2\right)^{\frac{1}{2}} + \frac{1}{R^2}\left(\fint_{\mathcal{C}_R} |\zeta|^2\right)^{\frac{1}{2}} \leq \frac{1}{C_0},$$

and such that, for each $R \in [R_1, R_2]$,

$$\left(\fint_{\mathcal{C}_R} |q|^2\right)^{\frac{1}{2}} \leq 2\left\langle |q|^2 \right\rangle^{\frac{1}{2}},$$

then any a-caloric function u satisfies $\mathrm{Exc}(R_1) \leq C_1 \left(\frac{R_1}{R_2}\right)^{2\alpha} \mathrm{Exc}(R_2)$.

[7] Theorem (Bella, Chiarini, F.)

Assume stationarity, ergodicity, and uniform ellipticity. Suppose that u is an a-caloric function which is strictly subquadratic in the sense that, for $\alpha \in (0,1)$,

$$\lim_{R\to\infty} \frac{1}{R^{1+\alpha}} \left(\fint_{\mathcal{C}_R} |u|^2 \right)^{\frac{1}{2}} = 0.$$

Then, there exists $c \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$ such that

$$u(x,t) = c + \xi \cdot x + \phi_\xi(x,t) \ \text{ for } \ (x,t) \in \mathbb{R}^{d+1}.$$

- Assuming a finite range dependence, Armstrong, Bordas, and Mourrat [2] have obtained quantitative estimates and higher order Liouville properties.

IV. Random parabolic setting

The first-order Liouville theorem follows from the large-scale regularity and the parabolic Caccioppoli inequality.

Parabolic Caccioppoli inequality

There exists $C > 0$ such that, for $\langle\cdot\rangle$-a.e. a, for every $R > 0$ and distributional solution u of the equation

$$\partial_t u = \nabla \cdot a\nabla u \ \text{ in } \ \mathcal{C}_R,$$

and for every $c \in \mathbb{R}$ and $\rho \in (0, \frac{R}{2})$,

$$\int_{\mathcal{C}_{R-\rho}} |\nabla u|^2 \leq \frac{C}{\rho^2} \int_{\mathcal{C}_R \setminus \mathcal{C}_{R-\rho}} |u - c|^2 .$$

IV. Random parabolic setting

Suppose that u is an a-caloric function satisfying, for some $\alpha \in (0,1)$,

$$\lim_{R\to\infty} \frac{1}{R^{1+\alpha}} \left(\fint_{\mathcal{C}_R} |u|^2 \right)^{\frac{1}{2}} = 0.$$

For each $r < R$ sufficiently large, the large-scale regularity and definition of excess (taking $\xi = 0$) prove that

$$\operatorname{Exc}(r;u) \lesssim \left(\frac{r}{R}\right)^{2\alpha} \operatorname{Exc}(R;u) \leq \left(\frac{r}{R}\right)^{2\alpha} \fint_{\mathcal{C}_R} |\nabla u|^2 \,,$$

and, from the the Caccioppoli inequality,

$$\operatorname{Exc}(r;u) \lesssim \liminf_{R\to\infty} \frac{r^{2\alpha}}{(2R)^{2+2\alpha}} \fint_{\mathcal{C}_{2R}} |u|^2 = 0.$$

Since, for each $r > 0$ sufficiently large,

$$\mathrm{Exc}(r; u) = \inf_{\xi \in \mathbb{R}^d} \fint_{B_R} (\nabla u - \xi - \nabla \phi_\xi) \cdot a \, (\nabla u - \xi - \nabla \varphi_\xi) = 0,$$

there exists $\xi \in \mathbb{R}^d$ such that

$$z(x,t) := u(x,t) - \xi \cdot x - \phi_\xi(x,t) \;\text{ in }\; \mathbb{R}^d \times (-\infty, \infty),$$

is constant in space. However, since z solves

$$\partial_t z = \nabla \cdot a \nabla z,$$

this implies that z is constant in time as well. Hence, for $c \in \mathbb{R}$,

$$u(x,t) = \xi \cdot x + \phi_\xi(x,t) + c \;\text{ in }\; \mathbb{R}^d \times (-\infty, \infty).$$

V. Ongoing/Future work

- quantifying the sublinear growth of $(\phi, \psi, \sigma, \zeta)$ assuming mixing conditions
- applications to quantitative homogenization (extending the results of [2] to non-symmetric systems)
- applications to the longtime homogenization of time-dependent wave equations (Benoit and Gloria [9])
- large-scale regularity and Liouville theorems in degenerate time-dependent environments in the setting of Andres, Chiarini, Deuschel, and Slowik [1]
- motivated by Bella, F., Otto [8] in the degenerate elliptic framework of Chiarini and Deuschel [11]

Thanks

Thank you.

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