Spatial critically branching particle systems with state-dependent branching rate

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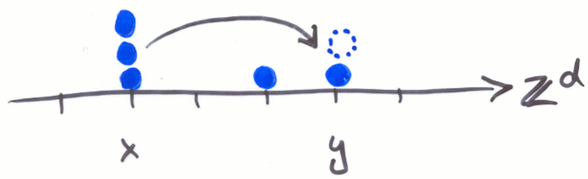
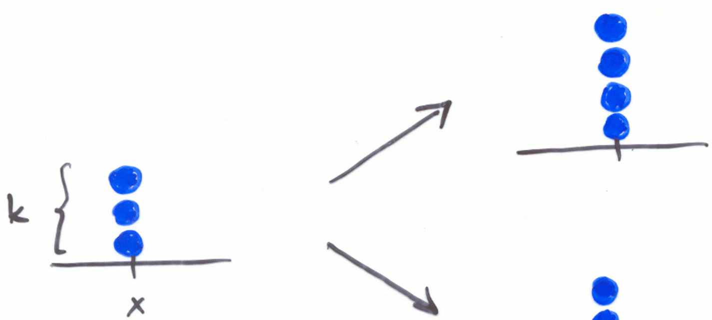
Particles live on $\mathbb{Z}^d$



at rate

$$p_{x,y} = p_{y-x}$$

while there are k part. at x :



at rate

$$\tfrac{1}{2}\sigma(k)$$

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$$\sigma : \mathbb{N}_0 \to \mathbb{R}_+ , \quad \sigma(0) = 0, \quad \overline{\lim_k} \frac{\sigma(k)}{k^2} < \infty .$$

$\xi_x(t)$ .... # part. at x at time t.

Assume: $\left(\xi_x(0)\right)_{x \in \mathbb{Z}^d}$ are i.i.d. Poisson(θ).

$\sigma(k) = C \cdot k$ corresponds to

independent branching random walks.

In this case, the long-time behaviour is well-known:

$$\hat{p}_{x,y} := \frac{1}{2}(p_{x,y} + p_{y,x})$$

1) $\hat{p}$ recurrent

$$\Rightarrow \quad P(\xi_x(t) = 0 \;\; \forall x \in B) \xrightarrow[t\to\infty]{} 1$$

$$\forall B \subset\subset \mathbb{Z}^d \qquad \text{"local extinction"}$$

2) $\hat{p}$ transient

$$\Rightarrow \quad \xi(t) \xrightarrow[t\to\infty]{\text{f.d.d.}} \xi(\infty),$$

$\xi(\infty)$ is an equilibrium,

$$E\,\xi_x(\infty) = \Theta, \qquad \mathrm{Var}\,\xi_x(\infty) < \infty$$

"persistence"

Key ingredient: local size-biasing

$$\hat{P}_{(x,t)}(\xi \in d\eta) = \frac{1}{\mathbb{E}\,\xi_x(t)}\,\eta_x\,P(\xi(t) \in d\eta)$$

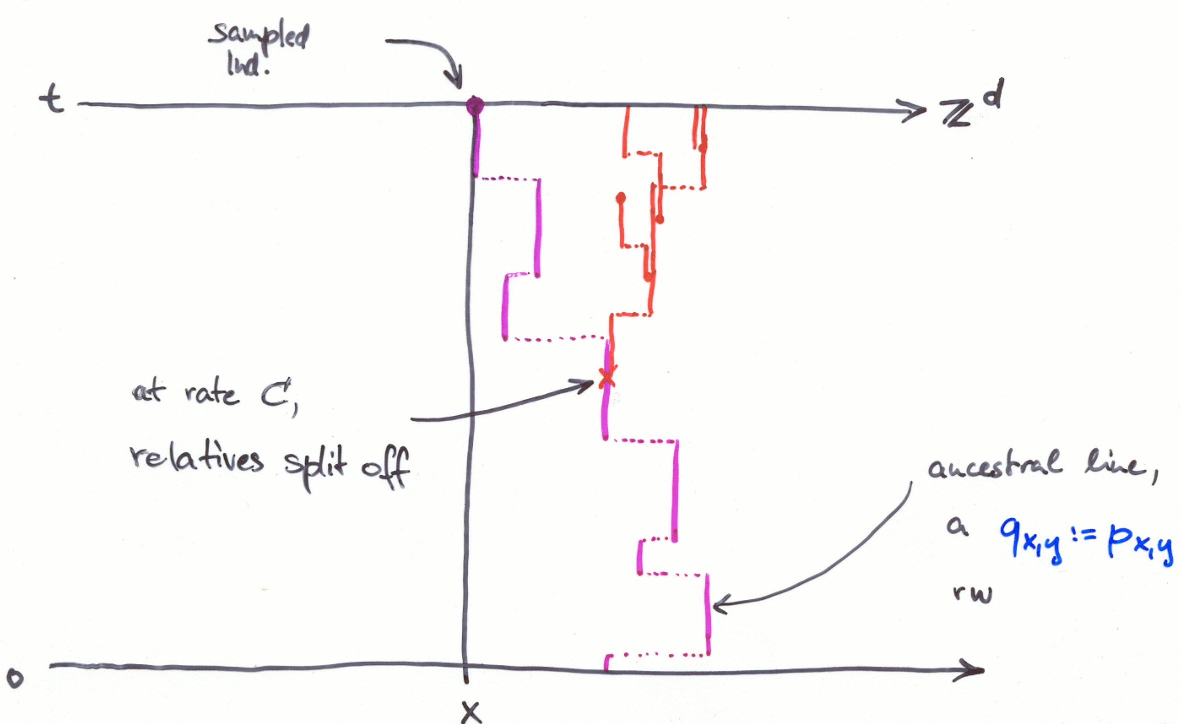
(Palm measure)

Lemma

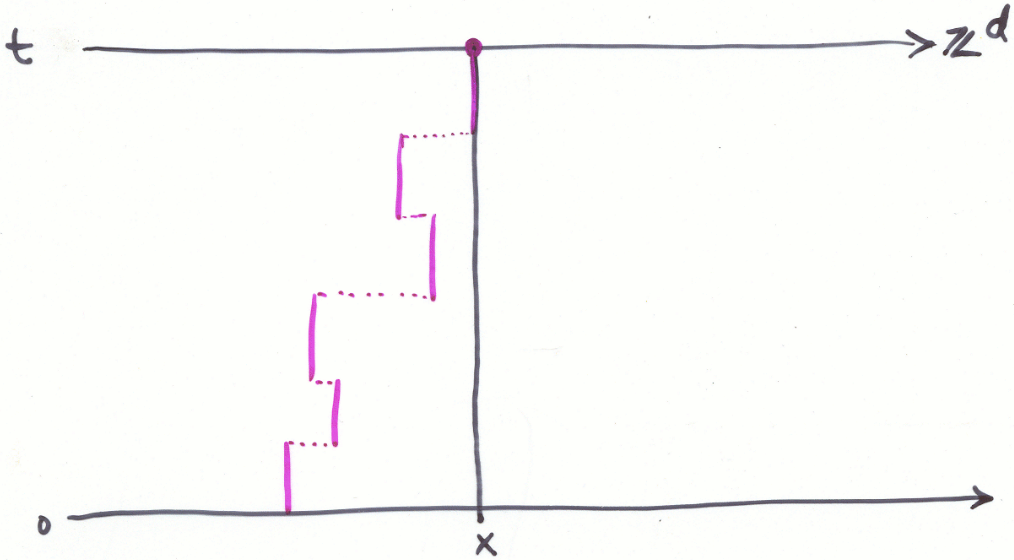
$$P(\xi_x(t) = 0) \xrightarrow[t\to\infty]{} 1$$

$$\iff \forall M: \quad \hat{P}_{(x,t)}(\xi_x \geq M) \xrightarrow[t\to\infty]{} 1.$$

Then, use a stochastic representation of ξ under $\hat{P}_{(x,t)}$ ("Kallenberg tree").



"Kallenberg tree" for general σ



Given , $\hat{\xi}$ evolves as ξ away from ,

while there are k particles on :

- a new part. is created at rate $\sigma(k+1) \cdot \frac{k+2}{2k+2}$
- a particle dies at rate $\sigma(k+1) \cdot \frac{k}{2k+2}$

"If branching is bad for you, more branching is worse."

Thm.

$\xi^{(1)}, \xi^{(2)}$ with same initial cond., same p.

$\xi^{(i)}$ uses $\sigma^{(i)}$, $i=1,2$,

$$\sigma^{(1)}(\cdot) \geq \sigma^{(2)}(\cdot)$$

Then

$$\mathbb{E}\, F(\xi^{(1)}(t)) \geq \mathbb{E}\, F(\xi^{(2)}(t))$$

for any bounded F with

$$F(\eta+\delta_x+\delta_y) - F(\eta+\delta_x) - F(\eta+\delta_y) + F(\eta) \geq 0$$

$\forall$ configurations η, $x, y \in \mathbb{Z}^d$.

Cor.

$\xi^{(2)}$ becomes locally extinct

$\Rightarrow \xi^{(1)}$ too.

$$\left(F(\xi) = \exp\left(-\sum_{x\in B} \xi_x\right)\right)$$

Note This is a "particle analogue" of Cox, Fleischmann, Greven, PTRF 1996.

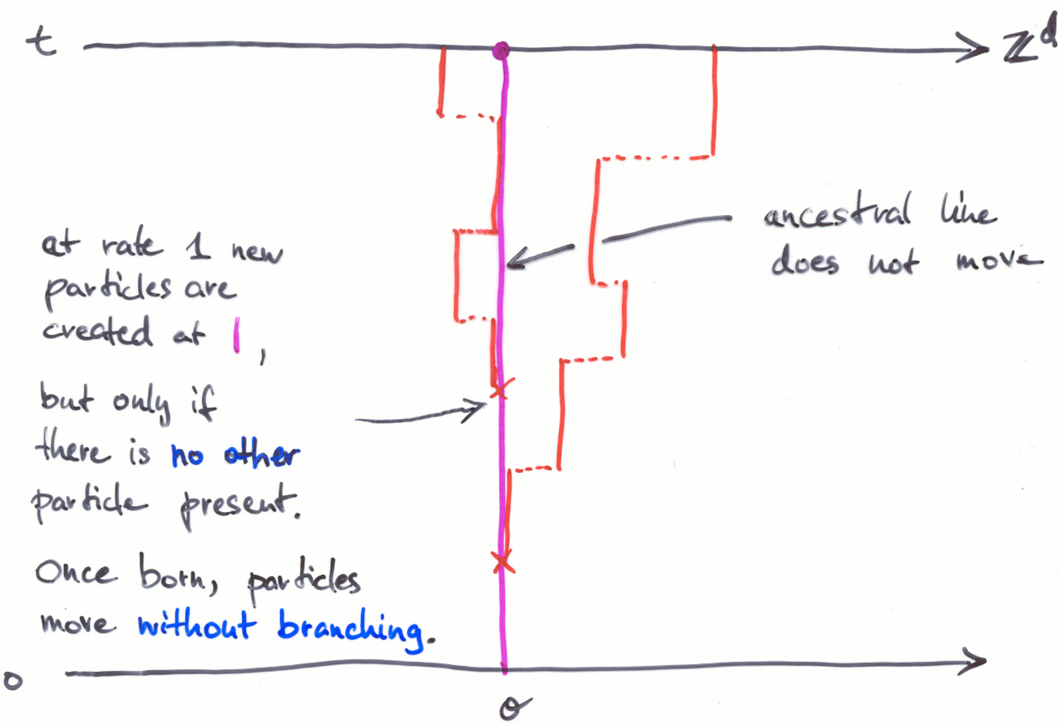
Question Can down-regulation of the branching rate enable survival in "low dimensions"?

Conjecture

No, local extinction is generic for recurrent $\hat{p}$.

"Evidence"

Consider caricature of the Kallenberg tree for $\sigma(k) = \mathbb{1}(k=1)$



Random walks with self-blocking immigration

Thm.

A system η of random walks with self-blocking immigration grows locally beyond all bounds, i.e.

$$\eta_\sigma(t) \xrightarrow[t\to\infty]{} \infty \quad \text{in probability}$$

if (and only if)

p is recurrent.

Idea of proof

Assume p recurrent, but $\eta_o(t)$ not growing

$\Rightarrow$ $\eta(t) \to \eta(\infty)$, an equilibrium.

But:

In $\eta(\infty)$, particles would arrive with positive rate at σ,

and a recurrent p is too slow to move them away

$\Rightarrow$ $\lightning$

Question Effect of up-regulated branching in "high dimensions"?

Thm

$\hat{p}$ transient.

1) If $\limsup_{k} \frac{\sigma(k)}{k^2} < 2\left(\int_0^\infty \hat{p}_{0,0}(t)\,dt\right)^{-1}$,

$\xi(t) \xrightarrow[t\to\infty]{} \xi(\infty)$ (in f.d.d.),

$\xi(\infty)$ is an equilibrium,

$\mathbb{E}\,\xi_x(\infty) = \Theta$, $\operatorname{Var} \xi_x(\infty) < \infty$,

$\operatorname{Cov}(\xi_x(\infty), \xi_y(\infty)) \xrightarrow[|x-y|\to\infty]{} 0$.

2) If $\sigma(k) = C\cdot k^2$

with $C < \sup\{\beta : \mathbb{E}[\exp(\beta\int_0^\infty 1(X(t)=X'(t)) \mid X] < \infty \text{ a.s.}\}$

then

$(\xi_x(t))_{t\geq 0}$ is uniformly integrable.

Note $\mathbb{E}\,\xi_x(t)^2$ may diverge!