Geometry, Conformal Field Theory, and String Theory

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Frobenius manifolds and integrable hierarchies of the topological type

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Main goal: to construct **universal hierarchy of integrable PDEs** over the moduli space of semisimple Frobenius manifolds of a given dimension n

(dim of the moduli space: $n(n-1)/2$)

Main observation (?): the defining relations in the theory of these hierarchies = universal identities in the theory of Gromov - Witten invariants (and their descendents).

Hamiltonian vector fields X_H:

$$\dot{x} = \{x, H\}$$

Integrability: $H = H_1,\ H_2,\ \ldots,$

$$\{H_i, H_j\} = 0$$

$\Rightarrow$ vector fields $X_i := X_{H_i}$ commute

$$[X_H, X_i] = 0, \quad [X_i, X_j] = 0$$

(commutative Lie algebra of symmetries).

Completeness:

$$\{F, H_i\} = 0 \quad \Rightarrow \quad X_F \in \text{span}(X_1, X_2, \ldots)$$

maximal Abelian subalgebras in the Lie algebra of Hamiltonian vector fields

Evolutionary PDEs as dynamical systems:

$$u_t = F(u, u_x, u_{xx}, \ldots), \quad u = (u^1, \ldots, u^n) \in M$$

vector field on the loop space

$$\mathcal{L}(M) = \{S^1 \to M\}$$

Cauchy data

$$u|_{t=0} = u_0(x)$$

Solution $u(x,t) =$ "integral curve" of the vector field beginning at the "point" $u_0(x) \in \mathcal{L}(M)$

	ODEs	PDEs
Functions	$H(u)$	$H = \int h(u; u_x, u_{xx}, \ldots)\, dx$
Differentials	$dH = \sum \frac{\partial H}{\partial u^i} du^i$	$\delta H = \int \frac{\delta H}{\delta u^i(x)} \delta u^i(x)\, dx$
		$\frac{\delta H}{\delta u^i(x)} = \sum (-1)^s \partial_x^s \frac{\partial h}{\partial u^{i,s}}, \quad u^{i,s} := \partial_x^s u^i$
Vector fields	$\dot{u}^i = F^i(u)$	$u_t^i = F^i(u; u_x, u_{xx}, \ldots)$
Poisson brackets	$\{f, g\} = \sum \{u^i, u^j\} \frac{\partial f}{\partial u^i} \frac{\partial g}{\partial u^j}$	$\{F, G\} = \iint \frac{\delta F}{\delta u^i(x)} \{u^i(x), u^j(y)\} \frac{\delta G}{\delta u^j(y)} dx dy$
Hamiltonian v. fields	$\dot{u}^i = \{u^i, H\}$	$u_t^i = \{u^i(x), H\}$
		$= \int \{u^i(x), u^j(y)\} \frac{\delta H}{\delta u^j(y)}\, dy$
Super P.B.	$\widehat{\pi} = \sum \{u^i, u^j\} \theta_i \theta_j$	$\widehat{\pi} = \iint \{u^i(x), u^j(y)\} \theta_i(x) \theta_j(y)\, dx\, dy$
	$\{\widehat{\pi}, \widehat{\pi}\} = \sum \frac{\partial \widehat{\pi}}{\partial \theta_i} \frac{\partial \widehat{\pi}}{\partial u^i}$	$\{\widehat{\pi}, \widehat{\pi}\} = \int \frac{\delta \widehat{\pi}}{\delta \theta_i(x)} \frac{\delta \widehat{\pi}}{\delta u^i(x)}\, dx$

Local Poisson brackets:

$$\{u^i(x), u^j(y)\} = \sum_k A_k^{ij}(u(x), u_x(x), \ldots)\delta^{(k)}(x-y)$$

or

$$\hat{\pi} = \int \sum_k A_k^{ij}(u, u_x, \ldots)\theta_i\theta_j^{(k)}\, dx, \quad \{\hat{\pi}, \hat{\pi}\} = \int \frac{\delta\hat{\pi}}{\delta\theta_i(x)}\frac{\delta\hat{\pi}}{\delta u^i(x)}\, dx = 0$$

P.B. of local functionals is a local functional

$$\{F, G\} = \int \sum_k \frac{\delta F}{\delta u^i(x)} A_k^{ij}(u, u_x, \ldots)\partial_x^k \frac{\delta G}{\delta u^j(x)}\, dx$$

Hamiltonian systems are evolutionary PDEs

$$u_t^i = \{u^i(x), H\} = \sum_k A_k^{ij}(u, u_x, \ldots)\partial_x^k \frac{\delta H}{\delta u^j(x)}$$

Problem

Classify maximal Abelian subalgebras of **local Hamiltonians**

$$\{H_i, H_j\} = 0, \quad H_i = \int h_i(u, u_x, \ldots)\, dx, \quad u = (u^1, \ldots, u^n) \in M^n$$

One of motivations

- X smooth projective

$$\mathcal{F}_X(\mathbf{t}; \epsilon) = \sum_{g \geq 0} \epsilon^{2g-2} \sum_k \frac{1}{k!} t^{\alpha_1, p_1} \ldots t^{\alpha_k, p_k} \int_{[\mathcal{M}_{g,k}(X)]^{\mathrm{virt}}} \mathrm{ev}_1^*(\phi_{\alpha_1}) \psi_1^{\mathrm{p}_1} \ldots \mathrm{ev}_{\mathrm{k}}^*(\phi_{\alpha_{\mathrm{k}}}) \psi_1^{\mathrm{p}_{\mathrm{k}}}$$

$$\phi_1 = 1,\, \phi_2, \ldots, \phi_n \quad \text{basis in} \quad H^*(X)$$

Witten's question: is

$$\tau(\mathbf{t}; \epsilon) := e^{\mathcal{F}_X}$$

a tau function of a hierarchy of integrable PDEs?

Examples

- $X = \mathrm{pt}\ \Rightarrow$ KdV

- $X = \mathbf{P}^1\ \Rightarrow$ (extended) Toda

New hierarchies constructed using properties of GW invariants?

Frobenius manifold M - multiplication on tangent bundle

$$a, b \in T_x M \ \mapsto \ a \cdot b \in T_x M$$

Commutative, associative, with a unity e and with invariant symmetric nondegenerate bilinear form on T^*M

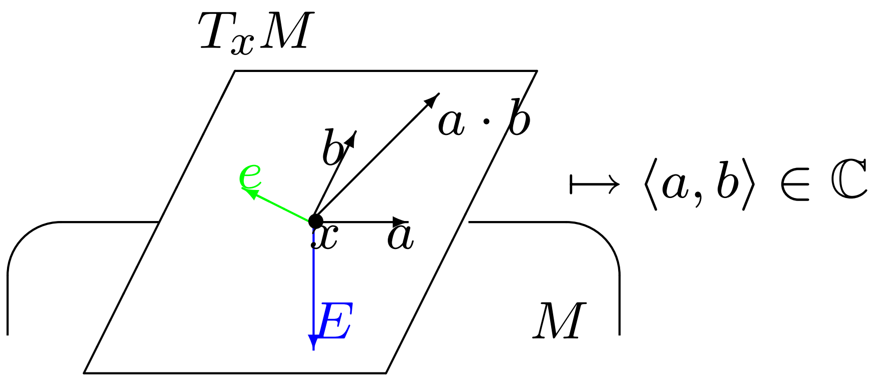
$$\langle a, b\rangle \in \mathbb{C}$$

that is constant in flat coordinates

$$\langle dx^i, dx^j\rangle = \eta^{ij}$$

Main axiom: the product in the flat coordinates given by triple derivatives of a function $F(x)$

$$\partial_i \cdot \partial_j = c_{ij}^k(x)\partial_k, \quad c_{ij}^k(x) = \eta^{kl}\frac{\partial^3 F(x)}{\partial x^i \partial x^j \partial x^l}$$



Other axioms:

- the unity e is constant in the flat coordinates

- quasihomogeneity: linear vector field E is defined

$$E = (A^i_j x^j + B^i)\partial_i$$

$$[e, E] = e, \quad \mathrm{Lie}_E F(x) = (3 - d)F(x) + \text{quadratic}$$

Recall examples

• Quantum cohomology, $M = QH^*(X)$ (genus zero Gromov - Witten invariants of X).

E.g., $QH^*(\mathbf{P}^1)$, dim $= 2$, $d = 1$

$$F = \frac{1}{2} u\, v^2 + e^u$$

$$\langle \partial_u, \partial_v \rangle = 1$$

$$e = \partial_v, \quad E = v\, \partial_v + 2\partial_u$$

- M =base of universal unfolding of an isolated singularity $f(x)$.

E.g., A_n singularity, $\dim = n$, $d = \frac{n-1}{n+1}$

$$f_a(x) = x^{n+1} + a_1 x^{n-1} + \ldots + a_n \in M$$

algebra

$$T_a M = \mathbb{C}[x]/\{f_a'(x) = 0\}, \quad e = 1$$

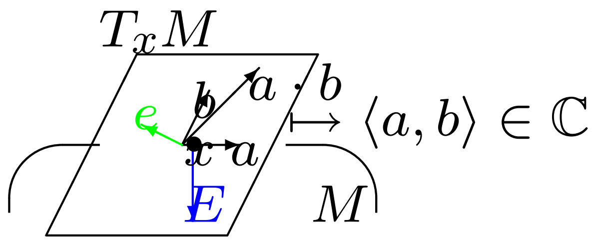
metric on $T_a M$

$$\langle \frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_j} \rangle = \frac{1}{n+1} \text{res}_{x=\infty} \frac{x^{n-i-j}}{f_a'(x)}$$

Euler vector field

$$E = \frac{1}{n+1} \left(2\, a_1 \frac{\partial}{\partial a_1} + 3\, a_2 \frac{\partial}{\partial a_2} + \ldots + (n+1)\, a_n \frac{\partial}{\partial a_n} \right)$$

Semisimplicity: the algebra T_aM is semisimple for generic $a \in M$.



On an arbitrary Frobenius manifold

- Canonical flat connection on $M \times \mathbb{C}^*$

$$\tilde{\nabla}_u v = \nabla_u v + z\, u \cdot v$$
$$\tilde{\nabla}_{d/dz} v = \partial_z v + E \cdot v - \frac{1}{z}(\nabla E)\, v$$

- Canonical coordinates near a semisimple point

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{ij}\, \frac{\partial}{\partial u_i}$$

- Another flat metric g

Systems of Integrable PDEs

Bihamiltonian systems of n PDEs:

$$u_t^i = A_j^i(u)u_x^j + \epsilon\left(B_j^i(u)u_{xx}^j + \frac{1}{2}C_{jk}^i(u)u_x^j u_x^k\right) + O(\epsilon^2)$$
$$= \{u^i(x), H_1\}_1 = \{u^i(x), H_2\}_2 \tag{1}$$

$u = (u^1, \ldots, u^n) \in M$ local coordinates (later: M=Frobenius manifold)

Hamiltonians are local functionals

$$H_k[u] = \int h_k(u; u_x, u_{xx}, \ldots; \epsilon)\, dx, \quad k = 1, 2.$$

Classify wrt the group of Miura-type transformations

$$u^i \mapsto \tilde{u}^i = f_0^i(u) + \epsilon\, f_1^i(u; u_x) + \epsilon^2\, f_2^i(u; u_x, u_{xx}) + O(\epsilon^3)$$

$$\deg f_m^i(u; u_x, \dots, u^{(m)}) = m, \qquad \det\left(\frac{\partial f_0^i(u)}{\partial u^j}\right) \neq 0$$

Semisimplicity: characteristic roots of $(A^i_j(u))$ distinct for generic u

Poisson brackets

$$\{u^i(x), u^j(y)\}_{1,2} = \eta^{ij}_{1,2}(u(x))\delta'(x-y) + \Gamma^{ij}_{k\,1,2}(u)u^k_x\delta(x-y) + O(\epsilon)$$

$$\det \eta^{ij}_{1,2}(u) \neq 0$$

Hierarchy = commuting Hamiltonian flows on the space of vector functions $u(x) = (u^1(x), \ldots, u^n(x))$ of the spatial variable x organized in n infinite chains

$$t = (t^{\alpha,p}), \quad \alpha = 1, \ldots, n, \quad p = 0,\, 1,\, 2, \ldots$$

$$\frac{\partial u}{\partial t^{\alpha,p}} = \{u(x), H_{\alpha,p}\}$$

$$H_{\alpha,p} = \int h_{\alpha,p}(u; u_x, \ldots; \epsilon)\, dx$$

$$\left[\frac{\partial}{\partial t^{\alpha,p}}, \frac{\partial}{\partial t^{\beta,q}}\right] = 0$$

$$\frac{\partial}{\partial t^{1,0}} = \frac{\partial}{\partial x}$$

Additional:

- triangular bihamiltonian recursion

$$\{\ .\ , H_{\alpha,p-1}\}_2 = \sum_{q\leq p} R_{\alpha,p}^{\beta,q} \{\ .\ , H_{\beta,q}\}_1$$

- existence of tau-function.

Hamiltonian densities

$$h_{\alpha,p}(u; u_x, \ldots; \epsilon) = \epsilon^2 \frac{\partial^2 \log \tau}{\partial x \partial t^{\alpha,p+1}}$$

In particular

$$u_\alpha = \epsilon^2 \frac{\partial^2 \log \tau}{\partial x \partial t^{\alpha,0}}, \quad \alpha = 1, \ldots, n.$$

(recall: $x = t^{1,0}$)

Tau-function $\tau(\mathbf{t};\epsilon)$ *depends* on the choice of a solution

$$u(x,\mathbf{t};\epsilon) = u_0(\mathbf{t}) + \epsilon\, u_1(x,\mathbf{t}) + \epsilon^2 u_2(x,\mathbf{t}) + \dots$$

Results (B.D., Y.Zhang)

Theorem 1 At $\epsilon = 0$
hierarchies $\leftrightarrow$ semisimple Frobenius manifolds M^n (or their degenerations)

Must also allow the operation of changing the spatial direction

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial t^{1,0}} \mapsto \frac{\partial}{\partial \tilde{x}} = \sum b^i \frac{\partial}{\partial t^{i,0}}$$

$\Rightarrow \frac{n(n-1)}{2}$ parameters* of dispersionless
(= no ϵ) integrable hierarchies with n dependent variables.

*moduli = holonomy of the flat connection $\tilde{\nabla}$
on $M^n \times \mathbb{C}^*$

Construction of the **Principal Hierarchy** associated with M

$$u_{t^{\alpha,p}} = \nabla\theta_{\alpha,p}(v)\cdot v_x = \{u(x), H_{\alpha,p}\}_1$$

$$H_{\alpha,p} = \int \theta_{\alpha,p+1}(v)\,dx$$

Here the functions

$$f_\alpha(v,z) = \sum_{p\geq 0}\theta_{\alpha,p}(v)z^{p+\mu_\alpha}, \qquad \alpha = 1,\dots,n$$

are such that

$$\tilde\nabla\, df_\alpha = 0$$

$\mu_1, \mu_2, \dots, \mu_n$ = eigenvalues of $\frac{d-2}{2} - \nabla E$ (assuming semisimplicity and nonresonancy of ∇E, $\mu_\alpha - \mu_\beta \notin \mathbb{Z}_{>0}$).

Bihamiltonian structure

$$\{u^\alpha(x), u^\beta(y)\}_1 = \eta^{\alpha\beta}\delta'(x-y)$$

$$\{u^\alpha(x), u^\beta(y)\}_2 = g^{\alpha\beta}(v(x))\delta'(x-y) + \Gamma^{\alpha\beta}_\gamma(v) v^\gamma_x \delta(x-y)$$

with

$$g^{\alpha\beta}(v) := E^\gamma(v) c^{\alpha\beta}_\gamma(v)$$

another flat metric on M

Theorem 2 The deformation space of a *given* Principal Hierarchy with n dependent variables is at most n-dimensional

Construction of the parameters c_1, ..., c_n for a bihamiltonian structure

$$\{w^\alpha(x), w^\beta(y)\}_1 = \eta^{\alpha\beta}\delta'(x-y) + \epsilon^2 A_1^{\alpha\beta}(w(x))\delta'''(x-y) + \dots$$

$$\{w^\alpha(x), w^\beta(y)\}_2 = g^{\alpha\beta}(w(x))\delta'(x-y) + \Gamma_\gamma^{\alpha\beta}(w)w_x^\gamma\delta(x-y)$$
$$+\epsilon^2 A_1^{\alpha\beta}(w(x))\delta'''(x-y) + \dots$$

- Canonical coordinates $u_1, \ldots, u_n$ on M: roots of

$$\det(g^{\alpha\beta}(v) - \lambda\, \eta^{\alpha\beta}) = 0$$

Main property:

$$\partial/\partial u_i \cdot \partial/\partial u_j = \delta_{ij} \partial/\partial u_i$$

The two metrics are diagonal in the canonical coordinates

$$\eta = \sum_{i=1}^{n} \eta_{ii}(u) du_i^2, \quad g = \sum_{i=1}^{n} \eta_{ii}(u) \frac{du_i^2}{u_i}.$$

- Define

$$B_k^{ij}(w) := \frac{\partial u^i}{\partial w^\alpha} \frac{\partial u^j}{\partial w^\beta} A_k^{\alpha\beta}(w), \quad k = 1, 2$$

- Then

$$c_i := \frac{1}{3}\eta_{ii}^2(w)(B_2^{ii}(w) - w_i B_1^{ii}(w)),\ i = 1, \ldots, n.$$

This is the complete set of invariants of the bihamiltonian structure with the given leading term at $\epsilon = 0$.

Uses results of S.-Q.Liu, Y.Zhang on deformations of bihamiltonian structures and also quasitriviality:

Example 1

Riemann wave $\mapsto$ KdV

$$v_t + v\,v_x = 0 \quad \mapsto \quad w_t + w\,w_x + \frac{\epsilon^2}{12} w_{xxx} = 0$$

The substitution Baikov, Gazizov, Ibragimov, 1989

$$w = v + \frac{\epsilon^2}{24}\partial_x^2 \left(\log v_x\right)$$
$$+\epsilon^4 \partial_x^2 \left(\frac{v^{IV}}{1152\, v_x^2} - \frac{7\, v_{xx} v_{xxx}}{1920\, v_x^3} + \frac{v_{xx}^3}{360\, v_x^4} \right) + O(\epsilon^6).$$

From Frobenius Manifolds to Integrable Systems of the Topological Type

Hierarchy of the topological type:

$$c_1 = c_2 = \ldots = c_n = \frac{1}{24}$$

Theorem 4 For any semisimple Frobenius manifold there exists unique integrable hierarchy of the topological type s.t. the additional flows

$$\frac{\partial \tau}{\partial s_m} = L_m \tau, \quad m \geq -1$$

act by (infinitesimal) symmetries

$$\left[\frac{\partial}{\partial t^{\alpha,p}}, \frac{\partial}{\partial s_m}\right] = 0.$$

Here

$$L_m = \sum \frac{\epsilon^2}{2} a_m^{\alpha,p;\beta,q} \frac{\partial^2}{\partial t^{\alpha,p} \partial t^{\beta,q}} + b_{m\,\alpha,p}^{\;\beta,q} t^{\alpha,p} \frac{\partial}{\partial t^{\beta,q}}$$
$$+ \frac{1}{2\epsilon^2} c^m_{\alpha,p;\beta,q} t^{\alpha,p} t^{\beta,q} + c_0 \delta_{m,0}$$

$$[L_m, L_n] = (m-n) L_{m+n}$$

$a_m^{\alpha,p;\beta,q}$, $b_{m\,\alpha,p}^{\;\beta,q}$, $c^m_{\alpha,p;\beta,q}$, c_0 are some constant coefficients depending on the Frobenius manifold.

Any *regular* solution to the hierarchy is obtained from the vacuum solution τ^{vac}

$$\frac{\partial \tau^{\text{vac}}}{\partial s_m} = 0, \text{ i.e. } L_m \tau^{\text{vac}} = 0, \quad m \geq -1$$

by a shift

$$\tau(\mathbf{t};\epsilon) = \tau^{\mathsf{vac}}(\mathbf{t} - \mathbf{t_0}(\epsilon);\epsilon), \quad \mathbf{t_0}(\epsilon) = \left(t_0^{\alpha,p}(\epsilon)\right)$$

Motivation: all known relations for the topological correlators reproduced for the topological solution specified by the shift

$$\tau^{\mathsf{top}} = \tau^{\mathsf{vac}}{}_{t^{1,1} \mapsto t^{1,1} - 1}$$

Proof based on

Lemma For any semisimple Frobenius manifold M^n there exists a unique solution

$$\Delta\mathcal{F} = \sum_{g\geq 1} \epsilon^{2g-2}\mathcal{F}_g(v; v_x, \ldots, v^{(3g-2)})$$

to the system of Virasoro constraints

$$L_m\left(\tau_0^{\text{vac}}(\mathbf{t})e^{\Delta\mathcal{F}}\right) = 0, \quad m \geq -1$$

(cf. Virasoro conjecture by T.Eguchi *et al.* and S.Katz).

The substitution

$$v_\alpha \mapsto w_\alpha = v_\alpha + \epsilon^2 \frac{\partial^2 \Delta\mathcal{F}}{\partial t^{1,0}\partial t^{\alpha,0}}$$

transforms the Principal Hierarchy associated with M^n to the hierarchy of the topological type associated with M^n.

Theorem 5 Let X be a smooth projective variety with $H^{\text{odd}}(X) = 0$ s.t.

- $QH^*(X)$ is semisimple
- Virasoro constraints hold true for the total GW potential

$$\mathcal{F}^X(\mathbf{t};\epsilon) = \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{F}_g^X(\mathbf{t})$$

(e.g., $X = \mathbf{P}^d$, due to Givental)
Then

$$\tau = \exp \mathcal{F}^X$$

is the tau-function of the topological solution to the hierarchy of the topological type associated with the Frobenius manifold

$$M^n = QH^*(X), \quad n = \dim H^*(X)$$

Examples of Integrable PDEs

Example 1 $n = 1$, trivial Frobenius manifold

$$F(v) = \frac{1}{6}v^3$$

$\Rightarrow$ KdV hierarchy

$$u_{t_0} = u_x$$
$$u_{t_1} = u\,u_x + \frac{\epsilon^2}{12}u_{xxx}$$
$$u_{t_2} = \frac{1}{2}u^2u_x + \frac{\epsilon^2}{12}(2u_xu_{xx} + u\,u_{xxx}) + \frac{\epsilon^4}{240}u^V$$
$$\ldots$$

Construction: Lax operator

$$L = \frac{\epsilon^2}{2}\frac{d^2}{dx^2} + u$$

$$\frac{\partial L}{\partial t_k} = [A_k, L], \quad k \geq 0, \quad A_k = \frac{2^{\frac{2k+1}{2}}}{(2k+1)!!} \left(L^{\frac{2k+1}{2}} \right)_+$$

Dispersionless limit $\epsilon = 0$,

$$L(x, \epsilon d/dx) \mapsto \lambda(x, p) = \frac{1}{2} p^2 + v(x)$$

(the symbol). The Frobenius manifold

$$M = \left\{ \frac{1}{2} p^2 + v \right\} = \mathbb{C}/W(A_1)$$

Vacuum tau-function

$$u = \epsilon^2 \partial_x^2 \log \tau$$

$$L_m \tau_{\mathrm{KdV}}^{\mathrm{vac}} = 0, \quad m \geq -1$$

has the form

$$\tau_{\mathrm{KdV}}^{\mathrm{vac}} = \frac{1}{(-t_1)^{1/24}} \exp\left\{ \frac{1}{\epsilon^2} \left[-\frac{t_0^3}{6\, t_1} - \frac{t_0^4 t_2}{24\, t_1^3} + O(t_0^5) \right] \right.$$
$$+ \left[\frac{t_0 t_2}{24\, t_1^2} - \frac{t_0^2 t_3}{48\, t_1^3} + \frac{t_0^2 t_2^2}{24\, t_1^4} + O(t_0^3) \right]$$
$$+\epsilon^2 \left[-\frac{t_4}{1152\, t_1^3} + \frac{29\, t_2 t_3}{5760\, t_1^4} - \frac{7\, t_2^3}{1440\, t_1^5} + O(t_0) \right]$$
$$\left. + O(\epsilon^4) \right\}.$$

Relationship between vacuum tau-function and the topological tau-function

Theorem (Kontsevich - Witten)

$$\log \tau_{\text{KdV}}^{\text{top}}(t_0, t_1, t_2, \ldots) = \log \tau_{\text{KdV}}^{\text{vac}}(t_0, t_1 - 1, t_2, \ldots)$$
$$= \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g(\mathbf{t})$$

where

$$\mathcal{F}_g = \sum \frac{1}{n!} t_{p_1} \ldots t_{p_n} \int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{p_1} \wedge \ldots \wedge \psi_n^{p_n}$$

Topological KdV tau-function

$$\log \tau_{\text{KdV}}^{\text{top}} = \frac{1}{\epsilon^2}\left(\frac{t_0^3}{6} + \frac{t_0^3 t_1}{6} + \frac{t_0^3 t_1^2}{6} + \frac{t_0^3 t_1^3}{6} + \frac{t_0^3 t_1^4}{6} + \frac{t_0^4 t_2}{24} + \frac{t_0^4 t_1 t_2}{8}\right.$$

$$\left. + \frac{t_0^4 t_1^2 t_2}{4} + \frac{t_0^5 t_2^2}{40} + \frac{t_0^5 t_3}{120} + \frac{t_0^5 t_1 t_3}{30} + \frac{t_0^6 t_4}{720} + \dots\right)$$

$$+ \left(\frac{t_1}{24} + \frac{t_1^2}{48} + \frac{t_1^3}{72} + \frac{t_1^4}{96} + \frac{t_0 t_2}{24} + \frac{t_0 t_1 t_2}{12} + \frac{t_0 t_1^2 t_2}{8} + \frac{t_0^2 t_2^2}{24}\right.$$

$$\left. + \frac{t_0^2 t_3}{48} + \frac{t_0^2 t_1 t_3}{16} + \frac{t_0^3 t_4}{144} + \dots\right)$$

$$+\epsilon^2\left(\frac{7 t_2^3}{1440} + \frac{7 t_1 t_2^3}{288} + \frac{29 t_2 t_3}{5760} + \frac{29 t_1 t_2 t_3}{1440} + \frac{29 t_1^2 t_2 t_3}{576} + \frac{5 t_0 t_2^2 t_3}{144}\right.$$

$$+ \frac{29 t_0 t_3^2}{5760} + \frac{29 t_0 t_1 t_3^2}{1152} + \frac{t_4}{1152} + \frac{t_1 t_4}{384} + \frac{t_1^2 t_4}{192} + \frac{t_1^3 t_4}{96} + \frac{11 t_0 t_2 t_4}{1440}$$

$$\left. + \frac{11 t_0 t_1 t_2 t_4}{288} + \frac{17 t_0^2 t_3 t_4}{1920} + \dots\right) + O(\epsilon^4).$$

Example 2 $n = 2$,

$$F(u,v) = \frac{1}{2} u\, v^2 + e^u$$

The Frobenius manifold

$$M^2 = \left\{ \lambda(p) = e^p + v + e^{u-p} \right\}$$

=symbol of the difference Lax operator

$$L = \Lambda + v + e^u \Lambda^{-1}, \quad \Lambda = e^{\epsilon \partial_x}$$

Extended Toda hierarchy
(G.Carlet, B.D., Y.Zhang)

$$\epsilon \frac{\partial L}{\partial t_k} = \frac{1}{(k+1)!} \left[(L^{k+1})_+, L \right]$$

$$\epsilon \frac{\partial L}{\partial s_k} = \frac{2}{k!} \left[\left(L^k (\log L - c_k) \right)_+, L \right]$$

$$c_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k}$$

$s_0 = x$

E.g.,

$$u_{t_0} = \frac{v(x) - v(x-\epsilon)}{\epsilon} = v_x - \frac{1}{2}\epsilon\, v_{xx} + O(\epsilon^2)$$

$$v_{t_0} = \frac{e^{u(x+\epsilon)} - e^{u(x)}}{\epsilon} = e^u u_x + \frac{1}{2}\epsilon\,(e^u)_{xx} + O(\epsilon^2)$$

(infinite order PDEs)

Eliminating v and defining

$$u_n = u(n\epsilon), \quad t = \frac{t_0}{\epsilon}$$

$\Rightarrow$ the standard **Toda lattice equation**

$$\ddot{u}_n = e^{u_{n-1} - u_n} - e^{u_n - u_{n+1}}.$$

times $s_1, s_2, \ldots$ are new.

Remark Interchanging time/space variables $x = s_0 \leftrightarrow t_0 = \tilde{x}$ transforms Toda $\leftrightarrow$ NLS

Application to $QH^*(\mathbf{P}^1)$

Claim (B.D., Y.Zhang, using A.Okounkov and R.Pandharipande) Substituting $s_1 \mapsto s_1 - 1$ into the vacuum tau-function of the extended Toda hierarchy gives the generating function of the Gromov - Witten invariants and their descendents of $\mathbf{P}^1$

$$F := \log \tau_{\text{Toda}}^{\text{vac}}(t_0, t_1, \dots; s_0, s_1 - 1, \dots; \epsilon)$$

$$= \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g(\mathbf{t}, \mathbf{s})$$

$$\mathcal{F}_g = \sum_m \frac{1}{m!} \sum t_{\alpha_1, p_1} \dots t_{\alpha_m, p_m} \langle \tau_{p_1}(\phi_{\alpha_1}) \dots \tau_{p_m}(\phi_{\alpha_m}) \rangle_{g, \beta}$$

$$\alpha = 1,\, 2, \quad p = 0,\, 2, \dots \qquad t_{1,p} = s_p, \quad t_{2,p} = t_p$$

$$\phi_1 = 1 \in H^0(\mathbf{P}^1), \quad \phi_2 \in H^2(\mathbf{P}^1), \quad \int \phi_2 = 1$$

$$\langle \tau_{p_1}(\phi_{\alpha_1}) \ldots \tau_{p_m}(\phi_{\alpha_m}) \rangle_{g,\beta}$$

$$= \int_{\bar{\mathcal{M}}_{g,m}(\mathbf{P}^1,\beta)} ev_1^*(\phi_{\alpha_1}) \wedge \psi_1^{p_1} \wedge \ldots \wedge ev_m^*(\phi_{\alpha_m}) \wedge \psi_m^{p_m}$$

Application to enumeration of fat graphs/triangulations

$$Z_N(\lambda;\epsilon) = \frac{1}{\mathrm{Vol}(U_N)} \int_{N\times N} e^{-\frac{1}{\epsilon}\mathrm{Tr}\, V(A)}\, dA$$

$$V(A) = \frac{1}{2}A^2 - \sum_{k\geq 3} \lambda_k A^k$$

as function of $N = x/\epsilon$, λ
is a tau-function of Toda lattice

Large N $\sim$ small ϵ expansion of

$$\tau(x, \mathbf{t}; \epsilon) = Z_N(\lambda; \epsilon)$$

$$x = \frac{N}{\epsilon}, \quad t_k = (k+1)!\lambda_{k+1}$$

has the form

$$\log \tau = \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g(x, \mathbf{t})$$

so the solution u, v admits regular expansion

$$u = \sum_{k \geq 0} \epsilon^k u_k(x, \mathbf{t})$$
$$v = \sum_{k \geq 0} \epsilon^k v_k(x, \mathbf{t})$$

For small λ the ϵ-expansion can be obtained by applying the saddle point method to

$$Z_N = \frac{1}{\mathrm{Vol}(U_N)} \int e^{-\frac{1}{\epsilon}\mathrm{Tr}V(A)} dA$$

$\Rightarrow \mathcal{F}_g(x, \mathbf{t})$ = generating function of numbers of fat graphs on genus g Riemann surfaces

Corresponds to the one-cut asymptotic distribution of the eigen-values of the large size Hermitean random matrix A

Claim: Substituting

$$\tau^{\text{vac}}_{\text{Toda}}(t_0, t_1, t_2, \ldots; s_0, s_1, s_2, \ldots; \epsilon)$$

$$t_0 = 0, \ t_1 = -1, \ t_k = (k+1)!\lambda_{k+1}, \ , k \geq 2$$
$$s_0 = x, \ s_k = 0, \ k \geq 1$$

one obtains

$$F := \log \tau^{\text{vac}}_{\text{Toda}}(0, -1, 3!\lambda_3, 4!\lambda_4, \ldots; x, 0, \ldots; \epsilon)$$

$$= \frac{x^2}{2\epsilon^2}\left(\log x - \frac{3}{2}\right) - \frac{1}{12}\log x + \sum_{g\geq 2}\left(\frac{\epsilon}{x}\right)^{2g-2}\frac{B_{2g}}{2g(2g-2)}$$

$$+ \sum_{g\geq 0}\epsilon^{2g-2}F_g(x; \lambda_3, \lambda_4, \ldots)$$

$$F_g(x; \lambda_3, \lambda_4, \ldots)$$

$$= \sum_n \sum_{k_1,\ldots,k_n} a_g(k_1, \ldots, k_n) \lambda_{k_1} \ldots \lambda_{k_n} x^h,$$

$$h = 2 - 2g - \left(n - \frac{|k|}{2} \right), \quad |k| = k_1 + \ldots + k_n,$$

and

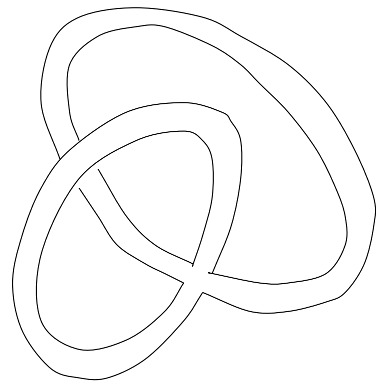
$$a_g(k_1, \ldots, k_n) = \sum_\Gamma \frac{1}{\# \operatorname{Sym} \Gamma}$$

where

Γ = a connected fat graph of genus g

with n vertices of the valencies $k_1, \ldots, k_n$.

E.g.: genus 1, one vertex, valency 4



$$F = \epsilon^{-2}\left[\frac{1}{2}x^2\left(\log x - \frac{3}{2}\right) + 6x^3{\lambda_3}^2 + 2x^3\lambda_4 + 216x^4{\lambda_3}^2\lambda_4 + 18x^4{\lambda_4}^2\right.$$

$$+288x^5{\lambda_4}^3 + 45x^4\lambda_3\lambda_5 + 2160x^5\lambda_3\lambda_4\lambda_5 + 90x^5{\lambda_5}^2 + 5400x^6\lambda_4{\lambda_5}^2 + 5x^4\lambda_6$$

$$+1080x^5{\lambda_3}^2\lambda_6 + 144x^5\lambda_4\lambda_6 + 4320x^6{\lambda_4}^2\lambda_6 + 10800x^6\lambda_3\lambda_5\lambda_6 + 27000x^7{\lambda_5}^2\lambda_6$$

$$\left.+300x^6{\lambda_6}^2 + 21600x^7\lambda_4{\lambda_6}^2 + 36000x^8{\lambda_6}^3\right]$$

$$-\frac{1}{12}\log x + \frac{3}{2}x{\lambda_3}^2 + x\lambda_4 + 234x^2{\lambda_3}^2\lambda_4 + 30x^2{\lambda_4}^2 + 1056x^3{\lambda_4}^3 + 60x^2\lambda_3\lambda_5$$

$$+6480x^3\lambda_3\lambda_4\lambda_5 + 300x^3{\lambda_5}^2 + 32400x^4\lambda_4{\lambda_5}^2 + 10x^2\lambda_6 + 3330x^3{\lambda_3}^2\lambda_6$$

$$+600x^3\lambda_4\lambda_6 + 31680x^4{\lambda_4}^2\lambda_6 + 66600x^4\lambda_3\lambda_5\lambda_6 + 283500x^5{\lambda_5}^2\lambda_6$$

$$+2400x^4{\lambda_6}^2+270000x^5\lambda_4{\lambda_6}^2+696000x^6{\lambda_6}^3$$

$$+\epsilon^2\left[-\frac{1}{240x^2}+240x{\lambda_4}^3+1440x\lambda_3\lambda_4\lambda_5+\frac{1}{2}165x{\lambda_5}^2+28350x^2\lambda_4{\lambda_5}^2\right.$$

$$+675x{\lambda_3}^2\lambda_6+156x\lambda_4\lambda_6+28080x^2{\lambda_4}^2\lambda_6+56160x^2\lambda_3\lambda_5\lambda_6+580950x^3{\lambda_5}^2\lambda_6$$

$$\left.+2385x^2{\lambda_6}^2+580680x^3\lambda_4{\lambda_6}^2+2881800x^4{\lambda_6}^3\right]+\ldots$$

Proof (B.D., T.Grava, in progress) uses **Toda equations** and the **large N expansion** for the Hermitean matrix integral

$$Z_N(\lambda;\epsilon) = \frac{1}{\mathrm{Vol}(U_N)} \int_{N\times N} e^{-\frac{1}{\epsilon}\mathrm{Tr}\, V(A)}\, dA$$

$$V(A) = \frac{1}{2}A^2 - \sum_{k\geq 3} \lambda_k A^k$$

(‘t Hooft; D.Bessis, C.Itzykson, J.-B.Zuber)

where one has to replace

$$N \mapsto \frac{x}{\epsilon}$$

Remark This is the topological solution for the (extended) non-linear Schrödinger hierarchy

Universality of critical behaviour

Commuting Hamiltonian PDEs

$$u_t + [f(u)]_x = 0$$

$$H = \int h(u)\, dx, \quad h'(u) = f(u).$$

Any other commutes

$$u_s + [g(u)]_x = 0, \quad (u_t)_s = (u_s)_t.$$

Gradient catastrophe at (x_0, t_0, u_0):

$$u_x = \infty$$

(see MOVIE1 ; for KdV $u_t + u\, u_x + \epsilon^2 u_{xxx} = 0$ see MOVIE2)

Hamiltonian regularization, mod $O(\epsilon^6)$

$$u_t + \partial_x \Big[f(u) + \frac{\epsilon^2}{24}\left(f'''(u)u_x^2 + 2f''(u)u_{xx}\right)$$

$$+\frac{\epsilon^4}{5760}\left(5f^{IV}(u)u_x^4 + 44f^{V}(u)u_x^2u_{xx} + 12f^{IV}(u)[4u_xu_{xxx} + 3u_{xx}^2] + 24f'''(u)u^{IV}(u)u_{xxxx}\right)\Big]$$

Claim. Near generic critcial point, $a := \left(d^3u/dx^3\right) \neq 0$ the solution behaves as follows:

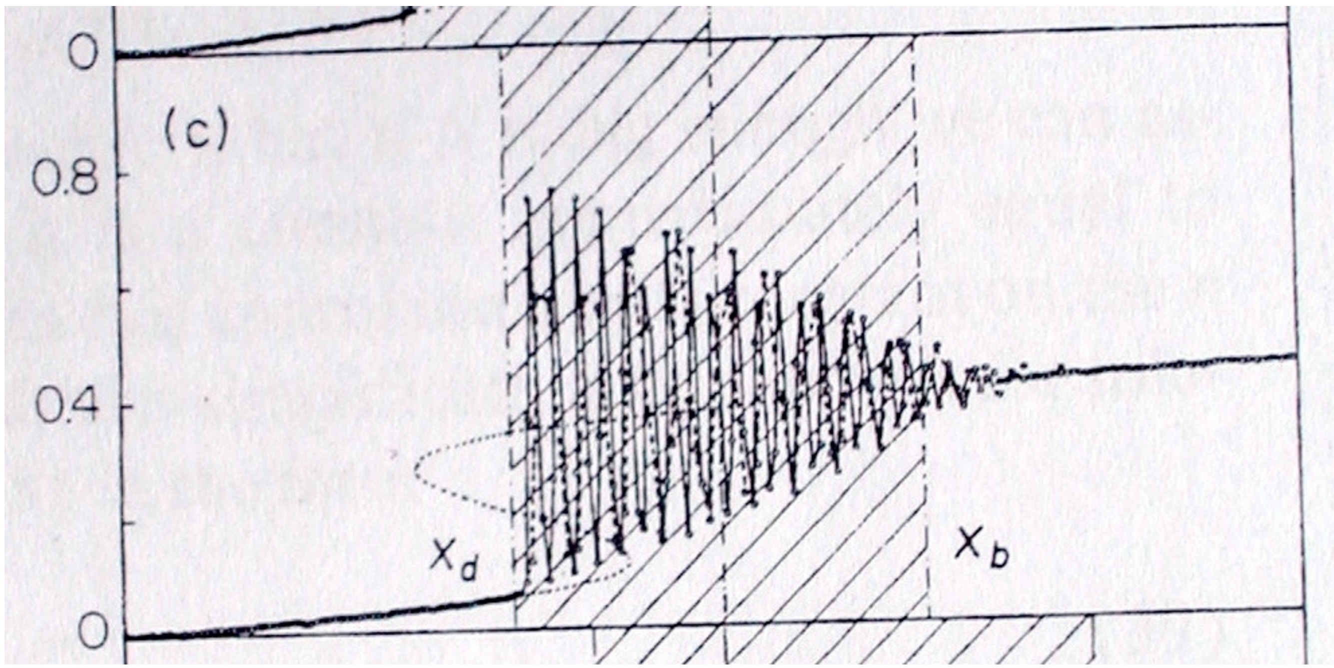
$$u(x,t;\epsilon) \simeq u_0 + \epsilon^{\frac{2}{7}} U\left(\frac{x - f_0'(t-t_0) - x_0}{\epsilon^{6/7}}, \frac{t-t_0}{\epsilon^{4/7}}\right) + O\left(\epsilon^{\frac{4}{7}}\right).$$

Here $U = U(X,T)$ is the **unique smooth solution** to the ODE

$$X = f_0' U\, T + a\left[\frac{1}{6}U^3 + \frac{1}{24}\left(U'^2 + 2U\,U''\right) + \frac{1}{240}U^{IV}\right]$$

(see MOVIE3)

Hermitean matrix model, **multicut case**: G gaps i in the asymptotic distribution of eigenvalues of random matrices $\Rightarrow$ singular behaviour of the correlation functions (terms $\sim e^{\frac{iat}{\epsilon}}$ arise)



(from Jurkiewicz, Phys. Lett. B, 1991)

Smoothed correlation functions: average out the singular terms

Question: Which integrable PDEs describe the large N expansion of *smoothed* correlation functions?

Claim: The full large N expansion of the smoothed correlation functions is given via the topological tau-function associated with the Frobenius structure M^n, $n = 2G + 2$ on the Hurwitz space of hyperelliptic curves

$$\mu^2 = \prod_{i=1}^{2G+2} (\lambda - u_i)$$

Recall the general construction: Frobenius structure on the Hurwitz space M^n = moduli of branched coverings

$$\lambda : \Sigma_G \to \mathbf{P}^1$$

fixed degree, genus G, ramification type at infinity, basis of a- and b-cycles (n = number of branch points $\lambda = u_i$ for generic covering).

Must choose a primary differential dp (say, holomorphic differential with constant a-periods)

Then, for any two vector fields ∂_1, ∂_2 on M^n the inner product

$$\langle \partial_1, \partial_2 \rangle = \sum_{i=1}^{n} \mathrm{res}_{\lambda = u_i} \frac{\partial_1(\lambda dp)\partial_2(\lambda dp)}{d\lambda}$$

for any three vector fields ∂_1, ∂_2, ∂_3 on M^n

$$\langle \partial_1 \cdot \partial_2, \partial_3 \rangle = -\sum_{i=1}^{n} \operatorname{res}_{\lambda=u_i} \frac{\partial_1(\lambda dp)\partial_2(\lambda dp)\partial_3(\lambda dp)}{d\lambda\, dp}$$

Example $G = 1$ (two-cut case). Here $n = 4$. Flat coordinates on the Hurwitz space of elliptic double coverings with 4 branch points are u, v, w, τ. Can describe by the superpotential (= symbol of Lax operator)

$$\lambda(p) = v + u\left(\log \frac{\theta_1(p-w|\tau)}{\theta_1(p+w|\tau)}\right)'$$

The Frobenius structure given by

$$F = \frac{i}{4\pi}\tau\, v^2 - 2\, u\, v\, w + u^2 \log\left[\frac{\pi}{u}\frac{\theta_1'(0|\tau)}{\theta_1(2w|\tau)}\right]$$

Recall

$$\log\left[\pi\,\frac{\theta_1'(0|\tau)}{\theta_1(x|\tau)}\right] = \log\sin\pi\, x + 4\sum_{m=1}^{\infty}\frac{q^{2m}}{1-q^{2m}}\frac{\sin^2\pi\, m\, x}{m}$$

$$q = e^{i\,\pi\,\tau}$$

Corresponding integrable hierarchy of the topological type for the functions u, v, w, τ, four infinite chains of times $t^{u,p}$, $t^{v,p}$, $t^{w,p}$, $t^{\tau,p}$. Then

$$Z \sim \tau^{\text{vac}}$$

$$\text{with}\, t^{w,1} \mapsto t^{w,1} - 1,\ t^{w,0} = 0,\ t^{w,k} = (k+1)!\lambda_{k+1}$$

$$t^{u,0} = x$$

other couplings = 0.

The *solution* is given via zero section of the Lagrangian manifold

$$\left\{p = d\,\Phi_{x,\,\lambda}\right\} \cap \{p = 0\}$$

$$\Phi_{x,\lambda} = x\,w - u\,v + u^2 P_1(2w|\tau)$$
$$+3\lambda_3 u\,\left[v^2 - 2\,u\,v\,P_1(2w|\tau) + u^2\left(P_1^2(2w|\tau) - P_2(2w|\tau) + 4\pi\,i(\log\eta(\tau))'\right)\right]$$
$$+2\lambda_4 u\,\left[2v^3 - 6u\,v^2 P_1(2w|\tau) + 6u^2 v\,\left(P_1^2(2w|\tau) - P_2(2w|\tau) + 4\pi\,i(\log\eta(\tau))'\right)\right.$$
$$\left.-u^3\left[P_3(2w|\tau) + 2P_1(2w|\tau)\left(P_1(2w|\tau)^2 - 3P_2(2w|\tau) + 12\pi\,i\,(\log\eta(\tau))'\right)\right]\right]$$
$$+\ldots$$

where

$$P_k(x|\tau) := \partial_x^k \log\theta_1(x|\tau), \quad k = 1, 2, 3$$

Canonical coordinates (branch points)

$$u_i = v - 2u\,[\log\theta_i(w|\tau)]', \quad i = 1, \ldots, 4.$$

ϵ^2-correction

$$\mathcal{F}_1 = -\frac{1}{6}\log u - \log \eta(2\tau) + \frac{1}{24}\sum_{i=1}^{4} \log u_i'$$

Further developments/problems

- (p, q) brackets, their normal form and deformations.

- Existence of the bihamiltonian structure for arbitrary $c_1(u_1)$, $\dots$, $c_n(u_n)$.

- Construction of the Lax operators; connections with infinite-dimensional Lie algebras

- Universal identities in the theory of integrable hierarchies and those in the theory of Gromov - Witten invariants and their descendents

- Other applications of new hierarchies