Fact B_n acts faithfully on free group F_n

$$\sigma_i \cdot x_j = \begin{cases} x_j & j \neq i, i+1 \\ x_{i+1} & j = i \\ x_{i+1}^{-1} x_i x_{i+1} & j = i+1 \end{cases}$$

Reason: $B_n = \pi_1(C_n / \mathrm{Sym}_n)$

$F_n = \pi_1(\mathbb{C} - n \text{ points})$

$C_{n+1} \longrightarrow C_n$ has fibre $\mathbb{C} - n$ points

so π_1 of base acts on π_1 of fibre

Claim Same thing happens when B_n is replaced with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and F_n is replaced with its profinite completion

Expectation This has something to do with rational conformal field theory

Field $\mathbb{K}$ A subset of $\mathbb{C}$ closed under addition, subtraction, multiplication, division

$\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ = Galois group of $\mathbb{K}$

= all symmetries σ of $\mathbb{K}$:

$$\sigma(xy) = \sigma(x)\,\sigma(y),$$
$$\sigma(x+y) = \sigma(x) + \sigma(y)$$

Example $\mathbb{Q}[i] = \mathbb{Q} + \mathbb{Q}i$ is a field

$$\mathrm{Gal}(\mathbb{Q}[i]/\mathbb{Q}) = \{\text{identity, complex conjugation}\}$$

Example cyclotomic field $\mathbb{Q}[e^{2\pi i/n}]$

= all $\mathbb{Q}$-polynomials evaluated at $e^{2\pi i/n}$

$$\mathrm{Gal}(\mathbb{Q}[e^{2\pi i/n}]/\mathbb{Q}) = \mathbb{Z}_n^\times$$
$$= \{1 \le \ell \le n \mid \gcd(\ell, n) = 1\}$$

$$\sigma_\ell \cdot \mathrm{poly}(e^{2\pi i/n}) = \mathrm{poly}(e^{2\pi i \ell/n}).$$

Example: $\overline{\mathbb{Q}}$ = all algebraic numbers in $\mathbb{C}$

$$\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \text{Absolute Galois group of } \mathbb{Q}.$$

RCFT has a Galois symmetry!

$\chi_M(\tau) = \mathrm{Tr}_M\, q^{L_0 - c/24}$ = Character of sector M

Get representation ρ of $SL_2(\mathbb{Z})$:

e.g. $$\chi_M(\tau+1) = \sum_N T_{MN}\, \chi_N(\tau)$$

$$\chi_M\left(\frac{-1}{\tau}\right) = \sum_N S_{MN}\, \chi_N(\tau)$$

$\rho\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = T$, $\rho\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = S$

Entries of all matrices $\rho(A)$, $A \in SL_2(\mathbb{Z})$, lie in cyclotomic field $\mathbb{Q}[e^{2\pi i/n}]$, where $T^n = I$.

So $\mathrm{Gal}(\mathbb{Q}[e^{2\pi i/n}]/\mathbb{Q}) = \mathbb{Z}_n^\times$.

For any $\ell \in \mathbb{Z}_n^\times$, there exists a permutation $M \mapsto M^{\sigma_\ell}$ of sectors = chiral primaries, and a choice of signs $\varepsilon_\ell = \pm 1$, such that

$$\sigma_\ell(S_{MN}) = \varepsilon_\ell(M)\, S_{M^{\sigma_\ell}, N} = \varepsilon_\ell(N)\, S_{M, N^{\sigma_\ell}}$$

$$\sigma_\ell(T_{MN}) = (T_{MN})^\ell$$

$$T_{M^{\sigma_\ell}, N^{\sigma_\ell}} = (T_{MN})^{\ell^2}$$

For any $A \in \Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}$

$\chi_M\left(\frac{a\tau+b}{c\tau+d}\right) = \chi_M(\tau)$ for all M.

Question How does this Galois action extend to rest of Moore-Seiberg (chiral) data?

Problem For S, T, there was a canonical basis (namely the characters). In general, the spaces of conformal blocks will have dim >1.

This Galois action is a discrete symmetry of the RCFT. If $\mathcal{H}$ = state-space of full CFT, write $\mathcal{H} = \bigoplus_{M,N} \mathcal{H}_{MN}\, M \otimes N$

Then $\mathcal{H}_{M^{G_\ell}, N^{G_\ell}} = \mathcal{H}_{M,N}$ for all G_ℓ, M, N.

profinite completion: $\hat{G} = \varprojlim G/N$, where limit is taken over all normal subgroups $N \lhd G$, with finite index in G.

It's a completion in the sense that $\mathbb{R}$ is a completion of $\mathbb{Q}$, by Cauchy sequences.

Characteristic example $\hat{\mathbb{Z}} = \prod_{p \text{ prime}} \hat{\mathbb{Z}}_p$

where $\hat{\mathbb{Z}}_p$ = p-adic integers

purpose #1 of profinite completion:

It integrates a sequence of structures into one collosal one.

e.g. $\hat{\mathbb{Z}}$ allows you to do simultaneously arithmetic in every $\mathbb{Z}/n\mathbb{Z}$.

purpose #2 of profinite completion:

It fills in holes ~~of~~ G.

e.g. most rationals don't have rational square roots, but half have real square roots.

Similarly for $\hat{\mathbb{Z}}_p$: e.g. $\sqrt{2} = 3 + 1\cdot 7 + 2 \cdot 7^2 + \cdots \in \hat{\mathbb{Z}}_7$

Abelianisation of $\Gamma_{\mathbb{Q}} = \Gamma_{\mathbb{Q}} / [\Gamma_{\mathbb{Q}} \Gamma_{\mathbb{Q}}]$
$\cong \hat{\mathbb{Z}}^\times$

$\bigcup_n \mathbb{Q}[e^{2\pi i/n}]$ is a field. Its Galois group is $\Gamma_{\mathbb{Q}} / [\Gamma_{\mathbb{Q}} \Gamma_{\mathbb{Q}}] = \hat{\mathbb{Z}}^\times$.

Cyclotomic character:

$$\chi^{cyclo} : \Gamma_{\mathbb{Q}} \to \hat{\mathbb{Z}}^\times$$

$\sigma \in \Gamma_{\mathbb{Q}}$ acts on any root ζ of 1 by
$\sigma \zeta = \zeta^{\chi^{cyclo}(\sigma)}$

Question Where does Galois action in RCFT come from?

Universal formula for Galois in RCFT:

$$\sigma.T = T^{\chi^{cyclo}(\sigma)}$$

$$\sigma.S = T^{\chi^{cyclo}(\sigma)} S\, T^{\chi^{cyclo}(\sigma^{-1})} S\, T^{\chi^{cyclo}(\sigma)} S^2$$
$$= \begin{pmatrix} \chi(\sigma) & 0 \\ 0 & \chi(\sigma^{-1}) \end{pmatrix} S$$

defines action of $\Gamma_{\mathbb{Q}}$ on $\widehat{SL_2(\mathbb{Z})}$

Looks bad, but very natural:

for any mod function $f(\tau) = \sum_{j=0}^{\infty} a_j q^j$ for $\Gamma(n)$ with coefficients in $\mathbb{Q}[e^{2\pi i/n}]$, $\sigma_\ell \in \mathbb{Z}_n^\times$ acts on f by

$$(\sigma.f)(\tau) = \sum_{j=0}^{\infty} (\sigma_\ell a_j) q^j$$
$$= f\left(\frac{a\tau+b}{c\tau+d}\right) \quad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

obeys $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix} \pmod n$

Also, this RCFT Galois action contains the Galois action on character tables of finite groups!

Some geometry

X = algebraic variety defined over $\mathbb{Q}$

= solutions to polynomials $p_i(z_1, \ldots, z_n)$ defined over $\mathbb{Q}$

Grothendieck Given basepoint $p \in X(\mathbb{Q})$,

$\Gamma_{\mathbb{Q}}$ acts on profinite completion $\widehat{\pi_1}(X, p)$

reason (sketch): Each finite-index normal subgroup $N \triangleleft \pi_1(X, p)$ corresponds to a finite (Galois) cover $X_N \to X$, with $\pi_1(X_N) = N$.

Each X_N is algebraically defined over $\overline{\mathbb{Q}}$, so $\Gamma_{\mathbb{Q}}$ permutes the finite-index normal subgroups N.

Example Taking $X = \mathbb{P}^1(\mathbb{C}) - \{1, 0, \infty\}$, get that $\Gamma_{\mathbb{Q}}$ acts on $\widehat{F_2}$.

This action is faithful (Belyi), so $\Gamma_{\mathbb{Q}}$ is subgroup of $Aut(\widehat{F_2})$.

Similarly, get action of $\Gamma_{\mathbb{Q}}$ on $\widehat{B_n}$, $\widehat{F_n}$.

Braided monoidal category

Category with tensor product, and (among other things) an associativity constraint

$$a_{UVW} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

and a commutativity constraint

$$c_{UV} : U \otimes V \to V \otimes U$$

obeying conditions (e.g. "pentagon", "hexagon") saying that any combination of a's and c's relating $U_1 \otimes U_2 \otimes \cdots \otimes U_n$ (bracketed anyway) and $U_{\pi 1} \otimes U_{\pi 2} \otimes \cdots \otimes U_{\pi n}$ (bracketed any way) must be equal.

Example In any braided monoidal category, any pure braid $\beta \in P_n$ acts on each set $\mathrm{Hom}(U_1 \otimes \cdots \otimes U_n, V)$.

Drinfeld Can we use $P_3 \times P_2$ to act on the associativity and commutativity constraints, to give us new braided monoidal category from an old one!

Linearising it, he asked:

do $\mathfrak{P}_3 \times \mathfrak{P}_2$ act on quasi-triangular quasi-Hopf algebras?

Easy answer: No!

We need e.g. pentagon, hexagon conditions to be satisfied, and this implies only 2 elements work: identity, and flip $R \leftrightarrow R_{21}$.

Drinfeld's idea Problem is $P_3 \times P_2$ is too small. So fill in holes by profinite completion!

New question: Can we use $\widehat{P}_3 \times \widehat{P}_2$ to get action on quasi-tri. quasi-Hopf algs?

$\widehat{P}_3 = \widehat{F}_2 \times \widehat{\mathbb{Z}}$, $\widehat{P}_2 = \widehat{\mathbb{Z}}$.

To get well-defined action, use formal power series. $\widehat{\mathbb{Q}}[[h]]$.

Solution $\widehat{GT}$ = Grothendieck-Teichmüller group.

Belyi $\Rightarrow$ $\Gamma_{\mathbb{Q}}$ is subgroup of $\widehat{GT}$.

Conjecture $\Gamma_{\mathbb{Q}} = \widehat{GT}$

Kassel–Turaev: found analogue for braided monoidal categories, by defining their "completions".

Example $\widehat{\text{ribbon}}$ has $\mathrm{Hom}(\emptyset, \emptyset)$ consisting of all finite linear combinations over $\hat{\mathbb{Q}}$ of framed oriented links in $\mathbb{R}^3$

$\therefore$ Get $\Gamma_{\mathbb{Q}}$-action on $\hat{\mathbb{Q}}$-span of those links.

Example Complex-conjugation sends L to its mirror reflection
(Knots often aren't isotopic to reflection — e.g. trefoil)

This $\Gamma_{\mathbb{Q}}$-action is identity for $[\Gamma_{\mathbb{Q}}\Gamma_{\mathbb{Q}}]$, so really get action of $\Gamma_{\mathbb{Q}}/[\Gamma_{\mathbb{Q}}\Gamma_{\mathbb{Q}}] = \hat{\mathbb{Z}}^{\times}$

Safe Guess: This action is the same as (or at least closely related to) the RCFT action.

Calculations by Bar-Natan, Le, Thurston should mean we can calculate Kassel–Turaev action on Hopf links, and compare with RCFT action on S.

Wild hope

Fact $A_1^{(1)}$ modular invariants fall into an A-D-E pattern.

Fact $A_2^{(1)}$ modular invariants are related to decompositions of Jacobians of Fermat curves into simple factors.

Question Are these coincidences related? Are they the first examples of ~~a~~ series of such correspondences?

Ihara (1986) developed a lot of this work relating profinite braid groups and $\Gamma_{\mathbb{Q}}$, etc. In his work Jacobians of Fermat curves also plays a key role.

Certainly, the main contact between $A_2^{(1)}$ modular invariants and Fermat involved the RCFT Galois action.

Could Ihara's work explain Fermat in $A_2^{(1)}$?