Claus Hertling (Mannheim) 29 July 05

Oscillating integrals and nilpotent orbits of twistor structures

---

Oscillating integrals (within Landau-Ginzburg models)

$\downarrow$

A generalisation of Hodge structures (Cecotti-Vafa '91, Sabbah July '05)

- - - -

Nilpotent orbits of this structure

$\downarrow$

A generalization of mixed Hodge structures

This catches some geometry in the renormalization group flow

P. Griffiths ($\geq$ 69): VHS

W. Schmid '73: PMHS and
nilpotent orbits of HS

S. Cecotti, C. Vafa ($\geq$ '91):
topological-antitopological fusion
$N = 2$ supersymmetric field theories

B. Dubrovin '92

M. Jimbo, T. Miwa, Y. Mori, M. Sato
($\geq$ '78): holonomic quantum fields
(semisimple case)

A weaker version is in the work of
C. Simpson ($\geq$ '88) on
harmonic bundles

C. Sabbah ($\geq$'01):
Polarizable twistor $\mathcal{D}$-modules

T. Mochizuki ($\geq$02):

Given:

Y an affine manifold, $\dim Y = n$.

$W: Y \to \mathbb{C}$ a holomorphic function with isolated singularities, tame.

Example 1: $Y = (\mathbb{C}^*)^n$,

$$W(x_1, \dots, x_n) = x_1 + \dots + x_n + \frac{1}{x_1 \cdot \ldots \cdot x_n},$$

(a Laurent polynomial, mirror partner of the quantum coh. of $\mathbb{P}^n$)

Milnor number $\mu = \dim \dfrac{\mathbb{C}[x_1, .. x_n, x_1^{-1}, .. x_n^{-1}]}{\left(\frac{\partial W}{\partial x_i}\right)}$

$= n+1$ here.

(= Witten index)

Example 2: $Y = \mathbb{C}^n$,

$$W(x_1, \dots, x_n) = x_1^8 + x_2^6 + x_3^2 + \dots + x_n^2 + \sum_{(i,j)} \overset{\text{fixed}}{\overset{\downarrow}{t_{ij}}} \cdot x_1^i x_2^j,$$

(i,j) with $\frac{i}{8} + \frac{j}{6} \leq 1$, $i \leq 6$, $j \leq 4$,

(a relevant + marginal deformation of the quasihomogeneous pol. $x_1^8 + x_2^6 + x_3^2 + \dots + x_n^2$)

$\mu = 35$.

$W : Y \to \mathbb{C}$ as above.

An oscillating integral:

a many valued holomorphic function

$$\begin{aligned} \mathbb{C}^* &\longrightarrow \mathbb{C} \\ z &\longmapsto [\omega]([\Gamma_z]) = \int_{\Gamma_z} e^{-W(x)/z} \cdot \omega , \end{aligned}$$

here

$\omega \in \Omega_Y^{n, alg}$,

$\Gamma_z \subset Y$ a Lefschetz thimble:

explanation below.

For simplicity, suppose

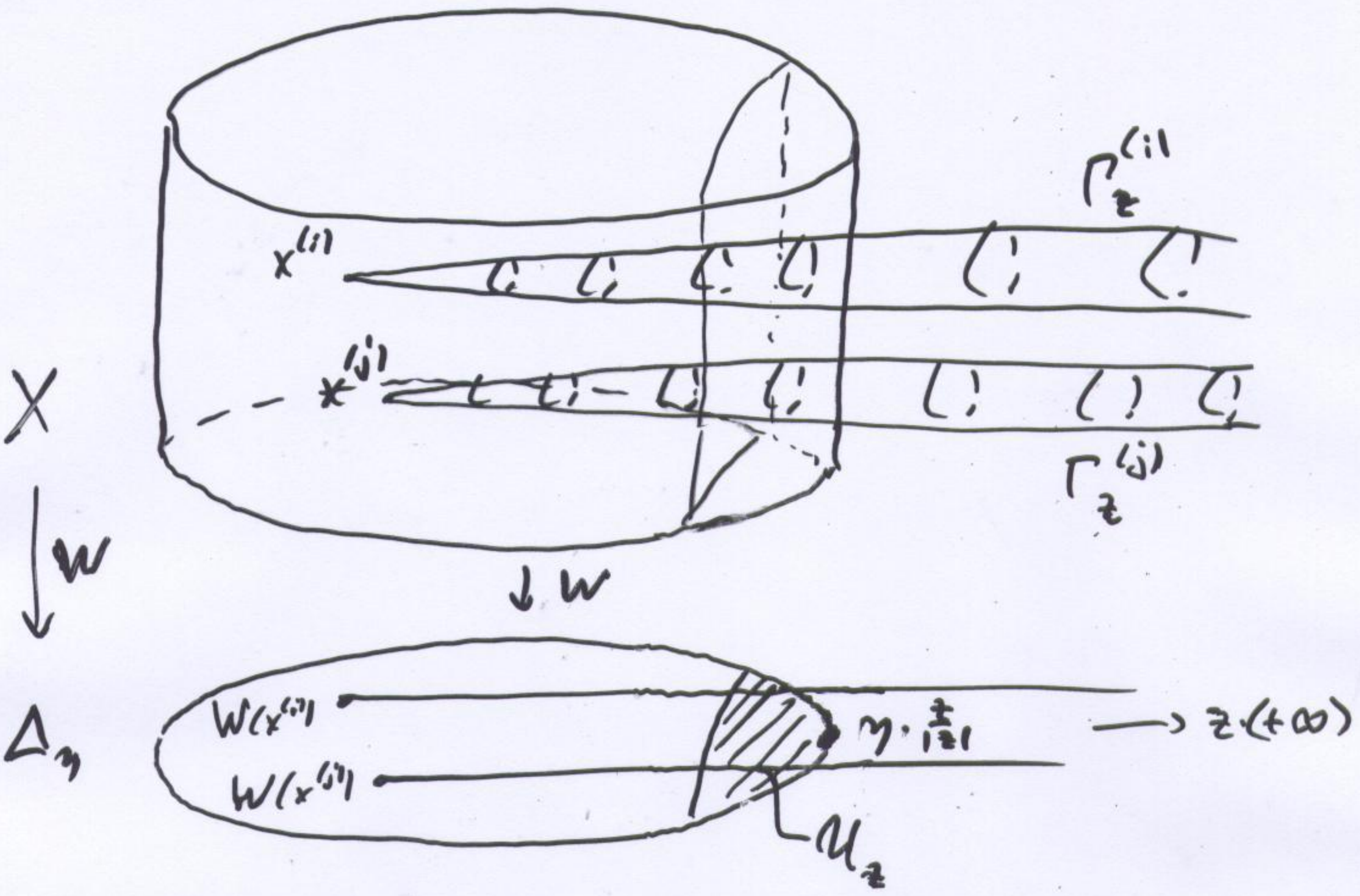
- that $Y = \mathbb{C}^n$
- that W has μ double points $x^{(1)}, \dots, x^{(\mu)} \in Y = \mathbb{C}^n$

Choose a large disk $\Delta_\eta \subset \mathbb{C}$, $\eta \gg 0$.

Choose a large ball $B \subset Y = \mathbb{C}^n$.

$$X := B \cap W^{-1}(\Delta_\eta) \xrightarrow{W} \Delta_\eta .$$

Then $x^{(1)}, \dots, x^{(\mu)} \in X$.



Lefschetz thimble

$$\Gamma_z^{(i)} = \bigcup_{\substack{\tau \in \text{path in } \Delta_\eta \\ \text{from } W(x^{(i)}) \\ \text{to } z\cdot(+\infty)}} (\text{vanishing cycle in } W^{-1}(\tau))$$

$\llcorner$ $(n-1)$-sphere

(F. Pham '83)

$$\Lambda_z := H_n(X, W^{-1}(\mathcal{U}_z), \mathbb{Z}) \cong \mathbb{Z}^\mu$$

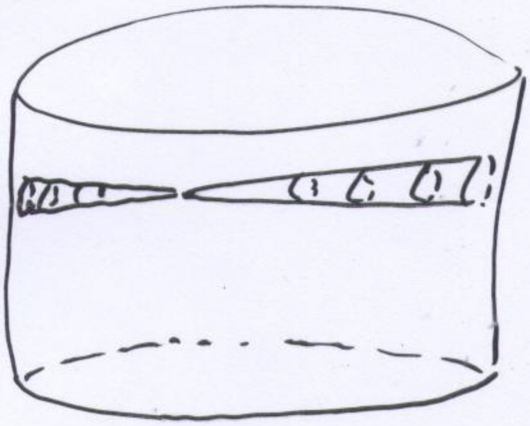
(relative homology group)

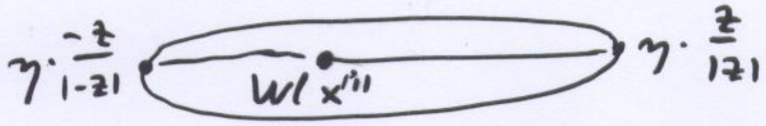
classes $[\Gamma_z^{(1)}], \dots, [\Gamma_z^{(\mu)}]$ $\quad$ $\mathbb{Z}$-basis of Λ_z.

Intersection form for Lefschetz thimbles

$$\langle\ ,\ \rangle : \quad \Lambda_z \times \Lambda_{-z} \quad : \quad \mathbb{Z}$$

$(-1)^n$ - symmetric, perfect pairing





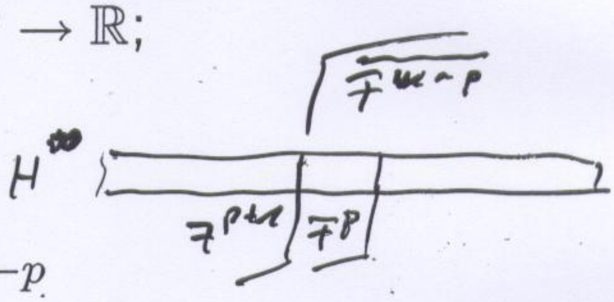
Definition 1: A polarized Hodge structure (PHS) of weight $w \in \mathbb{Z}$ is a tuple $(H^\infty, F^\bullet, H^\infty_\mathbb{R}, S)$ with

H^∞ a finite dim. $\mathbb{C}$-vector space;
$F^\bullet$ a decreasing filtration on H^∞;
$H^\infty_\mathbb{R} \subset H^\infty$ an $\mathbb{R}$-vector space with

$$H^\infty = H^\infty_\mathbb{R} \oplus i H^\infty_\mathbb{R};$$

S a $\mathbb{C}$-bilinear $(-1)^w$-symmetric nondegenerate pairing on H^∞ with $S : H^\infty_\mathbb{R} \times H^\infty_\mathbb{R} \to \mathbb{R}$;

such that



$F^\bullet$ and $\overline{F^{w-\bullet}}$ are opposite, i.e. $H^\infty = \oplus_p H^{p,w-p}$ with $H^{p,w-p} = F^p \cap \overline{F^{w-p}}$;

S gives a polarization, i.e. $\iff (F^p)^\perp = F^{w+1-p}$

$$S(F^p, F^{w+1-p}) = 0 \qquad \text{and}$$

the form $h_{Hodge} : H^\infty \times H^\infty \to \mathbb{C}$ with

$$h_{Hodge}(a,b) := i^{p-(w-p)} S(a, \overline{b})$$

for $a \in H^{p,w-p}, b \in H^\infty$, is hermitian and positive definite.

$z \in \mathbb{C}^*$

$$
\begin{array}{lll}
H_z := \mathrm{Hom}_{\mathbb{Z}}(\Lambda_z, \mathbb{C}) & & \cong \mathbb{C}^\mu \\
\cup & & \\
H_{\mathbb{R},z} & \mathbb{R} & \mathbb{R}^\mu \\
\cup & & \\
H_{\mathbb{Z},z} & \mathbb{Z} & \mathbb{Z}^\mu
\end{array}
$$

$$
\begin{array}{l}
H' := \bigcup_{z \in \mathbb{C}^*} H_z \\
\cup \\
H'_{\mathbb{R}} \\
\cup \\
H'_{\mathbb{Z}}
\end{array}
$$

$\mathbb{C}$-vector bundle with flat connection ∇

$\langle , \rangle \rightsquigarrow \langle , \rangle^* : H_{\mathbb{Z},z} \times H_{\mathbb{Z},-z} \to \mathbb{Z}$

$(-1)^n$-symmetric, nondegenerate

$$P := (-1)^{\frac{n(n-1)}{2}} \cdot \frac{1}{(2\pi i)^n} \cdot \langle , \rangle^* : H_z \times H_{-z} \to \mathbb{C}$$

Topological data:

$(H' \to \mathbb{C}^*, \nabla, H'_{\mathbb{R}} (, H'_{\mathbb{Z}}), P)$

Transcendent ingredient:

$\omega \in \Omega_Y^{n, alg}$,

$[\omega]$ is a global hol. section in H',

$$[\omega] \in \mathcal{O}(H'),$$

via

$$[\omega]([\Gamma_z]) = \int_{\Gamma_z} e^{-W(x)/z} \cdot \omega \, .$$

All $[\omega]$ together generate a

(free $\mathcal{O}_{\mathbb{C}}$-module of rank μ)

$= \mathcal{O}(H)$ with $H \to \mathbb{C}$

extension of $H' \to \mathbb{C}^*$

to a vector bundle on $\mathbb{C}$.

This extension generalizes

the Hodge filtration in a PHS.

(H, ∇) has a pole of order 2 at 0:

$$\partial_z \int_{\Gamma_z} e^{-W(x)/z} \cdot \omega = \frac{1}{z^2} \cdot \int_{\Gamma_z} e^{-W(x)/z} \cdot W(x) \cdot \omega$$

$$\nabla_{\partial_z} [\omega] = \frac{1}{z^2} [W(x) \cdot \omega]$$

Another fact:

$P : \mathcal{O}(H)_0 \times \mathcal{O}(H)_0 \longrightarrow z^n \cdot \mathcal{O}_{\mathbb{C},0}$,

nondegenerate.

$(H \to \mathbb{C}, \nabla, H'_{\mathbb{R}} \to \mathbb{C}, P^*)$

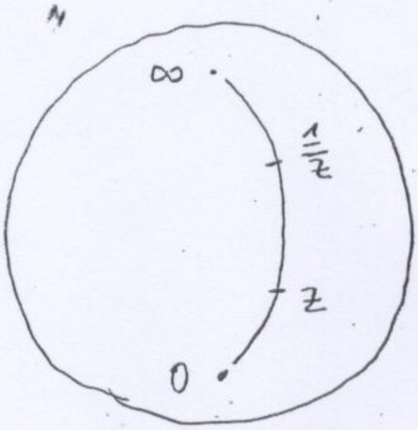
is a "(TERP)-structure" of weight n

(Twistor Extension Real Pairing)

(How to generalize $F^\bullet \to \overline{F^{w-\bullet}}$?)

Given a $(TERP)$-structure
$(H \to \mathbb{C}, \nabla, H'_{\mathbb{R}}, P)$ of weight n.

$\gamma : \mathbb{P}^1 \to \mathbb{P}^1, z \mapsto \frac{1}{\bar{z}}.$



Define

$$\begin{aligned} \tau \;:\; & H_z \to H_{\gamma(z)} \quad \text{a } \mathbb{C}\text{-antilinear isom.} \\ & a \mapsto \nabla\text{-flat shift to } H_{\gamma(z)} \text{ of } \overline{z^{-n} a}. \end{aligned}$$

$\tau^2 = \mathrm{id}$.

Glue $H \to \mathbb{C}$ and $\overline{\gamma^* H} \to \mathbb{P}^1 - \{0\}$ with τ to a bundle

$$\widehat{H} \to \mathbb{P}^1.$$

It has a pole of order ≤ 2 at ∞.

It has degree 0.

Theorem: (Cecotti-Vafa '91, Sabbah '05)

$\hat{H} \to \mathbb{P}^1$ is the trivial bundle.

["(tr.TERP)-structure";

this generalizes "$F^\cdot$ and $\overline{F^{w..}}$ opposite",

i.e. a Hodge structure]

Then

$\Gamma(\mathbb{P}^1, \mathcal{O}(\hat{H})) \cong H_0$ = fiber at 0

and $\tau : \circlearrowleft$ induces $\kappa : \circlearrowleft$

$\kappa : H_0 \to H_0$ is a $\mathbb{C}$-antilinear involution.

Define the $\mathbb{C}$-bilinear symmetric nondegenerate pairing

$$\begin{aligned} g \; : \; & H_0 \times H_0 \to \mathbb{C} \\ & (a,b) \mapsto z^{-w} P(\widetilde{a}, \widetilde{b}) \bmod z\mathcal{O}_{\mathbb{C},0} \end{aligned}$$

for $\widetilde{a}, \widetilde{b} \in \mathcal{O}(H)_0$
with $\widetilde{a}(0) = a, \widetilde{b}(0) = b$.

Define a nondegenerate pairing h on H_0,

$$h = g(\,.\,, \kappa\,.\,).$$

Nontrivial exercise: h is hermitian.

Theorem: (Cecotti-Vafa '91, Sabbah '05)

h is positive definite.

[Cecotti-Vafa: h ground state metric]

["(pos. def. tr. TERP)-structure";
it generalizes a polarized Hodge structure]

<u>Lemma 6</u>: Given a $(tr.TERP(n))$-structure.
There exist unique endomorphisms

$$\mathcal{U} : H_0 \to H_0 \quad \text{and} \quad \mathcal{Q} : H_0 \to H_0$$

s.t.

$$\nabla_{z\partial_z} = \frac{1}{z}\mathcal{U} - \mathcal{Q} + \frac{n}{2}\mathrm{id} - z\kappa\mathcal{U}\kappa$$

on $\Gamma(\mathbb{P}^1, \mathcal{O}(\hat{H})) \cong H_0$.

$(pos.def.tr.TERP)$-structure: $h(\mathcal{Q}a,b)=h(a,\mathcal{Q}b)$
$\mathcal{Q}$ is a hermitian endomorphism with eigenvalues in $\mathbb{R}$ symmetric around 0.

PHS: $\quad \mathcal{Q} \sim \bigoplus_p (p - \frac{w}{2})\mathrm{id}\,|_{H^{p,w-p}}.$

Cecotti-Fendley-Intriligator-Vafa '92:
$\mathcal{Q}$ is a *new supersymmetric index*.

Variation of $(TERP)$-structures
$=$ <u>tt^* geometry</u>:

$(H \to \mathbb{C}\times M, \nabla, H'_{\mathbb{R}}, P)$ with pole of Poincaré rank 1 along $\{0\} \times M$ ....

(This generalizes a variation of PHS.)

If $\hat{H}$ is trivial :

Rewrite it as structure on $H|_{\{0\}\times M}$:

real structure κ, hol. metric g, hermitian metric h, hermitian connection D, Higgs field C, endomorphisms $\mathcal{U}$ and $\mathcal{Q}$;

conditions: tt^* equations ...

$\rightsquigarrow \ (H|_{\{0\}\times M}, h, \nabla|_{\{1\}\times M})$

harmonic bundle

Fix a reference PHS $(H^\infty, H^\infty_\mathbb{R}, S, F_0^\bullet)$ of weight w.

$$\check{D} := \{ \text{ filtrations } F^\bullet \subset H^\infty \mid \\ \dim F^p = \dim F_0^p, \\ S(F^p, F^{w+1-p}) = 0\},$$

$\cup$ open

$$D := \{F^\bullet \in \check{D} \mid F^\bullet \text{ is part of a PHS}\}.$$

Fix a nilpotent endomorphism

$$N : H^\infty_\mathbb{R} \to H^\infty_\mathbb{R}$$

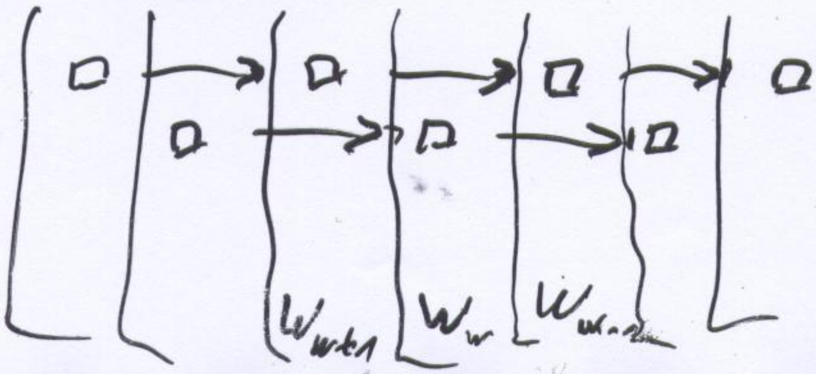
with $S(Na, b) + S(a, Nb) = 0$.

It induces a unique filtration $W_\bullet$ on $H^\infty_\mathbb{R}$ s.t.

$$N(W_l) \subset W_{l-2} \text{ and}$$

$$N^l : \mathrm{Gr}^W_{w+l} \to \mathrm{Gr}^W_{w-l}$$

is an isomorphism.



Definition : Fix $(H^{\infty}, H^{\infty}_{\mathbb{R}}, S)$.
Choose N as above and $F^{\bullet} \in \check{D}$.

a) The pair $(F^{\bullet}, N)$ is a *polarized mixed Hodge structure* (PMHS) *of weight* $w \in \mathbb{Z}$, if

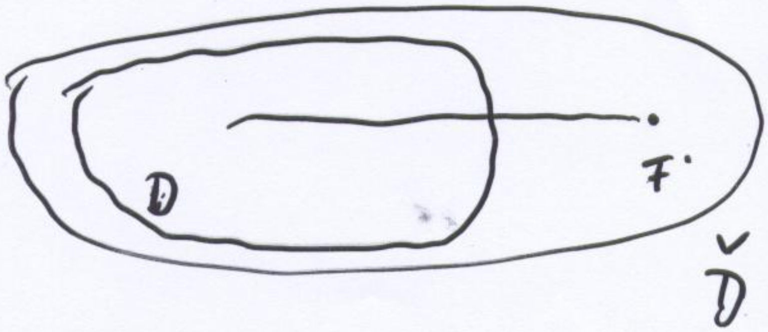
- $F^{\bullet} Gr^W_k$ is a Hodge structure of weight k,
- $N(F^p) \subset F^{p-1}$,
- and the induced Hodge structure on the primitive subspace

$$P_{w+l} := \ker(N^{l+1} : Gr^W_{w+l} \to Gr^W_{w-l-2})$$

is polarized by $S_l := S(., N^l .)$.

b) The pair $(F^{\bullet}, N)$ *gives rise to a nilpotent orbit* if

- $N(F^p) \subset F^{p-1}$ and
- $e^{i\xi N} F^{\bullet} \in D$ for $\xi \in \mathbb{C}$ with $\mathrm{Re}\,\xi \gg 0$.



Theorem :

The pair $(F^\bullet, N)$ is part of a PMHS
$\Longleftrightarrow$ it gives rise to a nilpotent orbit.

Proof:

$\Leftarrow$ Schmid's Sl_2-orbit theorem '73,
$\Rightarrow$ Cattani-Kaplan-Schmid '86.

given $(H^\infty, H^\infty_{\mathbb{R}}, S, N)$

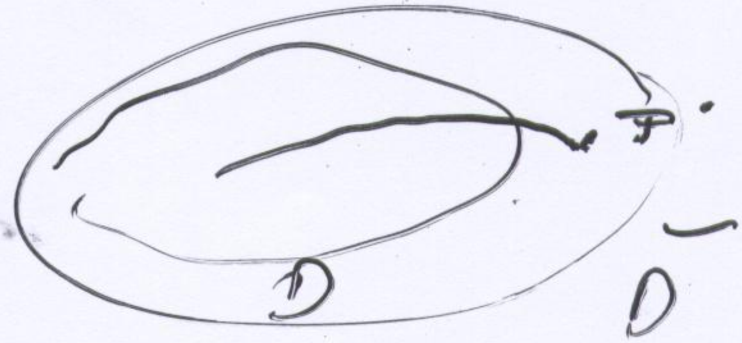
$\mathbb{C}$-v.s. $\quad N : H^\infty_{\mathbb{R}} \to H^\infty_{\mathbb{R}}$ nilp.

$F^\bullet$ looks like a Hodge filtration

except for

$\left\{ \begin{array}{l} \text{opposite} \\ \text{harm. metric } h \end{array} \right.$

$F^\bullet \in \check{D}$
$\quad\;\; \cup$
$\quad\;\; D$



<u>Conjecture</u> : Suppose a $(TERP)$-structure *without ramification* is given.

It is a $(mixed.TERP)$-structure
$\iff$ *it gives rise to a nilpotent orbit* of $(pos.def.tr.TERP)$-structures.

<u>Theorem</u>:

(a) $\Rightarrow$ is true

(b) $\Leftarrow$ is true if $\mathcal{U}$ is nilpotent
i.e. if (H, ∇) has a
reg. sing. pole at 0

(c) $\Leftarrow$ is true if rank $H = 2$

Proof: (a) Case $\mathcal{U}$ nilpotent:
Hertling '03
(Crelle vol. 555 (2003), Theorem 7.20.)
with
Cattani-Kaplan-Schmid '86
+ additional estimates.

Case $\mathcal{U}$ semisimple: Dubrovin '92,
Riemann boundary value problem,
solution with a
singular integral equation.

New solution (Hertling): with a Toeplitz
operator (à la Malgrange '83).

General case: combination of both.

(b) Application of a result of Mochizuki '03:
Thm 12.1 in math.DG/0312230
(on page 221, 358 pages)
tame harmonic bundle on Δ^*
$\rightsquigarrow$ a polarized mixed twistor structure

Rank 2 and $\mathcal{U}$ semisimple:

Given a semisimple $(pos.def.tr.TERP)$-structure of rank 2. Then with respect to a distinguished basis of
$\Gamma(\mathbb{P}^1, \hat{H}) \cong H_0$

$$\text{(matrix of } h) = \begin{pmatrix} \cosh\frac{\alpha}{2} & -i\sinh\frac{\alpha}{2} \\ i\sinh\frac{\alpha}{2} & \cosh\frac{\alpha}{2} \end{pmatrix}.$$

Given a nilpotent orbit of $(pos.def.tr.TERP)$-structures:
$\alpha = \alpha(r)$ is a nonsingular real solution of the radial sinh-Gordon equation ($\sim$ Painlevé III)

$$(\partial_r^2 + \frac{1}{r}\partial_r)\alpha(r) = 4\sinh\alpha(r).$$

McCoy-Tracy-Wu '77 and
Its-Novokshenov '86 $\Rightarrow$
for all such solutions *real structure* and *Stokes structure* are compatible. $\square$

For $r > 0$ define $\pi_r : \mathbb{C} \to \mathbb{C},\ z \mapsto \frac{1}{r}z.$

$(r \sim e^{\mathrm{Re}\,\xi})$

<u>Definition</u>:
A $(TERP)$-structure $(H, \nabla, H'_{\mathbb{R}}, P)$ *gives rise to a nilpotent orbit* if
$\pi_r^*(H, \nabla, H'_{\mathbb{R}}, P)$ is a $(pos.def.tr.TERP)$-structure for $r \gg 0$.

Definition (not precise):
A $(mixed.TERP)$-structure is a $(TERP)$-structure s.t.:

(a) (condition *without ramification*) $(\mathcal{O}(H)_0, \nabla)$ is formally isomorphic to a sum

$$\bigoplus_{i=1}^{k} e^{-u_i/z} \otimes \mathcal{R}_i$$

where $u_i \in \mathbb{C}$ with $u_i \neq u_j$ for $i \neq j$ are the eigenvalues of

$$\mathcal{U} := [z\nabla_{z\partial_z}] : H_0 \to H_0$$

and $\mathcal{R}_i$ are free $\mathcal{O}_{\mathbb{C},0}$-modules with flat connection with regular singularity at 0;

(b) real structure and Stokes structure are *compatible* ~~(def. next two slides);~~

(c) the regular singular pieces give rise to PMHS's.

What happens for $r \to 0$?

Given a $(TERP)$-structure $(H, \nabla, H'_{\mathbb{R}}, P)$, Sabbah defined a tuple

$$(H^\infty, H^\infty_{\mathbb{R}}, S, M_s, N, F^\bullet_{Sabbah})$$

(from the behaviour at $z = \infty$).

Theorem: This tuple is a PMHS
$\iff \pi_r^*(H, \nabla, H'_{\mathbb{R}}, P)$ is a $(pos.def.tr.TERP)$-structure for $r > 0$ close to 0.

Proof: $\Rightarrow$ Cattani-Kaplan-Schmid + additional estimates.

$\Leftarrow$ Application of Mochizuki's theorem.

Renormalization group flow:

1-par-family $\qquad r \cdot W : Y \to \mathbb{C}$

$\qquad 0 < r < \infty$

$r \to 0$: UV-limit ; if the limit exists, it gives a quasihomogeneous fct W_0

$r \to \infty$: IR-limit .

$$(TERP)(r \cdot W) = \pi_r^* \big((TERP)(W) \big) .$$

Theorem of Cecotti-Vafa / Sabbah

$\rightsquigarrow$ for all r $\quad (TERP)(r \cdot W)$ is a (pos. def. tr. TERP)-structure

$$\pi_r : \mathbb{C} \to \mathbb{C}$$
$$z \mapsto \frac{1}{r} \cdot z$$

$\rightsquigarrow$ for $r \to \infty$ nilpotent orbit
"$\Leftarrow$" of conjecture

$\Rightarrow$

a (mixed. TERP)-structure,

i.e. • PMHS's associated to the local singularities of W,

• Stokes and real structure are compatible.

[Both are well known facts.]

$\rightsquigarrow$ for $r \to 0$ "Sabbah-nilpotent orbit theorem for this case

$\Rightarrow$

a PMHS associated to W.

[1998 Sabbah constructed a closely related MHS in a different way.]