Branes in the Poisson sigma model and deformation quantization

Giovanni Felder, ETH Zurich
joint work with Alberto S. Cattaneo

Durham, July 2005

Table of contents

`math.QA/0309180`

`math.QA/0501540`

Deformation quantisation-1

Let A be a commutative algebra with 1 over $k = \mathbb{R}$ or $\mathbb{C}$. A formal associative deformation of the product in A is an associative $k[[\epsilon]]$-bilinear product $\star$ on $A[[\epsilon]]$ with unit $1 \in A$ and which reduces to the product $\cdot$ in A modulo ϵ.

Such a product is uniquely determined by bilinear maps $P_i : A \times A \to A$ appearing in the product of $f, g \in A$:

$$f \star g = f \cdot g + \epsilon P_1(f,g) + \epsilon^2 P_2(f,g) + \cdots\,, \quad (\epsilon = i\hbar)$$

Deformation quantisation-1

Let A be a commutative algebra with 1 over $k=\mathbb{R}$ or $\mathbb{C}$. A formal associative deformation of the product in A is an associative $k[[\epsilon]]$-bilinear product $\star$ on $A[[\epsilon]]$ with unit $1 \in A$ and which reduces to the product $\cdot$ in A modulo ϵ.

Such a product is uniquely determined by bilinear maps $P_i : A \times A \to A$ appearing in the product of $f, g \in A$:

$$f \star g = f \cdot g + \epsilon P_1(f,g) + \epsilon^2 P_2(f,g) + \cdots \,, \quad (\epsilon = i\hbar)$$

Equivalence relation: $\star \simeq \star'$ if $D(f \star g) = D(f) \star' D(g)$,

$$D(f) = f + \epsilon D_1(f) + \epsilon^2 D_2(f) + \cdots$$

Deformation quantisation-2

Basic fact: If $\star$ is an associative deformation of the product then A with $\{f, g\} = \frac{1}{\epsilon}(f \star g - g \star f) \bmod \epsilon$ is a Poisson algebra:

$\{\ ,\ \}$ is a Lie bracket on A obeying $\{f \cdot g, h\} = f\{g, h\} + \{f, h\}g$.

General problem: classify all deformations $\star$ up to equivalence.

Deformation quantisation-2

Basic fact: If $\star$ is an associative deformation of the product then A with $\{f,g\} = \frac{1}{\epsilon}(f \star g - g \star f) \bmod \epsilon$ is a Poisson algebra:

$\{\ ,\ \}$ is a Lie bracket on A obeying $\{f \cdot g, h\} = f\{g,h\} + \{f,h\}g$.

General problem: classify all deformations $\star$ up to equivalence.

Special case of deformation quantisation (star-products): $A = C^\infty(M)$. Require P_j, D_j to be differential operators in each argument.

In this case, Poisson brackets are given by bivector fields $\pi = \pi^{ij}\partial_i \wedge \partial_j \in \Gamma(M, \wedge^2 TM)$: $\{f,g\} = \pi^{ij}\partial_i f \partial_j g$, obeying $[\pi,\pi] = 0$

Deformation quantisation-3

Theorem (Kontsevich) *There is a bijection*

$$\{\pi \in \Gamma(M, \wedge^2 TM)[[\epsilon]] \text{ Poisson}\}/D \to \{\textit{Star-products on } M\}/\simeq$$

where $D = \exp(\epsilon\Gamma(M, TM)[[\epsilon]])$ *is the group of formal diffeomorphisms.*

$\mathbb{R}$ –

—

Deformation quantisation-3

Theorem (Kontsevich) *There is a bijection*

$$\{\pi \in \Gamma(M, \wedge^2 TM)[[\epsilon]] \text{ Poisson}\}/D \to \{\textit{Star-products on } M\}/\simeq$$

where $D = \exp(\epsilon\Gamma(M, TM)[[\epsilon]])$ *is the group of formal diffeomorphisms.*

Explicit formula for $M = \mathbb{R}^d$, $\frac{1}{2}\{f, g\} = \pi^{ij}\partial_i f \partial_j g$

$$f \star g = fg + \epsilon\pi^{ij}\partial_i f \partial_j g + \frac{\epsilon^2}{3}\pi^{il}\partial_l \pi^{jk}\partial_i\partial_j f \partial_k g + \cdots$$

Deformation quantisation-3

Theorem (Kontsevich) *There is a bijection*

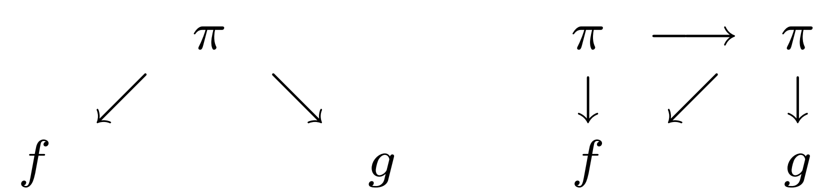
$$\{\pi \in \Gamma(M, \wedge^2 TM)[[\epsilon]] \text{ Poisson}\}/D \to \{\textit{Star-products on } M\}/\simeq$$

where $D = \exp(\epsilon\Gamma(M, TM)[[\epsilon]])$ *is the group of formal diffeomorphisms.*

Explicit formula for $M = \mathbb{R}^d$, $\frac{1}{2}\{f, g\} = \pi^{ij}\partial_i f \partial_j g$

$$f \star g = fg + \epsilon\pi^{ij}\partial_i f \partial_j g + \frac{\epsilon^2}{3}\pi^{il}\partial_l \pi^{jk}\partial_i\partial_j f \partial_k g + \cdots$$

Corresponding graphs :



Deformation quantisation-4

$\mathcal{G}_{n,2}$ the set of graphs with vertices $1, \ldots, n$ of the first kind (two outgoing edges) and $\bar{1}, \bar{2}$ of the second kind (no outgoing edges).

$$f \star g = fg + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \sum_{\Gamma \in \mathcal{G}_{n,2}} w_\Gamma P_\Gamma(f, g)$$

Deformation quantisation-4

$\mathcal{G}_{n,2}$ the set of graphs with vertices $1, \ldots, n$ of the first kind (two outgoing edges) and $\bar{1}, \bar{2}$ of the second kind (no outgoing edges).

$$f \star g = fg + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \sum_{\Gamma \in \mathcal{G}_{n,2}} w_\Gamma P_\Gamma(f, g)$$

To each $\Gamma \in \mathcal{G}_{n,2}$ there corresponds a bidifferential operator P_Γ as above and a weight

$$w_\Gamma = \frac{1}{(2\pi)^{2n}} \int_{H_+^n} \prod_{(i,j) \in E_\Gamma} d\phi(z_i, z_j), \qquad \phi(z, w) = \frac{1}{2i} \ln \frac{(z-w)(z-\bar{w})}{(\bar{z}-w)(\bar{z}-\bar{w})}$$

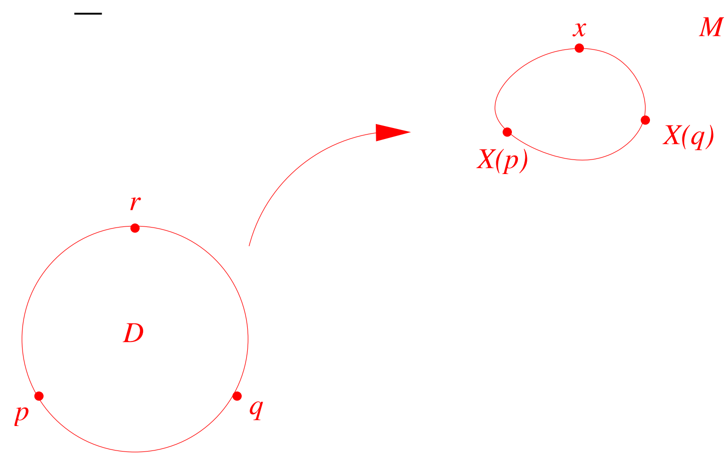
The integration is over points $z_1, \ldots, z_n \in H_+ = \{\mathrm{Im}\, z > 0\}$ corresponding to vertices i of the first kind, with $z_{\bar{1}} = 0$, $z_{\bar{2}} = 1$ for vertices of the second kind.

Poisson sigma model-1

Path integral formula for a star-product on a Poisson manifold (M, π)

$$f \star g(x) = \int_{X(r)=x} e^{\frac{i}{\hbar} S(\widehat{X})} f(X(p)) g(X(q)) d\widehat{X}$$

The integration is over bundle maps $\widehat{X} = (X, \eta) \colon TD \to T^*M$ with base map $X \colon D \to M$ and fiber map $\eta \in \Omega^1(D, X^*T^*M)$.



Poisson sigma model-1

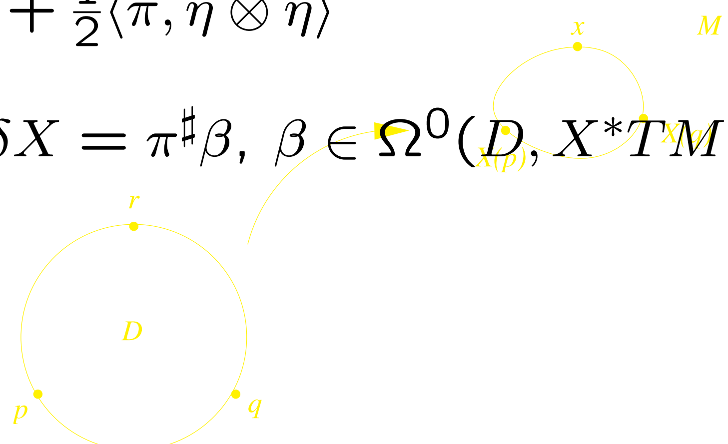
Path integral formula for a star-product on a Poisson manifold (M, π)

$$f \star g(x) = \int_{X(r)=x} e^{\frac{i}{\hbar}S(\widehat{X})} f(X(p))g(X(q))d\widehat{X}$$

The integration is over bundle maps $\widehat{X} = (X, \eta)\colon TD \to T^*M$ with base map $X\colon D \to M$ and fiber map $\eta \in \Omega^1(D, X^*T^*M)$.

$S(X, \eta) = \int_D \langle dX, \eta \rangle + \frac{1}{2}\langle \pi, \eta \otimes \eta \rangle$

Gauge symmetry: $\delta X = \pi^\sharp \beta$, $\beta \in \Omega^0(D, X^*TM)$



Poisson sigma model-1

Path integral formula for a star-product on a Poisson manifold (M, π)

$$f \star g(x) = \int_{X(r)=x} e^{\frac{i}{\hbar} S(\hat{X})} f(X(p)) g(X(q)) d\hat{X}$$

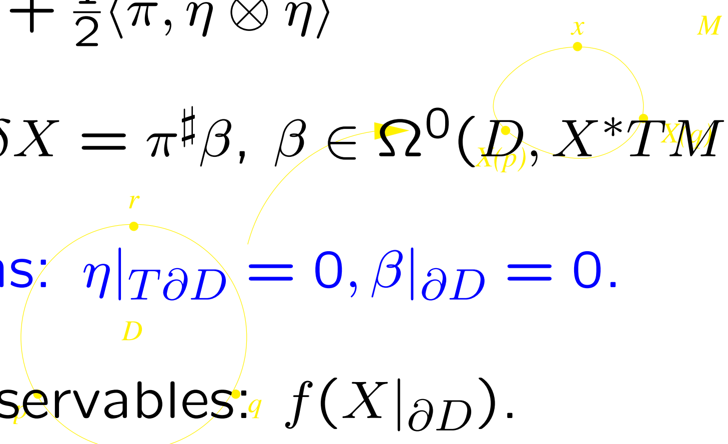
The integration is over bundle maps $\hat{X} = (X, \eta) \colon TD \to T^*M$ with base map $X \colon D \to M$ and fiber map $\eta \in \Omega^1(D, X^*T^*M)$.

$S(X, \eta) = \int_D \langle dX, \eta \rangle + \frac{1}{2} \langle \pi, \eta \otimes \eta \rangle$

Gauge symmetry: $\delta X = \pi^\sharp \beta$, $\beta \in \Omega^0(D, X^*TM)$

Boundary conditions: $\eta|_{T\partial D} = 0, \beta|_{\partial D} = 0.$

Gauge invariant observables: $f(X|_{\partial D})$.



Poisson sigma model-2

The Kontsevich formula arises as the Feynman perturbation series around the classical solution $X(u) = x$, $\eta(u) = 0$.

$\mathbb{R}$

Poisson sigma model-2

The Kontsevich formula arises as the Feynman perturbation series around the classical solution $X(u) = x$, $\eta(u) = 0$.

If we set $X^i(u) = x^i + \xi^i(u)$, the relavant propagator is

$$\langle \eta_i(z)\xi^j(w)\rangle = \delta_i^j G(z,w), \quad G(z,w) = \frac{1}{2\pi} d_z\phi(z,w).$$

It is (up to sign) the Green function of the de Rham differential on the upper half plane with boundary condition $\iota_{\partial/\partial x} G(x,w) = 0, x \in \mathbb{R}$:

$$d_z G(z,z_0) = -\delta_{z_0}(z)$$

Poisson sigma model-2

The Kontsevich formula arises as the Feynman perturbation series around the classical solution $X(u) = x$, $\eta(u) = 0$.

If we set $X^i(u) = x^i + \xi^i(u)$, the relavant propagator is

$$\langle \eta_i(z)\xi^j(w)\rangle = \delta_i^j G(z,w), \quad G(z,w) = \frac{1}{2\pi} d_z\phi(z,w).$$

It is (up to sign) the Green function of the de Rham differential on the upper half plane with boundary condition $\iota_{\partial/\partial x} G(x,w) = 0, x \in \mathbb{R}$:

$$d_z G(z,z_0) = -\delta_{z_0}(z)$$

The other component $d_w\phi(z,w)$ appearing in Kontsevich's formula comes from ghosts after BV gauge fixing.

Questions:

More general boundary conditions (D-branes) for the Poisson sigma model?

Quantisation of algebras of functions on singular manifolds?

Modules of the algebra with star-product?

Questions:

More general boundary conditions (D-branes) for the Poisson sigma model?

Admissible Dirichlet boundary conditions are on coisotropic submanifolds

Quantisation of algebras of functions on singular manifolds?

Modules of the algebra with star-product?

Questions:

More general boundary conditions (D-branes) for the Poisson sigma model?

Admissible Dirichlet boundary conditions are on coisotropic submanifolds

Quantisation of algebras of functions on singular manifolds?

Examples are given by Hamiltonian reduction

Modules of the algebra with star-product?

Questions:

More general boundary conditions (D-branes) for the Poisson sigma model?

Admissible Dirichlet boundary conditions are on coisotropic submanifolds

Quantisation of algebras of functions on singular manifolds?

Examples are given by Hamiltonian reduction

Modules of the algebra with star-product?

They are associated to coisotropic submanifolds

Coisotropic submanifolds-1

Let (M, π) be a Poisson manifold, $C \subset M$ a submanifold, I_C the ideal in $C^\infty(M)$ of functions vanishing on C.

C is called coisotropic if I_C is a Poisson subalgebra.

Coisotropic submanifolds-1

Let (M, π) be a Poisson manifold, $C \subset M$ a submanifold, I_C the ideal in $C^\infty(M)$ of functions vanishing on C.

C is called coisotropic if I_C is a Poisson subalgebra.

Hamiltonian vector fields $\{h, \cdot\}$, $h \in I_C$ are then tangent to C and form an integrable distribution. The corresponding foliation is called the characteristic foliation of C. The space of leaves is the reduced phase space $\underline{C}$. If it is smooth it inherits a Poisson structure.

Coisotropic submanifolds-2

Even if $\underline{C}$ is not smooth, the 'algebra of smooth funtions' $C^\infty(\underline{C})$ on it is defined:

$$C^\infty(\underline{C}) = N(I_C)/I_C, \quad N(I_C) = \{f \in C^\infty(M) \,|\, \{f, I_C\} \subset I_C\}.$$

It is a Poisson algebra.

Coisotropic submanifolds-2

Even if $\underline{C}$ is not smooth, the 'algebra of smooth funtions' $C^\infty(\underline{C})$ on it is defined:

$$C^\infty(\underline{C}) = N(I_C)/I_C, \quad N(I_C) = \{f \in C^\infty(M) \,|\, \{f, I_C\} \subset I_C\}.$$

It is a Poisson algebra.

Dirac's terminology: suppose C is defined by equations $h_i(x) = 0$, $i = 1, \ldots, r$. Then I_C is generated by the constraints h_i and C is coisotropic if h_i are first class constraints, namely

$$\{h_i, h_j\}(x) = \sum_{k=1}^{r} \lambda_{ij}^k(x) h_k(x),$$

for some functions λ_{ij}^k on M. The foliation is spanned by the hamiltonian flows of the constraints h_i.

Coisotropic submanifolds-3

Examples of coisotropic submanifolds

1. Lagrangian submanifolds of symplectic manifolds are coisotropic.

$\mathfrak{g}$ $\mathfrak{g}$

$\mathfrak{h}$ $\mathfrak{g}$

$\mathfrak{h}$

$\mathfrak{g}$ $\mathfrak{h}$ $^{\mathfrak{h}}$

$\mathfrak{g}$

Examples of coisotropic submanifolds

1. Lagrangian submanifolds of symplectic manifolds are coisotropic.

2. Graphs of Poisson maps $(M, \pi) \to (M', \pi')$ are coisotropic (in $(M \times M', (-\pi) \oplus \pi')$).

$\mathfrak{g}$ $\mathfrak{g}$

$\mathfrak{h}$ $\mathfrak{g}$

$\mathfrak{h}$

$\mathfrak{g}$ $\mathfrak{h}$ $^{\mathfrak{h}}$

$\mathfrak{g}$

Coisotropic submanifolds-3

Examples of coisotropic submanifolds

1. Lagrangian submanifolds of symplectic manifolds are coisotropic.

2. Graphs of Poisson maps $(M, \pi) \to (M', \pi')$ are coisotropic (in $(M \times M', (-\pi) \oplus \pi'))$.

3. If $\mathfrak{g}$ is a Lie algebra, $\mathfrak{g}^*$ is a Poisson manifold (such that the bracket of linear functions is the Lie bracket). If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra then $C = \mathfrak{h}^\perp$ is coisotropic and $C^\infty_{\text{polynomial}}(\underline{C}) = S(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}$.

$\mathfrak{g}$

Coisotropic submanifolds-3

Examples of coisotropic submanifolds

1. Lagrangian submanifolds of symplectic manifolds are coisotropic.

2. Graphs of Poisson maps $(M, \pi) \to (M', \pi')$ are coisotropic (in $(M \times M', (-\pi) \oplus \pi'))$.

3. If $\mathfrak{g}$ is a Lie algebra, $\mathfrak{g}^*$ is a Poisson manifold (such that the bracket of linear functions is the Lie bracket). If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra then $C = \mathfrak{h}^\perp$ is coisotropic and $C^\infty_{\text{polynomial}}(\underline{C}) = S(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}$.

4. If $\mu\colon M \to \mathfrak{g}^*$ is an equivariant moment map, then $C = \mu^{-1}(0)$ is coisotropic and $\underline{C} = \mu^{-1}(0)/G = M//G$ is the symplectic quotient.

Coisotropic submanifolds-4

5. $C = M$ is a coisotropic submanifold of itself. The characteristic foliation is trivial (every point is a leaf) and thus $\underline{C} = C$.

Coisotropic submanifolds-4

5. $C = M$ is a coisotropic submanifold of itself. The characteristic foliation is trivial (every point is a leaf) and thus $\underline{C} = C$.

6. A one-point set $C = \{x_0\}$ is coisotropic iff the Poisson bivector field vanishes there. Again $\underline{C} = C$ in this case.

Coisotropic submanifolds-4

5. $C = M$ is a coisotropic submanifold of itself. The characteristic foliation is trivial (every point is a leaf) and thus $\underline{C} = C$.

6. A one-point set $C = \{x_0\}$ is coisotropic iff the Poisson bivector field vanishes there. Again $\underline{C} = C$ in this case.

7. All submanifolds of codimension 1 are coisotropic. For example, constant energy hypersurfaces of a hamiltonian system are coisotropic and the foliation is given by the trajectories.

The algebra of cochains of N^*C

There is a differential graded commutative algebra $A = A(C, \pi)$ canonically associated with a coisotropic submanifold C, whose cohomology in degree 0 is $C^\infty(\underline{C})$. It is the Lie algebroid cochain complex of the conormal bundle N^*C.

The algebra of cochains of N^*C

There is a differential graded commutative algebra $A = A(C, \pi)$ canonically associated with a coisotropic submanifold C, whose cohomology in degree 0 is $C^\infty(\underline{C})$. It is the Lie algebroid cochain complex of the conormal bundle N^*C.

As an algebra $A = \Gamma(C, \wedge NC)$ is the algebra of sections of the exterior algebra of the normal bundle $NC = T_C M/TC$. It comes with a differential

$$C^\infty(C) \xrightarrow{\delta} \Gamma(C, NC) \xrightarrow{\delta} \Gamma(C, \wedge^2 NC) \xrightarrow{\delta} \cdots$$

such that for $f \in C^\infty(C)$,

$$\delta f = \pi^\sharp d\tilde{f}, \quad \tilde{f} \in C^\infty(M), \quad \tilde{f}|_C = f$$

Lie algebroid cohomology algebra

The cohomology of the differential graded algebra $(A = \Gamma(C, \wedge NC), \delta)$ is a graded commutative algebra

$$H^{\cdot}(N^*C) = \mathrm{Ker}(\delta)/\mathrm{Im}(\delta).$$

Lie algebroid cohomology algebra

The cohomology of the differential graded algebra $(A = \Gamma(C, \wedge NC), \delta)$ is a graded commutative algebra

$$H^{\cdot}(N^*C) = \mathrm{Ker}(\delta)/\mathrm{Im}(\delta).$$

Special cases:

$H^0(N^*C) = C^\infty(\underline{C})$

$H^1(N^*C)$ =infinitesimal deformations of the coisotropic embedding of C modulo Hamiltonian deformations.

P_∞-brackets

The Poisson bracket on M induces a Poisson bracket on $H^0(N^*C) = C^\infty(\underline{C})$. However this bracket does not come from a Poisson bracket on the algebra of cochains $A = \Gamma(C, \wedge NC)$.

P_∞-brackets

The Poisson bracket on M induces a Poisson bracket on $H^0(N^*C) = C^\infty(\underline{C})$. However this bracket does not come from a Poisson bracket on the algebra of cochains $A = \Gamma(C, \wedge NC)$.

Rather one has a 'homotopy Poisson algebra' or P_∞-algebra, namely a sequence of higher brackets $\lambda_n : \wedge^n A \to A$ of degree $2 - n$, which are derivations in each argument and obey the generalized Jacobi identity

$$\sum_{p+q=n} \frac{(-1)^{pq}}{p!q!} \lambda_{p+1}(\lambda_q(a_1, \ldots, a_q), a_{q+1}, \ldots, a_n) \pm \text{permutations} = 0.$$

P_∞-brackets-2

Here is how these brackets arise:

P_∞-brackets-2

Here is how these brackets arise: around a point of C choose coordinates $x^i, i = 1, \ldots, k$ on C and transversal coordinates x^μ. Define naively a skew-symmetric bracket on $C^\infty(C)$ by

$$\{f, g\} = \sum_{i,j=1}^{k} \pi^{ij} \partial_i f \partial_j g$$

P_∞-brackets-2

Here is how these brackets arise: around a point of C choose coordinates $x^i, i = 1, \ldots, k$ on C and transversal coordinates x^μ. Define naively a skew-symmetric bracket on $C^\infty(C)$ by

$$\{f, g\} = \sum_{i,j=1}^{k} \pi^{ij} \partial_i f \partial_j g$$

This bracket obeys the Leibniz rule but the Jacobi identity holds only up to homotopy

$$\{\{f, g\}, h\} + \{\delta f, g, h\} + \text{cycl} = 0.$$

for functions f, g, h on C with

$$\{\xi, g, h\} = \partial_\mu \pi^{ij} \xi^\mu \partial_i g \partial_j h, \quad \xi = \xi^\mu \partial_\mu \in \Gamma(C, \wedge^1 TC).$$

In general, higher brackets depend on the choice of embedding of NC into a tubular neighbourhood of C.

Quantisation programme

	Cochain complex	Cohomology
Semiclassical		
Quantum		

Quantisation programme

	Cochain complex	Cohomology
Semiclassical		$H^0(A,\delta) = C^\infty(\underline{C})$, a Poisson algebra
Quantum		

Quantisation programme

	Cochain complex	Cohomology
Semiclassical	$(A,(\lambda_n)_{n\geq 1})$, a P_∞-algebra, $\lambda_1 = \delta$	$H^0(A,\delta) = C^\infty(\underline{C})$, a Poisson algebra
Quantum		

Quantisation programme

	Cochain complex	Cohomology
Semiclassical	$(A,(\lambda_n)_{n\geq 1})$, a P_∞-algebra, $\lambda_1=\delta$	$H^0(A,\delta) = C^\infty(\underline{C})$, a Poisson algebra
Quantum	$(A[[\epsilon]],(\mu_n)_{n\geq 1})$, an A_∞-algebra with $\mu_1/\epsilon=\delta(\epsilon)=\delta+O(\epsilon)$	$H^0(A[[\epsilon]],\delta(\epsilon))$ an associative algebra

Quantisation-1

Theorem *Let $C \subset M$ be coisotropic. The P_∞-algebra $A = \Gamma(C, \wedge NC)$ can be quantised as an A_∞-algebra. Thus there are products $\mu_n : A^{\otimes n} \to A[[\epsilon]]$ of degree $2-n$, $n = 0, 1, 2, \ldots$ such that*

$$\sum \pm \mu_k(\mathrm{id}^{\otimes \ell} \otimes \mu_{n-k} \otimes \mathrm{id}^{\otimes \ell'}) = 0.$$

Moreover, μ_n is of degree $2-n$, $\mu_0 = O(\epsilon^2)$, $\mu_1 = \epsilon\delta + O(\epsilon^2)$, $\mu_2 =$ product in $A + O(\epsilon)$, $\mu_j = O(\epsilon)$, $j \geq 3$.

Quantisation-1

Theorem *Let $C \subset M$ be coisotropic. The P_∞-algebra $A = \Gamma(C, \wedge NC)$ can be quantised as an A_∞-algebra. Thus there are products $\mu_n : A^{\otimes n} \to A[[\epsilon]]$ of degree $2-n$, $n = 0, 1, 2, \ldots$ such that*

$$\sum \pm \mu_k(\mathrm{id}^{\otimes \ell} \otimes \mu_{n-k} \otimes \mathrm{id}^{\otimes \ell'}) = 0.$$

Moreover, μ_n is of degree $2-n$, $\mu_0 = O(\epsilon^2)$, $\mu_1 = \epsilon\delta + O(\epsilon^2)$, $\mu_2 =$ product in $A + O(\epsilon)$, $\mu_j = O(\epsilon)$, $j \geq 3$.

Here μ_0 is an 'anomaly' (i.e., a nuisance). The coisotropy condition implies that $\mu_0 = 0$ at the classical ($\epsilon = 0$) level, but quantum corrections could arise.

Quantisation-1

Theorem *Let $C \subset M$ be coisotropic. The P_∞-algebra $A = \Gamma(C, \wedge NC)$ can be quantised as an A_∞-algebra. Thus there are products $\mu_n : A^{\otimes n} \to A[[\epsilon]]$ of degree $2-n$, $n = 0, 1, 2, \ldots$ such that*

$$\sum \pm \mu_k(\mathrm{id}^{\otimes \ell} \otimes \mu_{n-k} \otimes \mathrm{id}^{\otimes \ell'}) = 0.$$

Moreover, μ_n is of degree $2-n$, $\mu_0 = O(\epsilon^2)$, $\mu_1 = \epsilon\delta + O(\epsilon^2)$, $\mu_2 =$ product in $A + O(\epsilon)$, $\mu_j = O(\epsilon)$, $j \geq 3$.

Here μ_0 is an 'anomaly' (i.e., a nuisance). The coisotropy condition implies that $\mu_0 = 0$ at the classical ($\epsilon = 0$) level, but quantum corrections could arise.

It is an open problem to find an example where the anomaly cannot be removed

Quantisation-2

If $\mu_0 = 0$ then the first few equations are $(a, b, c \in A = \Gamma(C, \wedge NC))$.

$\mu_1 \circ \mu_1(a) = 0,$

Quantisation-2

If $\mu_0 = 0$ then the first few equations are $(a, b, c \in A = \Gamma(C, \wedge NC))$.

$\mu_1 \circ \mu_1(a) = 0$,

$\mu_1 \circ \mu_2(a, b) = \mu_2(\mu_1(a), b) + (-1)^{|a|}\mu_2(a, \mu_1(b))$,

Quantisation-2

If $\mu_0 = 0$ then the first few equations are $(a, b, c \in A = \Gamma(C, \wedge NC))$.

$\mu_1 \circ \mu_1(a) = 0$,

$\mu_1 \circ \mu_2(a, b) = \mu_2(\mu_1(a), b) + (-1)^{|a|}\mu_2(a, \mu_1(b))$,

$\mu_2(\mu_2(a, b), c) - \mu_2(a, \mu_2(b, c)) = \mu_1 \circ \mu_3(a, b, c) + \mu_3(\mu_1(a), b, c) +$
$(-1)^{|a|}\mu_3(a, \mu_1(b), c) + (-1)^{|a|+|b|}\mu_3(a, b, \mu_1(c))$

Quantisation-2

If $\mu_0 = 0$ then the first few equations are $(a, b, c \in A = \Gamma(C, \wedge NC))$.

$\mu_1 \circ \mu_1(a) = 0,$

$\mu_1 \circ \mu_2(a, b) = \mu_2(\mu_1(a), b) + (-1)^{|a|}\mu_2(a, \mu_1(b)),$

$\mu_2(\mu_2(a, b), c) - \mu_2(a, \mu_2(b, c)) = \mu_1 \circ \mu_3(a, b, c) + \mu_3(\mu_1(a), b, c) + (-1)^{|a|}\mu_3(a, \mu_1(b), c) + (-1)^{|a|+|b|}\mu_3(a, b, \mu_1(c))$

Thus μ_1/ϵ is a differential deforming δ, μ_2 is a chain map $A \otimes A \to A$ and is associative up to the homotopy μ_3, so it induces an associative product on the cohomology of $(A[[\epsilon]], \mu_1/\epsilon)$.

Quantisation-3

Further properties:

If $H^2(N^*C) = 0$ the anomaly μ_0 can be removed recursively by a 'shift'. Geometrically this means that the anomaly disappears if we deform appropriately the coisotropic submanifold in M.

$\mathfrak{h}$ $\mathfrak{g}$ $\mathfrak{h}$ $\mathfrak{g}$

Quantisation-3

Further properties:

If $H^2(N^*C) = 0$ the anomaly μ_0 can be removed recursively by a 'shift'. Geometrically this means that the anomaly disappears if we deform appropriately the coisotropic submanifold in M.

Even if $H^2(N^*C) \neq 0$ it is possible that $\mu_0 = 0$. For example $\mu_0 = 0$ for $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ for any Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of a finite dimensional Lie algebra.

Quantisation-3

Further properties:

If $H^2(N^*C) = 0$ the anomaly μ_0 can be removed recursively by a 'shift'. Geometrically this means that the anomaly disappears if we deform appropriately the coisotropic submanifold in M.

Even if $H^2(N^*C) \neq 0$ it is possible that $\mu_0 = 0$. For example $\mu_0 = 0$ for $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ for any Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of a finite dimensional Lie algebra.

If $\mu_0 = 0$, then $H^0(A[[\epsilon]], \delta(\epsilon))$ is an associative algebra with product induced by μ_2. It is a flat deformation of $C^\infty(\underline{C})$ if $H^1_\pi(N^*C) = 0$.

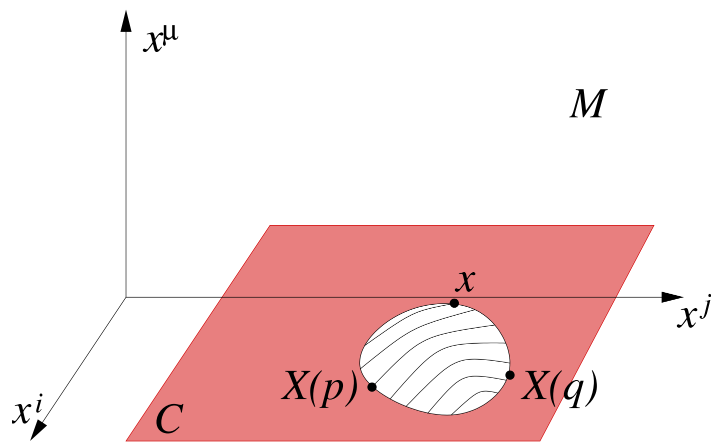
Poisson sigma model description of the quantisation

Local description: Let $x^1, \ldots, x^d$ be local coordinates around a point of the k-dimensional submanifold C, such that C is given by the equations

$$x^\mu = 0, \qquad \mu = k+1, \ldots, d.$$

Dirichlet boundary conditions ($1 \leq i \leq k$, $k < \mu \leq d$) $X^\mu|_{\partial D} = 0$, $\eta_i|_{T\partial D} = \beta_i|_{\partial D} = 0$.

The original boundary conditions correspond to $C = M$, a 'space-filling brane'.



Observables

Local observables are associated with sections of $\wedge NC$

$$a = a^{\mu_1\cdots\mu_n}(x)\frac{\partial}{\partial x^{\mu_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_n}}\in\Gamma(C,\wedge^n NC).$$

$$O_a(u) = a^{\mu_1\cdots\mu_n}(X(u))\beta_{\mu_1}(u)\cdots\beta_{\mu_n}(u)+\cdots\,,\qquad u\in\partial D.$$

A_∞-products are expectation values of products of such observables

$$\mu_m(a_1,\ldots,a_m) = \int_{0<u_2<\cdots<u_{m-1}<1}\langle O_{a_1}(0)O_{a_2}(u_2)\cdots O_{a_m}(1)\rangle,$$

in a Feynman expansion around $X^i(u)=x^i$, $\beta_\mu(u)=\partial/\partial x^\mu$.

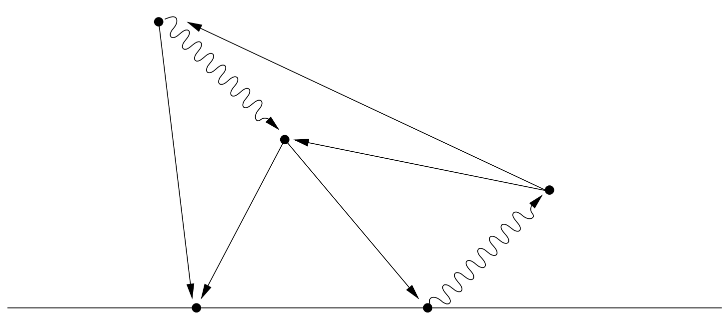
To give a mathematical proof of the theorem one defines the A_∞-products by the Feynman expansion and checks the associativity graph by graph.

Feynman rules

The terms of the products μ_m are labeled by graphs with two types of edges

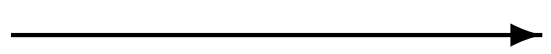
$$z \longrightarrow w \qquad d\phi(z,w)\frac{\partial}{\partial x^i} \qquad\qquad z \rightsquigarrow w \qquad d\phi(w,z)\frac{\partial}{\partial x^\mu}$$

Vertices of the first kind have two outgoing edges and correspond to transversal/parallel components of the Poisson bivector field.

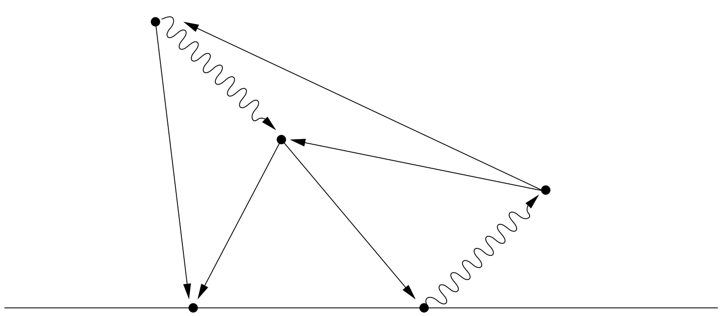


Feynman rules

The terms of the products μ_m are labeled by graphs with two types of edges

z  w $\qquad d\phi(z,w)\frac{\partial}{\partial x^i}$ $\qquad\qquad z$  w $\qquad d\phi(w,z)\frac{\partial}{\partial x^\mu}$

Vertices of the first kind have two outgoing edges and correspond to transversal/parallel components of the Poisson bivector field.



Example: A graph contributing to $\mu_2(a,b)$ with $a \in C^\infty(C)$, $b = b^\mu\partial_\mu \in \Gamma(C, \wedge^1 NC)$ corresponding to the bidifferential operator $\partial_p\pi^{i\mu}\partial_q\pi^{jk}\partial_\nu\pi^{pq}\partial_i\partial_j a\partial_k b^\nu$.

Kontsevich's formality theorem-1

Two differential graded Lie algebras:

Multivector fields

$$\mathcal{T}(M) = \oplus_{i \geq -1} \Gamma(M, \wedge^{i+1} TM),$$

with Nijenhuis–Schouten bracket (= Lie bracket on vector fields, extended by the Leibniz rule) and zero differential.

Multidifferential operators:

$$\mathcal{D}(M) = \oplus_{i \geq -1} \mathrm{Hom}_{\mathrm{diff}}(A^{\otimes i+1}, A), \quad A = C^{\infty}(M).$$

with Gerstenhaber bracket and Hochschild differential.

Kontsevich's formality theorem-1

Two differential graded Lie algebras:

Multivector fields

$$\mathcal{T}(M) = \oplus_{i \geq -1} \Gamma(M, \wedge^{i+1} TM),$$

with Nijenhuis–Schouten bracket (= Lie bracket on vector fields, extended by the Leibniz rule) and zero differential.

Multidifferential operators:

$$\mathcal{D}(M) = \oplus_{i \geq -1} \mathrm{Hom}_{\mathrm{diff}}(A^{\otimes i+1}, A), \quad A = C^{\infty}(M).$$

with Gerstenhaber bracket and Hochschild differential.

Kontsevich's formality theorem-2

It is an old result (HKR) that the cohomology of $\mathcal{D}(M)$ is isomorphic as a graded Lie algebra to $\mathcal{T}(M)$ but the HKR isomorphism is not induced by an isomorphism at the level of dgla's.

$\rightsquigarrow$

Kontsevich's formality theorem-2

It is an old result (HKR) that the cohomology of $\mathcal{D}(M)$ is isomorphic as a graded Lie algebra to $\mathcal{T}(M)$ but the HKR isomorphism is not induced by an isomorphism at the level of dgla's.

Theorem (Kontsevich)

There is an L_∞-quasiisomorphism $U\colon \mathcal{T}(M) \rightsquigarrow \mathcal{D}(M)$ whose first order component U_1 is the HKR quasiisomorphism.

Thus U is given by a sequence of 'Taylor components' U_n : $\wedge^n\mathcal{T}(M) \to \mathcal{D}(M)[1-n]$ obeying a sequence of quadratic relations.

Maurer Cartan equations

Let $\mathfrak{g}$ be a differential graded Lie algebra. The equation

$$da + \frac{1}{2}[a, a] = 0$$

for $a \in \mathfrak{g}^1$ is called the Maurer–Cartan equation. If $\mathfrak{g}^0$ is nilpotent the group $G = \exp(\mathfrak{g}^0)$ acts on the space MC of solutions of the Maurer–Cartan equations by gauge transformations. (If $\mathfrak{g}^0$ is not nilpotent, replace $\mathfrak{g}$ by $\epsilon\mathfrak{g}$ and work over formal power series in ϵ.) L_∞-quasiisomorphisms induce isomorphisms between moduli spaces MC/G.

Maurer Cartan equations

Let $\mathfrak{g}$ be a differential graded Lie algebra. The equation

$$da + \frac{1}{2}[a, a] = 0$$

for $a \in \mathfrak{g}^1$ is called the Maurer–Cartan equation. If $\mathfrak{g}^0$ is nilpotent the group $G = \exp(\mathfrak{g}^0)$ acts on the space MC of solutions of the Maurer–Cartan equations by gauge transformations. (If $\mathfrak{g}^0$ is not nilpotent, replace $\mathfrak{g}$ by $\epsilon\mathfrak{g}$ and work over formal power series in ϵ.) L_∞-quasiisomorphisms induce isomorphisms between moduli spaces MC/G.

$\pi \in MC(\mathcal{T}(M)) \subset \Gamma(M, \wedge^2 TM) \Longleftrightarrow \pi$ is a Poisson bivector field.

Maurer Cartan equations

Let $\mathfrak{g}$ be a differential graded Lie algebra. The equation

$$da + \frac{1}{2}[a, a] = 0$$

for $a \in \mathfrak{g}^1$ is called the Maurer–Cartan equation. If $\mathfrak{g}^0$ is nilpotent the group $G = \exp(\mathfrak{g}^0)$ acts on the space MC of solutions of the Maurer–Cartan equations by gauge transformations. (If $\mathfrak{g}^0$ is not nilpotent, replace $\mathfrak{g}$ by $\epsilon\mathfrak{g}$ and work over formal power series in ϵ.) L_∞-quasiisomorphisms induce isomorphisms between moduli spaces MC/G.

$\pi \in MC(\mathcal{T}(M)) \subset \Gamma(M, \wedge^2 TM) \Longleftrightarrow \pi$ is a Poisson bivector field.

$P \in MC(\mathcal{D}(M)) \subset \mathrm{Hom}_{\mathrm{diff}}(C^\infty \otimes C^\infty(M), C^\infty(M)) \Longleftrightarrow f \cdot g + P(f, g)$ is an associative product.

The relative formality theorem-1

The relative case: $C \subset M$ a submanifold.

Relative multivector fields (multivector fields in a formal neighbourhood of C)

$$\mathcal{T}(M, C) = \varprojlim \mathcal{T}(M)/I_C^n \mathcal{T}(M)$$

Relative multidifferential operators

$$\mathcal{D}(M, C) = \oplus_j \mathcal{D}^n(M, C),$$

$$\mathcal{D}^j(M, C) = \prod_{p+q=j+1} \mathrm{Hom}^p_{\mathrm{diff}}(A^{\otimes q}, A).$$

The relative formality theorem-2

Theorem

There is an L_∞-quasiisomorphism $U\colon \mathcal{T}(M,C) \rightsquigarrow \mathcal{D}(M,C)$. Maurer–Cartan elements in $\mathcal{T}(M,C)$ are P_∞-structures on C, Maurer–Cartan elements in $\mathcal{D}(M,C)$ are A_∞-deformations of the product in A.

The relative formality theorem-2

Theorem

There is an L_∞*-quasiisomorphism* $U : \mathcal{T}(M, C) \rightsquigarrow \mathcal{D}(M, C)$*. Maurer–Cartan elements in* $\mathcal{T}(M, C)$ *are* P_∞*-structures on* C*, Maurer–Cartan elements in* $\mathcal{D}(M, C)$ *are* A_∞*-deformations of the product in* A*.*

Two possible proofs: direct or reduce by 'Fourier transform' $\mathcal{T}(M, C) \simeq \mathcal{T}(N^*[1]C)$ to Kontsevich's theorem on the supermanifold $N^*[1]C$. In any case the components U_n of the local formula L_∞-quasiisomorphism are given by the same type of Feynman graphs as above but with more general vertices

Several D-branes: bimodules-1

Suppose $C_1, C_2 \subset M$ are anomaly-free (i.e., such that $\mu_0 = 0$) coisotropic submanifolds, A_1, A_2 the corresponding A_∞-algebras. Fix a point in the intersection $C_1 \cap C_2$ where the intersection is clean, i.e., locally looking like the intersection of subspaces of a vector space.

Several D-branes: bimodules-1

Suppose $C_1, C_2 \subset M$ are anomaly-free (i.e., such that $\mu_0 = 0$) coisotropic submanifolds, A_1, A_2 the corresponding A_∞-algebras. Fix a point in the intersection $C_1 \cap C_2$ where the intersection is clean, i.e., locally looking like the intersection of subspaces of a vector space.

Then the perturbative expansion of the Poisson sigma model with Dirichlet boundary conditions C_1 on one half of the circle ∂D and C_2 on the other half, gives structure maps

$$A_1^{\otimes p} \otimes M_{12} \otimes A_2^{\otimes q} \to M_{12}[1-p-q],$$

$M_{12} = \Gamma(C_1 \cap C_2, \wedge N_{12})[[\epsilon]]$. These maps obey A_∞-type associativity relations: M_{12} is an A_∞-bimodule over A_1 and A_2.

Several D-branes: bimodules-2

In particular M_{12} has a differential and there is a left action of A_1 up to homotopy $A_1 \otimes M_{12} \to M_{12}$, a right action of A_2 and the actions commute up to homotopy.

It follows that the cohomology $H(M_{12})$ is a $H(A_1) - H(A_2)$ bimodule.

Several D-branes: bimodules-2

In particular M_{12} has a differential and there is a left action of A_1 up to homotopy $A_1 \otimes M_{12} \to M_{12}$, a right action of A_2 and the actions commute up to homotopy.

It follows that the cohomology $H(M_{12})$ is a $H(A_1) - H(A_2)$ bimodule. Important special case: $C_1 = M$, $C_2 = C$. Then A_1 is an associative algebra (the Kontsevich algebra) and $H(M_{12})$ is (in particular) a module over A_1.

Several D-branes: bimodules-3

The construction may be extended to several coisotropic submanifolds by dividing ∂D into arcs and imposing different boundary conditions on different arcs.

Several D-branes: bimodules-3

The construction may be extended to several coisotropic submanifolds by dividing ∂D into arcs and imposing different boundary conditions on different arcs.

At the cohomology level we have the following result:

let x_0 be a point of clean intersection of anomaly-free coisotropic $C_1, \ldots, C_n \subset M$. Then the Poisson sigma model gives:

$H(A_i) - H(A_j)$-bimodules $H(M_{ij})$.

Homomorphisms of bimodules $\phi_{ijk} \colon H(M_{ij}) \otimes_{H(A_j)} H(M_{jk}) \to H(M_{ik})$

Associativity relations $\phi_{ikl} \circ \phi_{ijk} = \phi_{ijl} \circ \phi_{jkl}$