3-Colouring of links

Given any link diagram, colour all arcs red, blue, green, so that at each crossing, either 1 or 3 colours are used.

e.g. 

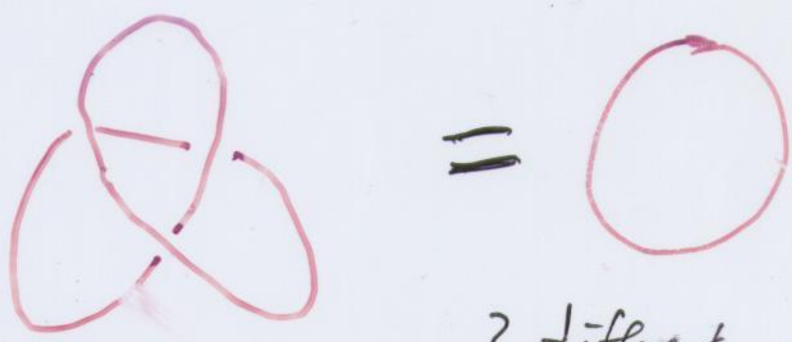
good good bad

Fact The number of 3-colourings is a link-invariant.

e.g. trefoil

3x3 different 3-colourings

e.g. trivial knot



3 different 3-colourings

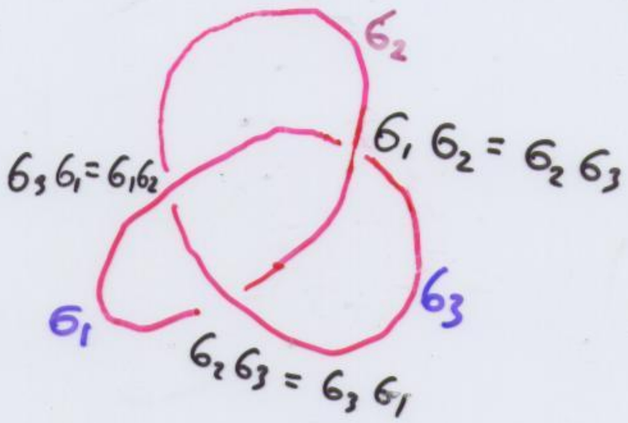
$\therefore$ trefoil $\neq$ trivial

Fact Counting 3-colourings is essentially counting group homomorphisms

$$\pi_1(\mathbb{R}^3 - \text{link}) \to \mathrm{Sym}_3$$

Reason Wirtinger presentation of $\pi_1(\mathbb{R}^3 - \text{link})$

e.g.



$$\pi_1(\mathbb{R}^3 - \text{trefoil}) = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_3\sigma_1 = \sigma_1\sigma_2,\ \sigma_2\sigma_3 = \sigma_3\sigma_1,\ \sigma_1\sigma_2 = \sigma_2\sigma_3 \rangle$$
$$= \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$
$$= B_3$$

braid group on 3 strands

red = $(12) \in \mathrm{Sym}_3$

green = $(23) \in \mathrm{Sym}_3$

blue = $(31) \in \mathrm{Sym}_3$

Definition A **modular form** f of weight k and multiplier μ for $SL_2(\mathbb{Z})$ is a holomorphic function $f: \mathbb{H} \to \mathbb{C}$ $\quad (\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\})$,

$k \in \mathbb{Q}$, $\quad \mu : SL_2(\mathbb{Z}) \to \mathbb{C}^\times$,

"f holomorphic at cusps $\tau \in \mathbb{Q} \cup \infty$",

and $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$$\boxed{f\left(\frac{a\tau+b}{c\tau+d}\right) = \mu\begin{pmatrix} a & b \\ c & d \end{pmatrix} (c\tau+d)^k f(\tau)}$$

Example

"Eisenstein series"

$$E_k(\tau) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} (m\tau+n)^{-k}$$

for any even $k \geq 4$ is a modular form of weight k and multiplier $\mu \equiv 1$.

Whenever $k \in \mathbb{Z}$, μ is a representation of $SL_2(\mathbb{Z})$.

When $k \notin \mathbb{Z}$, μ is only a projective representation of $SL_2(\mathbb{Z})$.

Example $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$

"Dedekind eta"

$$\left(q = e^{2\pi i \tau} \right)$$

is a modular form for $SL_2(\mathbb{Z})$ of weight $k=\frac{1}{2}$ and multiplier

"Dedekind sum"

$$\mu \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \exp\left[\pi i \left(\frac{a+d}{12c} - \frac{1}{c} - \sum_{j=1}^{c-1} \frac{j}{c} \left(\frac{dj}{c} - \left\lfloor \frac{dj}{c} \right\rfloor - \frac{1}{2} \right) \right) \right]$$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $c > 0$.

"homogeneous space"

Modern philosophy Realise $\mathbb{H} = SL_2(\mathbb{R}) / SO_2(\mathbb{R})$

In particular, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = y^{-\frac{1}{2}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where $x + iy = \frac{ai+b}{ci+d}$, $e^{i\theta} = \frac{d-ic}{|d-ic|}$.

Dumb idea Lift f from $\mathbb{H}$ to $SL_2(\mathbb{R})$:

$$\phi_f \begin{pmatrix} a & b \\ c & d \end{pmatrix} = f\left(\frac{ai+b}{ci+d}\right)$$

Result: ϕ_f is invariant with respect to $SO_2(\mathbb{R})$

ϕ_f is covariant with respect to $SL_2(\mathbb{Z})$

Good idea Lift f with twist :

$$\phi_f \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ci+d)^{-k} f\left(\frac{ai+b}{ci+d}\right)$$

Result: ϕ_f is covariant with respect to $SO_2(\mathbb{R})$

ϕ_f is invariant with respect to $SL_2(\mathbb{Z})$

The Point Everything is much more conceptual; arbitrariness removed.

e.g. weight K = Dynkin label of $SO_2(\mathbb{R})$-rep.

e.g. "$SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ by $\frac{a\tau+b}{c\tau+d}$"
becomes
"$SL_2(\mathbb{Z})$ acts on $SL_2(\mathbb{R})$ by matrix mult."

A modular form is thus related to a unitary rep of $SL_2(\mathbb{R})$ on $L^2\left(SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})\right)$;

ϕ_f is highest-weight vector for $SO_2(\mathbb{R})$.

Generalisation to other Lie groups is clear.

Anything similar has such a realisation!

Jacobi forms, Siegel forms, ...

Tempting to speculate: Any relation to Quantum field theory?

Wigner:
- particle in 3+1 dimensions corresponds to a unitary rep of $\widetilde{SO_{3,1}}(\mathbb{R}) \cong SL_2(\mathbb{C})$.
- particle in 2+1 dimensions corresponds to unitary rep of $\widetilde{SO_{2,1}}(\mathbb{R}) \simeq SL_2(\mathbb{R})$.

Lax-Phillips Scattering:
- poles of scattering matrix corresponds to zeros of Riemann zeta
- get new proof of Selberg trace formula for SL_2

Moral: Think of a modular form as living on $X = SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$.

Topologically, $SL_2(\mathbb{R})$ is interior of solid torus.

Topologically, X is $\mathbb{R}^3 -$ trefoil !

Question: Can we see the trefoil in e.g. RCFT ?

Now, Universal cover $\tilde{Y} \to Y$ is more fundamental than Y.

If Y isn't simply-connected, then monodromy = "global anomaly" is given by $\pi_1(Y)$ = distance of Y from $\tilde{Y}$.

Example: Spinors are projective reps of $SO_{3,1}$, but true reps of $SL_2(\mathbb{C})$.

$\pi_1 = 2$

Example: Cauchy integral formula = residue $\times$ winding number

$\pi_1 = \mathbb{Z}$

Recall: $\pi_1(X) = B_3$

$$\tilde{X} = \widetilde{SL_2(\mathbb{R})} \xrightarrow{\mathbb{Z}} SL_2(\mathbb{R}) \xrightarrow{SL_2(\mathbb{Z})} SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{B_3}$$

B_3 is central extension of $SL_2(\mathbb{Z})$ by $\mathbb{Z}$:

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

"reduced Burau rep at $w=-1$"

$$PSL_2(\mathbb{Z}) = B_3/\langle z \rangle, \quad SL_2(\mathbb{Z}) = B_3/\langle z^2 \rangle$$

($z = (\sigma_1\sigma_2)^3$ generates centre)

So: Most fundamental "symmetry" of a modular form is B_3, not $SL_2(\mathbb{Z})$.

Rather obvious in hindsight: μ is projective rep of $SL_2(\mathbb{Z})$, so centrally extend. It will be true rep of B_3.

($k \in \frac{1}{2}\mathbb{Z}$: metaplectic group suffices)

Example: Dedekind eta $\eta(\tau)$.

Action on τ by B_3 done via Burau.

$SL_2(\mathbb{Z})$: $\mu\begin{pmatrix} a & b \\ c & d \end{pmatrix} =$ Dedekind sum

B_3: $\mu(\beta) = \exp\left[\frac{2\pi i}{24} \deg \beta\right]$

Example:

$$\left.\begin{aligned}
\theta_1(\tau, z) &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau (n+\frac{1}{2})^2 + 2\pi i (z+\frac{1}{2})(n+\frac{1}{2})} \\
\theta_2(\tau, z) &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau (n+\frac{1}{2})^2 + 2\pi i z (n+\frac{1}{2})} \\
\theta_3(\tau, z) &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2 + 2\pi i z n} \\
\theta_4(\tau, z) &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2 + 2\pi i (z+\frac{1}{2}) n}
\end{aligned}\right\}$$

vector-valued Jacobi form for $k = \frac{1}{2}$

μ is 4-dim projective rep of $SL_2(\mathbb{Z})$.

$= \exp\left[\frac{2\pi i}{8} \deg \beta\right] \otimes$ true rep of $SL_2(\mathbb{Z})$.

Moral #1:

Especially when there are conceptual obscurities, lift "modular form" from $\mathbb{H}$ to $SL_2(\mathbb{R})$.

Example String amplitudes in plane-wave background with RR flux (Green, Gaberdiel, Takayanagi, ...)

$$\eta^{(m \mp \frac{\tau}{i})}\left(\frac{-1}{\tau}\right) = \eta^{(m)}(\tau)$$

Example 1-point functions on torus in string theory = characters of VOAs

(maybe helpful in Norton's generalised Moonshine, or understanding better the modular behaviour of non rational CFTs (logarithmic?))

Moral #2 Especially when weight K is not an integer, its B_3 not $SL_2(\mathbb{Z})$ which acts.

Hint: Whenever you see a "Maslov index" in context of modularity, then the math is urging you to do this.

Example Turaev's book in modular tensor categories.

Example Monstrous moonshine

Best guess for what is special about Monster is 6-transposition property

Relation of it to modularity involves B_3 action on $M \times M$.

Eary Guess The missing idea in Moonshine is B_3

Example Conformal field theory

1-pt function on torus:

$$\chi_M(\tau, v, z) = e^{-2\pi i \tau \frac{c}{24}} \mathrm{Tr}_M\, Y(v, z)$$

$$= z^{h_M} \mathrm{Tr}_M\, O(v)\, e^{2\pi i \tau \left(L_0 - \frac{c}{24}\right)}$$

$$G_1 \cdot \chi_M(\tau, v, z) = \chi_M(\tau+1, v, z)\, z^{h_M/12}$$

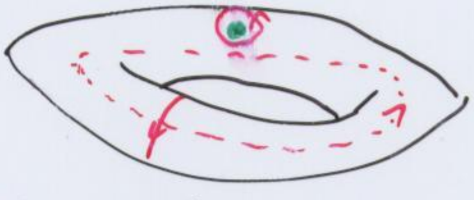
$$G_2 \cdot \chi_M(\tau, v, z) = \chi_M\left(\frac{\tau}{\tau-1}, \frac{v}{\tau-1}, z\right) z^{\frac{h_M}{12}}$$

In CFT we really have enhanced surfaces, e.g. torus with marked point and choice of local coordinate at marked point.

Needed for sewing surfaces together.

Virasoro acts on enhanced moduli space.

B_3 is enhanced mapping class group $\widehat{\Gamma}_{1,1}$



generated by Dehn twists.

WARNING: This B_3 action unrelated to B_n monodromy of e.g. quantum groups

Example $\mathcal{N} = 4, 2, 1$ SUSY Yang-Mills instantons.

Correlation function calculations can involve modular forms (Nick Dorey, ...).

$SL_2(\mathbb{Z})$-action here is S-duality.

Guess: S-duality, at least in these special cases, can be naturally extended to $\mathbb{B}_3$ from $SL_2(\mathbb{Z})$. (fractional instantons?)