On Microstructures with Sign Changing Properties

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Motivation

- Given $\Omega \subset \mathbb{R}^d$, $d \geq 1$, a smooth bounded domain and $\tilde{\Omega}$ an open neighbourhood of Ω.
- Given $\nabla u_d \in C^{2,\alpha}\left(\overline{\tilde{\Omega}}\right)$, where u is the solution of

$$\operatorname{div}\left(a \nabla u_d\right) = 0 \text{ in } \tilde{\Omega},$$

can we find a (with $\int_\Omega a \mathrm{d}x$ given and $a \geq I_d$)?

Motivation

- Given $x_0 \in \partial\Omega$, consider the flow $X'(t, x_0) = \nabla u_d(X(t, x_0))$, $t \in \mathbb{R}$ and $X(0, x_0) = x_0$. You find Briane, Milton, Treibergs '14

$$\log a(X(t, x_0)) - \log a(x_0) = \int_0^{t_0} \Delta u_d(t, x_0)\, \mathrm{d}t.$$

  So a is known, provided any point $z \in \Omega$ is attained as $X(t, x_0)$ for some t, x_0.

- If $(u_1, \ldots, u_d)$ is a diffeomorphism on $\tilde{\Omega}$, that is the case. So we want to know whether

$$\log|\det(\nabla u_1, \ldots, \nabla u_d)| < K < \infty \text{ in } \Omega.$$

Optimal Bounds / G-Closure

What are the possible effective tensors for N-mixtures?
$N = 2$, OK (Hashin-Strikman Bounds), not $N \geq 3$.
Briane-Nesi 2004 : $\det P_\varepsilon > 0$ for laminates
where

$$\begin{cases} \operatorname{div}(a_\varepsilon \nabla u_\varepsilon) = 0 & \text{in } \Omega, \\ u_a = \underline{\phi} & \text{on } \partial\Omega, \end{cases}$$

we have

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H^1(\Omega),$$

$$a_\varepsilon \nabla u_\varepsilon \rightharpoonup a^* \nabla u_0 \text{ weakly in } L^2(\Omega)$$

and

$$\nabla u_\varepsilon - P_\varepsilon \nabla u_0 \to 0 \text{ strongly in } L^1(\Omega)$$

Motivation

Q: When is the sign of the determinant imposed *a priori* ?

Theorem (Rado–Kneser–Choquet)

Let $D \subseteq \mathbb{C}$ be a bounded convex domain whose boundary is a Jordan curve ∂D. Let $\Phi\colon \partial B(0,1) \to \partial D$ be a homeomorphism of $\partial B(0,1)$ onto ∂D and f be defined as

$$\begin{cases} \Delta f = 0 & \text{in } B(0,1), \\ f = \Phi & \text{on } \partial B(0,1). \end{cases}$$

Then $J_f(z) \neq 0$ for all $z \in B(0,1)$.

For a-harmonic functions, generalized G. Alessandrini 1986, & R. Magnanini 1994, & V. Nesi '01-'15 : $\mathrm{Det} Df \in A_\infty$.

Good boundary conditions?

The Rado–Kneser–Choquet Theorem fails when $d = 3$.

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Lemma

If $a \in W^{1,\infty}$, and $\nabla \ln a \in L^\infty(\Omega)$ is small enough, then if

$$\begin{cases} \Delta \underline{u} = 0 & \text{in } \Omega, \\ \underline{u} = \underline{\phi} & \text{on } \partial\Omega, \end{cases}$$

satisfies $\det(\nabla \underline{u}) > 1$ *in* Ω *and* $u \in C^1\left(\bar{\Omega}\right)$, *then*

$$\begin{cases} \mathrm{div}\,(a \nabla \underline{u_a}) = 0 & \text{in } \Omega, \\ \underline{u_a} = \underline{\phi} & \text{on } \partial\Omega, \end{cases}$$

satisfies

$$\det\left(\nabla \underline{u_a}\right) > C\left(\Omega, \|\nabla \ln a\|_\infty\right) > 0 \text{ in } \Omega.$$

Complex Geometric Solutions

CGO Solutions (Faddeev, Calderón , Sylvester-Uhlmann '87, Bal-Uhlmann '09, Bal-Bonnetier-Monard-Triki '13):

Assume $a \in H^{\frac{d}{2}+3+\varepsilon}(\Omega)$.

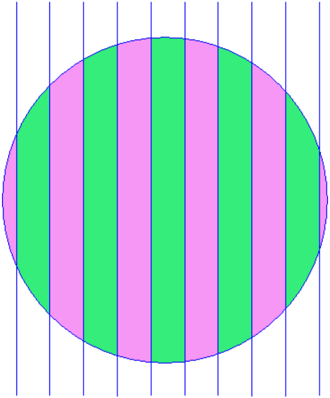
There exists a non-empty open set $\mathbf{G} \subset (H^{\frac{1}{2}}(\partial\Omega))^4$ of quadruples of boundary conditions such that for any $G = (g_1, g_2, g_3, g_4) \in \mathbf{G}$, there exists an open cover of Ω of the form $\{\Omega_{2i-1}, \Omega_{2i}\}_{1\leq i\leq N}$ and a constant $c_0 > 0$ such that

$$\pm \inf_{x\in\Omega_{2i-1}} \det(\nabla u_1, \nabla u_2, \nabla u_4) \geq c_0$$

and

$$\pm \inf_{x\in\Omega_{2i}} \det(\nabla u_1, \nabla u_2, \nabla u_3) \geq c_0, \quad 1 \leq i \leq N.$$

Complex Geometric Solutions



$$N^3 \exp(N(2x_2 + x_3))(\cos(Nx_1) \text{ or } \sin(Nx_1) + O(1/N))$$

Runge Approximation Property

$$L := \operatorname{div}(a\nabla\cdot)\,,\; a \in W^{1,\infty}(D_2, \mathbb{R}^{\frac{d(d+1)}{2}}),\; a \geq I_d.$$

Suppose $D_1 \Subset D_2$, bounded, open Lipschitz sets and $D_2 \setminus D_1$ connected.

$$S_1 := \{h \in H^1(D_1) : Lw = 0 \text{ in } D_1\}$$
$$S_2 := \{h \in H^1(D_2) : Lw = 0 \text{ in } D_2\}.$$

Theorem (Lax, Malgrange (56), Rüland & Salo (18))

For any $\varepsilon > 0$ and any $h \in S_1$ there exists $u \in S_2$ such that

$$\|h - u|_{D_1}\|_{L^2(D_1)} \leq \varepsilon.$$

Unique Continuation implies Runge Approximation.

Runge Approximation Property

Since $a \in W^{1,\infty}$, $x_1, \ldots, x_d$ are approximate local solutions, so $\frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$ gives

Theorem (Bal, Uhlmann (13))

Take $\Omega' \Subset \Omega$, given $a \in C^{0,1}(\overline{\Omega}, \mathbb{R}^{\frac{d(d+1)}{2}})$, there exists N, r, such that

$$\Omega' \subset \cup_{i=1}^{N} B(x_i, r) \ \textit{and} \ \det\left(\nabla u^1_{(i)}, \ldots, \nabla u^d_{(i)}\right) > \frac{1}{2} \ \textit{in} \ B(x_i, r)$$

where $u^k_{(i)}$ satisfy solutions of $Lu^k_{(i)} = 0$ in Ω.

(which gives a set of $N \times d$ boundary conditions)

Runge Approximation Property

Set

$$E(\Omega') = \left\{ \underline{\phi} \in H^{\frac{1}{2}}(\partial\Omega)^{2d} : \mathrm{rank}(\nabla \underline{u})(x) = d \, \forall x \in \overline{\Omega'} \right\}$$

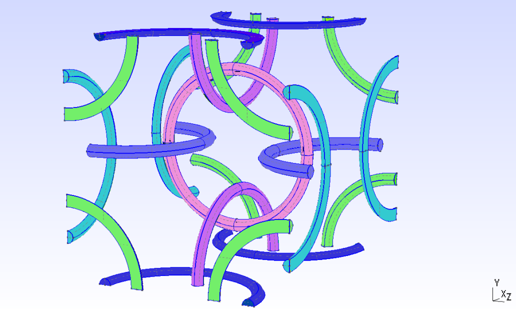
where

$$\begin{cases} L\underline{u} = 0, & \text{in } \Omega \\ \underline{u} = \underline{\phi} & \quad \text{on } \partial\Omega. \end{cases}$$

Theorem (Alberti, C.)

$E(\Omega')$ is open and dense in $H^{\frac{1}{2}}(\partial\Omega)^{2d}$.

The large case



For periodic boundary conditions. Briane, Milton & Nesi '04.

$$\pm \det(P(y)) > C, \text{ in } B_{\pm},$$

where $P(y)$ is the periodic corrector

The large case

Let

$$\Delta \underline{u} = 0 \text{ in } \Omega, \quad \underline{u} = \underline{\phi} \text{ on } \partial\Omega$$

and

$$\operatorname{div}\left(a(nx)\nabla \underline{u}_n\right) = 0 \text{ in } \Omega \quad \underline{u}_n = \underline{\phi} \text{ on } \partial\Omega$$

Theorem (C. '15)

Let

$$A := \left\{\phi \in H^{\frac{1}{2}}\left(\partial\Omega; \mathbb{R}^d\right) \,:\, \det\left(\nabla \underline{u}\right) > \lambda \left\|\phi\right\|^d_{H^{\frac{1}{2}}(\partial\Omega)} \text{ in } B_\rho(x_0)\right\}.$$

$\exists\, n(\rho, \Omega, \Omega', \lambda)$, $\tau > 0$, *and*
$B_\pm \subset B(x_0, \rho)$ *with* $|B_\pm| > \tau |B(x_0, \rho)|$ *such that*

$$\forall \phi \in A, \pm \det\left(\nabla \underline{u}_n\right)(x) > \tau\lambda \left\|\phi\right\|^d_{H^{\frac{1}{2}}(\partial\Omega)} \text{ on } B_\pm,$$

Proof

Use the Avellaneda & Lin Method and regularity theory to show

Lemma (Li & Nirenberg '03, Ben Hassen & Bonnetier '06)

There exists a constant $C > 0$, independent of n such that

$$\begin{aligned} \|\nabla \underline{u}_n\|_{L^\infty(\Omega')} &\leq C\|\phi\|_{H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)}, \\ \|P\left(\cdot\right)\|_{L^\infty(\Omega')} &\leq C, \end{aligned}$$

and

$$\|\nabla \underline{u}_n - P\left(n\cdot\right)\nabla \underline{u}\|_{L^\infty(\Omega')} \leq \frac{1}{n^{1/3}} C\|\phi\|_{H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)}.$$

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Then

$$\det(\nabla \underline{u}_n(x)) = \det(P(nx))\det(\nabla \underline{u}(x)) + R_n(x)$$

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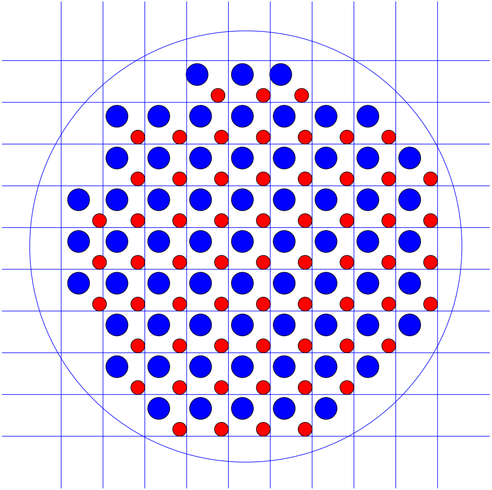
Then

$$\det(\nabla \underline{u}_n(x)) = \det(P(nx))\det(\nabla \underline{u}(x)) + R_n(x)$$

Now use the sign changing properties of $\det \nabla \underline{u}_\#$.

Proof



Many boundary condition do not help

Corollary

Take $\varphi_1, \ldots, \varphi_N$ *in* $H^{\frac{1}{2}}(\partial\Omega)$ *for some* N. $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ *such that for any* $B_\varepsilon \subseteq \Omega'$,

- *there exists* $x_1 \in B_\varepsilon$ *such that*

$$\max_{1 \le i,j,k \le N} \left|\det\left(\left[\nabla u_i(x_1), \nabla u_j(x_1), \nabla u_k(x_1)\right]\right)\right| \le \varepsilon,$$

  where $div(a(nx)\nabla u_i) = 0$ *in* Ω *and* $u_i = \varphi_i$ *on* $\partial\Omega$.

- *for every* $1 \le i, j, k \le N$ *such that the harmonic extension of* $(\varphi_i, \varphi_j, \varphi_k)$ *has maximal rank in* Ω *there exists* $x_2 \in B_\varepsilon$ *such that*

$$\det\left(\left[\nabla u_i(x_2), \nabla u_j(x_2), \nabla u_k(x_2)\right]\right) = 0.$$

New microstructures

Theorem

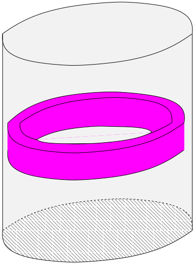
Given $\Omega = (-1,1)^d$, let $Q \subset \Omega$ be a non-empty, open, connected C^1 domain, symmetric with respect to the origin in all variables and such that $B(0,\varepsilon) \cap Q = \emptyset$ for some $\varepsilon > 0$ and $\Omega \setminus Q$ is connected. Then there exists $\delta > 0$ such that

$$a = 1 + \delta \chi_Q$$

is a sign changing micro-structure.

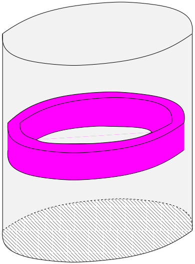
With Luc Nguyen and Shaun Chen Yang Ong.

Sketch of Proof



In a cylindrical geometry, with a small ring and b.c. $(r\cos\theta, r\sin\theta, z)$.

Sketch of Proof



In a cylindrical geometry, with a small ring and b.c. $(r\cos\theta, r\sin\theta, z)$.

$$u_x = f(r,z)\cos\theta, \quad u_y = f(r,z)\sin\theta, \quad u_z = g(r,z)$$

and for all θ

$$\det P(r,\theta,z) = \frac{f}{r}\left(\partial_z g \partial_r f - \partial_r g \partial_z f\right)$$

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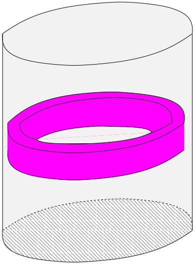

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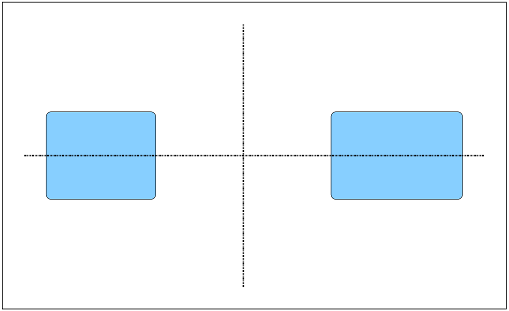
By symmetry f is odd in r, even in z, g is odd in z, even in r.

New microstructures

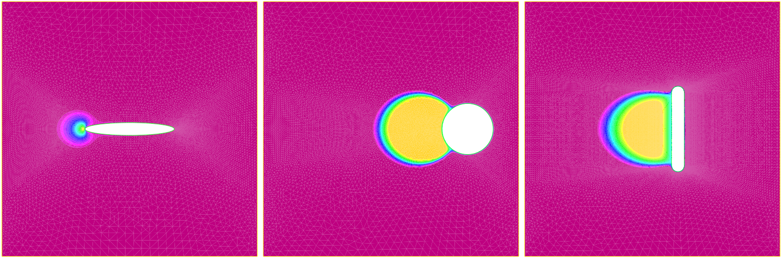


$$\det P(r,0,0) = \frac{f}{r}\partial_z g \partial_r f$$

Hoptf's Maximum principle and oddness to conclude.
On going work : quantitative estimates (with Luc Nguyen and Shaun Chen Yang Ong).

New microstructures

Computation made with FreeFem++



Homogenization Question

Suppose $u_n \in H_0^1(\Omega)$

$$\operatorname{div}\left(a\left(nx\right)\nabla u_n\right) = f,$$

If

$$\nabla u_n \in L^p(\Omega) \text{ (uniformly in } n)$$

is

$$\nabla u_n - P\left(n\cdot\right)\nabla u_0 \in L^{p'}(\Omega), \text{ (uniformly in } n) \text{ ?}$$

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First non-trivial example: 2 dimensional laminate ($\rightsquigarrow$ Milton, Faraco-Astala Meyers' exponent)