Stochastic Homogenisation in Carnot groups

Federica Dragoni

Cardiff University

joint with **Nicolas Dirr**, **Paola Mannucci** and **Claudio Marchi.**

Plan of the talk

- Introduction of the known coercive case.
- A non coercive Hamilton-Jacobi equation: the horizontal gradient in Carnot groups and anisotropic rescaling.
- The associated variational problem.
- The effective Lagrangian as limit of a constrained variational problem.
- Approximation argument by piecewise $\mathcal{X}$-lines.
- Sketch of the proof for the convergence result.

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Homogenization of Hamilton-Jacobi equations.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the Hamilton-Jacobi problem:

$$\begin{cases} u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon, \omega\right) = 0, \ x \in \mathbb{R}^N, \omega \in \Omega, t > 0 \\ u^\varepsilon(0,x) = g(x). \end{cases} \tag{1}$$

Theorem (Souganidis 1999 and Rezakhanlou-Tarver 2000)

Under suitable assumptions, the (unique) viscosity solutions $u^\varepsilon(t,x,\omega)$ of problems (1) converge locally uniformly in x and t and a.s. in ω to a deterministic limit function $u(t,x)$.
Moreover the limit function u can be characterised as the (unique) viscosity solution of a deterministic effective Hamilton-Jacobi problem of the form:

$$\begin{cases} u_t + \overline{H}(Du) = 0, \ x \in \mathbb{R}^N, t > 0 \\ u(0,x) = g(x). \end{cases} \tag{2}$$

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Assumptions:

- $p \mapsto H(x,p,\omega)$ is convex in p, $\forall\,(x,\omega) \in \mathbb{R}^N \times \Omega$
- there exist $C_1 > 0,\ \gamma > 1$ such that

$$C_1^{-1}(|p|^\gamma - 1) \leq H(x,p,\omega) \leq C_1(|p|^\gamma + 1),\ \forall\,(x,p,\omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega$$

- there exists $m : [0,+\infty) \to [0,+\infty)$ continuous, monotone increasing, with $m(0^+) = 0$ such that $\forall\ x,y,p \in \mathbb{R}^N, \omega \in \Omega$

$$|H(x,p,\omega) - H(y,p,\omega)| \leq m\big(|x-y|\,(1+|p|)\big)$$

- for all $p \in \mathbb{R}^N$ the function $(x,\omega) \mapsto H(x,p,\omega)$ is stationary, ergodic random field on $\mathbb{R}^N \times \Omega$ w.r.t. the unitary translation operator.

Idea of the proof.

Use of the variational formula for the solutions: For all $\varepsilon > 0$, the viscosity solution of (1) is given by

$$u^{\varepsilon}(t,x,\omega) = \inf_{y \in \mathbb{R}^N} \left[g(y) + L^{\varepsilon}(x,y,t,\omega) \right],$$

where

$$L^{\varepsilon}(x,y,t,\omega) = \inf_{\xi} \int_0^t L\left(\frac{\xi(s)}{\varepsilon}, \dot{\xi}(s), \omega \right) ds$$

and $\xi \in W^{1,\infty}((0,t))$ such that $\xi(0) = y$ and $\xi(t) = x$,
and where $L = H^*$ is the Legendre-Fenchel transform of the H, i.e.

$$L(q) = \sup_{p \in \mathbb{R}^N} \{ p \cdot q - H(p) \}.$$

Key property: $H = L^*$ if and only if H convex.

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1. (Uniform in x and $t > 0$) convergence of L^ε via Sub-additive Ergodic Theorem.
2. In particular $L^\varepsilon(x, y, t, \omega) \to t\overline{L}\left(\frac{x-y}{t}\right)$; so one can find the effective Lagrangian as limit of the variational problem.
3. Then $u^\varepsilon(t, x, \omega) \to \inf_y \left[g(y) + t\overline{L}\left(\frac{x-y}{t}\right)\right] =: u(t, x)$.
4. Whenever the effective Lagrangian is convex, by Hopf-Lax formula u (as above) is the (unique) viscosity solution of

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Goal

To work with Hamiltonians coercive only w.r.t. some prescribed directions:

$$\begin{cases} u_t + H(x, \sigma(x)Du, \omega) = 0, \ x \in \mathbb{R}^N, t > 0 \\ u(0,x) = g(x), \end{cases}$$

where $\sigma(x)Du$ is a subgradient (in Carnot group); that means $\sigma(x)$ is a $m \times n$ matrix satisfying the Hörmander condition.

Main model: $H(x, \sigma(x)Du, \omega) = \frac{1}{2}\left|\sigma(x)Du\right|^2 + V(x,\omega) =$

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Viscosity solutions via variational formula

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for some $\alpha : [0,t] \to \mathbb{R}^m$ measurable.
In that case we call ξ horizontal curve and α horizontal velocity of the horizontal curve ξ and we write $\alpha = \alpha^{\xi}$.
Hörmander condition $\Rightarrow$ for every x and y, $L(x,y,t,\omega) \neq +\infty$.

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Carnot groups

- Carnot group: is a (non-commutative) nilpotent Lie group with a stratified Lie algebra.
- Any Carnot group can be identified with $\mathbb{R}^N$ with a non commutative polynomial group operation.
- **Example:** 1-dimensional Heisenberg group $\mathbb{R}^3$ with the group law

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{-x_2 y_1 + x_1 y_2}{2} \right)$$

- The Left-invariant vector fields spanning the first layer satisfy the Hörmander condition.

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Homogenization in Carnot groups vs homogenization in $\mathbb{R}^N$ (Euclidean)

- (isotropic) scaling in $\mathbb{R}^N$ vs (anisotropic) dilations.
- In general: $\lambda(x \circ y) \neq (\lambda x \circ \lambda y)$
- E.g. variational formula for the rescaled Hamiltonian?
  If ξ horizontal, in general $\frac{\xi}{\varepsilon}$ is not horizontal.
- Dilations δ_λ induced by the stratification of the algebra:
  E.g. in Heisenberg: $\delta_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$.
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ε-problem in Carnot groups.

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Theorem (Dirr-D.-Mannucci-Marchi 2017)

Under suitable assumptions, the (unique) viscosity solutions $u^\varepsilon(t,x,\omega)$ of problems (4) converge locally uniformly in x and t and a.s. in ω to a deterministic limit function $u(t,x)$, that can be characterised as the (unique) viscosity solution of a deterministic effective Hamilton-Jacobi problem of the form:

$$\begin{cases} u_t + \overline{H}(\sigma(x)Du) = 0, \ x \in \mathbb{R}^N, t > 0 \\ u(0,x) = g(x). \end{cases} \tag{5}$$

ε-problem in Carnot groups.

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Assumptions:

Set $q = \sigma(x)p \in \mathbb{R}^m$, for all $p \in \mathbb{R}^N$

- $q \mapsto H(x, q, \omega)$ is convex in q, $\forall\, (x, \omega) \in \mathbb{R}^N \times \Omega$
- there exist $C_1 > 0$, $\gamma > 1$ such that

$$C_1^{-1}(|q|^\gamma - 1) \leq H(x, q, \omega) \leq C_1(|q|^\gamma + 1),\ \forall\, (x, q, \omega) \in \mathbb{R}^N \times \mathbb{R}^m \times \Omega$$

- there exists $m : [0, +\infty) \to [0, +\infty)$ continuous, monotone increasing, with $m(0^+) = 0$ such that $\forall\, x, y, p \in \mathbb{R}^n, \omega \in \Omega$ $|H(x, q, \omega) - H(y, q, \omega)| \leq m\left(\left\|y^{-1} \circ x\right\|_h (1 + |q|^\gamma)\right)$, with $\|x\|_h$ homogeneous norm (note locally $\|x\|_h \leq |x|^{1/k}$ for some natural number $k \geq 2$)
- for all $q \in \mathbb{R}^m$ the function $(x, \omega) \mapsto H(x, q, \omega)$ is stationary, ergodic random field on $\mathbb{R}^N \times \Omega$ w.r.t. the unitary translation operator.

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Model

$$H(x,p,\omega) = a(x,\omega)|\sigma(x)p|^{\beta} + V(x,\omega)$$

with $\beta > 1$, and $V(x,\omega)$ bounded and uniformly continuous while $a(x,\omega)$ bounded, uniformly continuous, and bounded away from zero.
The ε-problems are

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Variational formula for the viscosity solutions

Set $L = H^*$, then

$$u^\varepsilon(t,x,\omega) = \inf_{y\in\mathbb{R}^N}\left[g(y) + L^\varepsilon(x,y,t,\omega)\right]$$

where

$$L^\varepsilon(x,y,t,\omega) = \inf_\xi \int_0^t L\left(\delta_{\frac{1}{\varepsilon}}(\xi(s)), \alpha^\xi(s), \omega\right) ds$$

- $\xi \in W^{1,\infty}((0,t))$ horizontal curve s.t. $\xi(0) = y$, $\xi(t) = x$, $x, y \in \mathbb{R}^n$, i.e. $\dot{\xi}(t) = \sum_{i=1}^m \alpha_i(t) X_i(\xi(t))$, a.e. $t > 0$.
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Convergence of a "constrained" L^ε

We apply the Sub-additive Ergodic Theorem to the following minimising problem:

$$\inf_{\xi} \int_a^b L(\xi(s), \alpha^{\xi}(s), \omega)\, ds$$

where $\xi - l_q \in W_0^{1,+\infty}((a,b))$ and $l_q(s)$ is the horizontal curve (starting from the origin) with constant horizontal velocity $\alpha(s) = q$. We call the horizontal curves with constant horizontal velocity $\mathcal{X}$-lines.

E.g. In Heisenberg:

$$\begin{cases} \xi_1(t) = x_1^0 + q_1\, t \\ \xi_2(t) = x_2^0 + q_2\, t \\ \xi_3(t) = x_3^0 + \dfrac{-q_1 x_2^0 + q_2 x_1^0}{2}\, t \end{cases}$$

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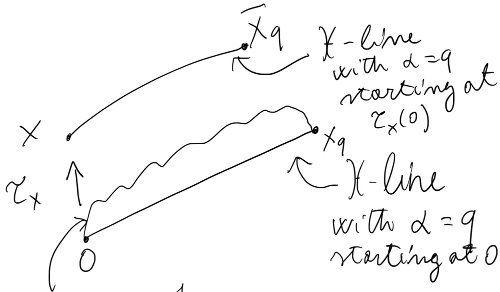
Independence on ω

We have ergodicity on $\mathbb{R}^N$ while we have translation invariance only for a one-parameter subgroup.

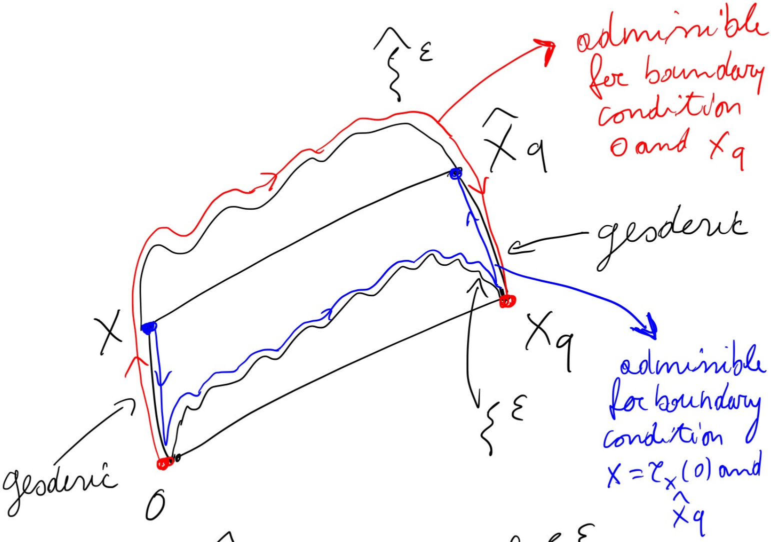
We can show that this is enough to deduce a.s. independence on ω.

CLAIM: $\bar{L}(q, \tau_x \omega) = \bar{L}(q, \omega)$

($\Rightarrow$ const a.s. by ergodicity)



ξ^ε s.t. $\int_0^1 L^\varepsilon(\delta_{\frac{1}{\varepsilon}}(\xi^\varepsilon), \alpha^{\xi^\varepsilon}, \omega)\, ds \approx \bar{L}(q, \omega)$



energy of $\hat{\xi}_\varepsilon \approx$ energy of ξ^ε

Homogenization for the constrained variational problem.

Definition

Call V_x the set of all the points reachable from x with a constant horizontal velocity curve.

By using the Subadditive Ergodic Theorem, the Ergodic Theorem, uniform estimates on L^ε etc....

Theorem (Dirr-D.-Mannucci-Marchi 2017)

If $y \in V_x$ then, as $\varepsilon \to 0^+$, $L^\varepsilon(x, y, t, \omega) \to t\overline{L}\left(\frac{\pi_m(x)-\pi_m(y)}{t}\right)$, locally uniformly in x, y, t and a.s. ω (where $\pi_m(x)$ is the projection of x on the first m components).

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What we have so far....

The constrained variational problem defines the effective Lagrangian. By proving that the effective Lagrangian $\overline{L}$ is convex (non trivial), we can define the effective Hamiltonian $\overline{H} := \overline{L}^*$ and so deduce the effective problem.

By uniform convergence we can deduce the following result:

$$v^\varepsilon(x,t,\omega) := \inf_{y\in V_x} \left[g(y) + L^\varepsilon(x,y,t,\omega)\right] \to \inf_{y\in V_x}\left[g(y) + t\overline{L}\left(\frac{\pi_m(x)-\pi_m(y)}{t}\right)\right]$$

Note: v^ε do not solve the ε-problems and in general the right-hand side does not solve the limit problem either!

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The Hörmander conditions and the unconstrained variational problem.

Heuristic idea: Assume that $\alpha(s)$ is smooth, then we can approximate by piece-wise constant functions in L^1.
This means that there exists $\alpha^\pi : [0, t] \to \mathbb{R}^m$ piecewise constant functions such that as $|\pi| \to 0$

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where ξ^π is the piecewise $\mathcal{X}$-line horizontal curve with horizontal velocity α^π and by π we indicate a partition of the interval $[0, t]$.

Theorem (Dirr-D.-Mannucci-Marchi 2017)

$$L^\varepsilon(x, y, t, \omega) \to \inf_\alpha \int_0^t \overline{L}(\alpha(s))\, ds, \quad as\ \varepsilon \to 0^+,$$

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The right-hand side is called Hopf-Lax function in Carnot groups.

Theorem (Balogh-Calogero-Pini 2014)

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Idea of the proof for the convergence of $L^\varepsilon(x, y, t, \omega)$.

$$L^\varepsilon(x, y, t, \omega) = \inf_\xi \int_0^t L\left(\delta_{\frac{1}{\varepsilon}}(\xi(s)), \alpha^\xi(s), \omega\right) ds \to \inf_\alpha \int_0^t \overline{L}\,(\alpha(s))\, ds$$

Convergence of minimisers:

- For the Upper bound we consider the Γ-realising sequence.
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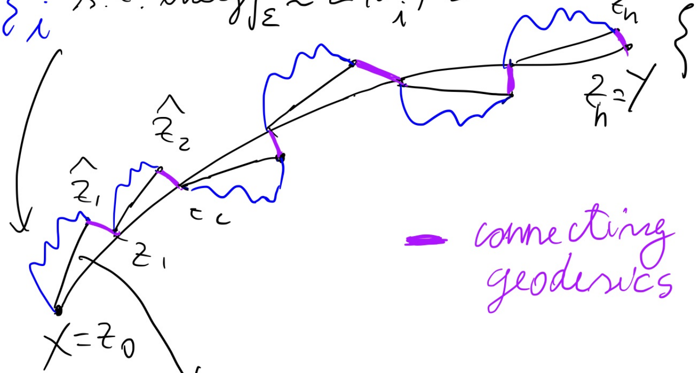
Upper bound:

ξ_i^ε s.t. $\text{energy}_\varepsilon \simeq \bar{L}(\alpha_i^\pi) \cdot \Delta t$



ξ_1^π horiz curve with const. velocity α_1^π

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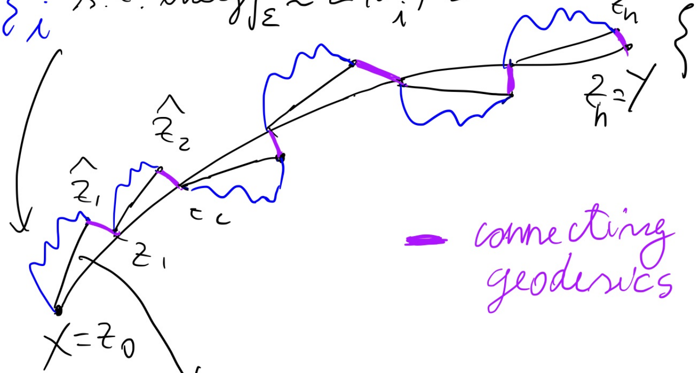

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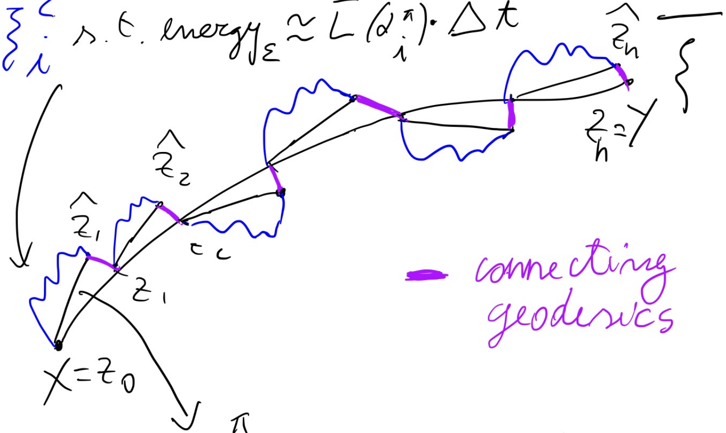

ξ_1^{π} horiz curve with const. velocity α_1^{π}

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Step 1 Take a sequence ξ^ε converging to the infimum on l.h.s., take the limit $\overline{\xi}$ of ξ^ε (up to subsequence)

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Step 3 Show that minimisers with boundary condition on a piece of a $\mathcal{X}$-line and the corresponding piece of ξ^ϵ have almost same energy.

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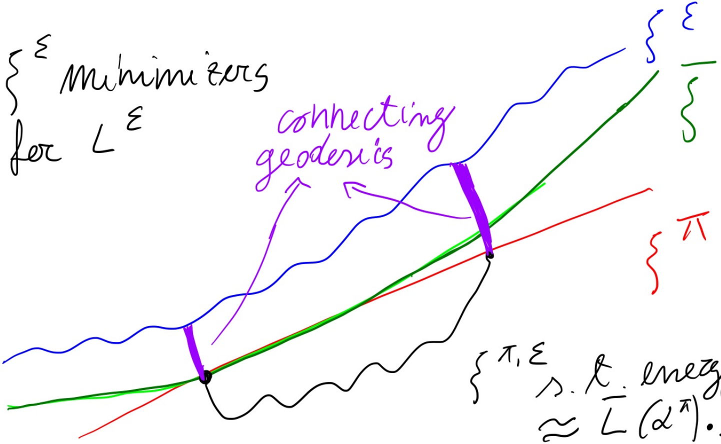
ξ^{ε} minimizers for L^{ε}



$\xi^{\pi,\varepsilon}$ s.t. $\text{energy}_{\varepsilon} \simeq \bar{L}(\alpha^{\pi})\cdot\Delta t$

$\xi^{\varepsilon} \xrightarrow[\text{up to subs.}]{} \bar{\xi}$ & ξ^{π} piecewise H-line approx of $\bar{\xi}$

Thanks for your attention!