# B-FIELDS, GERBES AND GENERALIZED GEOMETRY

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## PART 1

(i) BACKGROUND

(ii) GERBES

(iii) GENERALIZED COMPLEX STRUCTURES

(iv) GENERALIZED KÄHLER STRUCTURES

# BACKGROUND

# BASIC SCENARIO

- manifold $M^n$
- replace $T$ by $T \oplus T^*$
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- manifold $M^n$

- replace $T$ by $T \oplus T^*$

- inner product of signature $(n, n)$

$$(X + \xi, X + \xi) = -i_X \xi$$

- skew adjoint transformations:

$$\mathrm{End}\, T \oplus \Lambda^2 T^* \oplus \Lambda^2 T$$

- .... *in particular* $B \in \Lambda^2 T^*$

## TRANSFORMATIONS

- exponentiate $B$:

  $X + \xi \mapsto X + \xi + i_X B$

- $B \in \Omega^2$ ... *the B-field*

- natural group $\text{Diff}(M) \ltimes \Omega^2(M)$

# SPINORS

- Take $S = \Lambda^* T^*$

- $S = S^{ev} \oplus S^{od}$

- Define Clifford multiplication by

$$\begin{aligned}(X+\xi)\cdot\varphi &= i_X\varphi + \xi\wedge\varphi \\ (X+\xi)^2\cdot\varphi &= i_X\xi\varphi = -(X+\xi, X+\xi)\varphi\end{aligned}$$

- $\exp B(\varphi) = (1 + B + \frac{1}{2}B\wedge B + \ldots)\wedge\varphi$

## DERIVATIVES

- Lie bracket:

  $$2i_{[X,Y]}\alpha = d([i_X, i_Y]\alpha) + 2i_X d(i_Y\alpha) - 2i_Y d(i_X\alpha) + [i_X, i_Y]d\alpha$$

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- $A = X + \xi$, $B = Y + \eta$ use Clifford multiplication $A \cdot \alpha$ to define a bracket $[A, B]$:

  $$2[A, B] \cdot \alpha = d((A \cdot B - B \cdot A) \cdot \alpha) + 2A \cdot d(B \cdot \alpha) -$$

  $$-2B \cdot d(A \cdot \alpha) + (A \cdot B - B \cdot A) \cdot d\alpha$$

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- *COURANT bracket*

  $$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)$$

Apply a 2-form $B$...

- $d \mapsto e^{-B} d e^{B} = d + dB$

- $[X + \xi, Y + \eta] \mapsto [X + \xi, Y + \eta] - 2 i_X i_Y dB$

- Diff$(M) \ltimes \Omega^2_{closed}(M)$ preserves inner product, exterior derivative and Courant bracket.

# GENERALIZED GEOMETRIC STRUCTURES

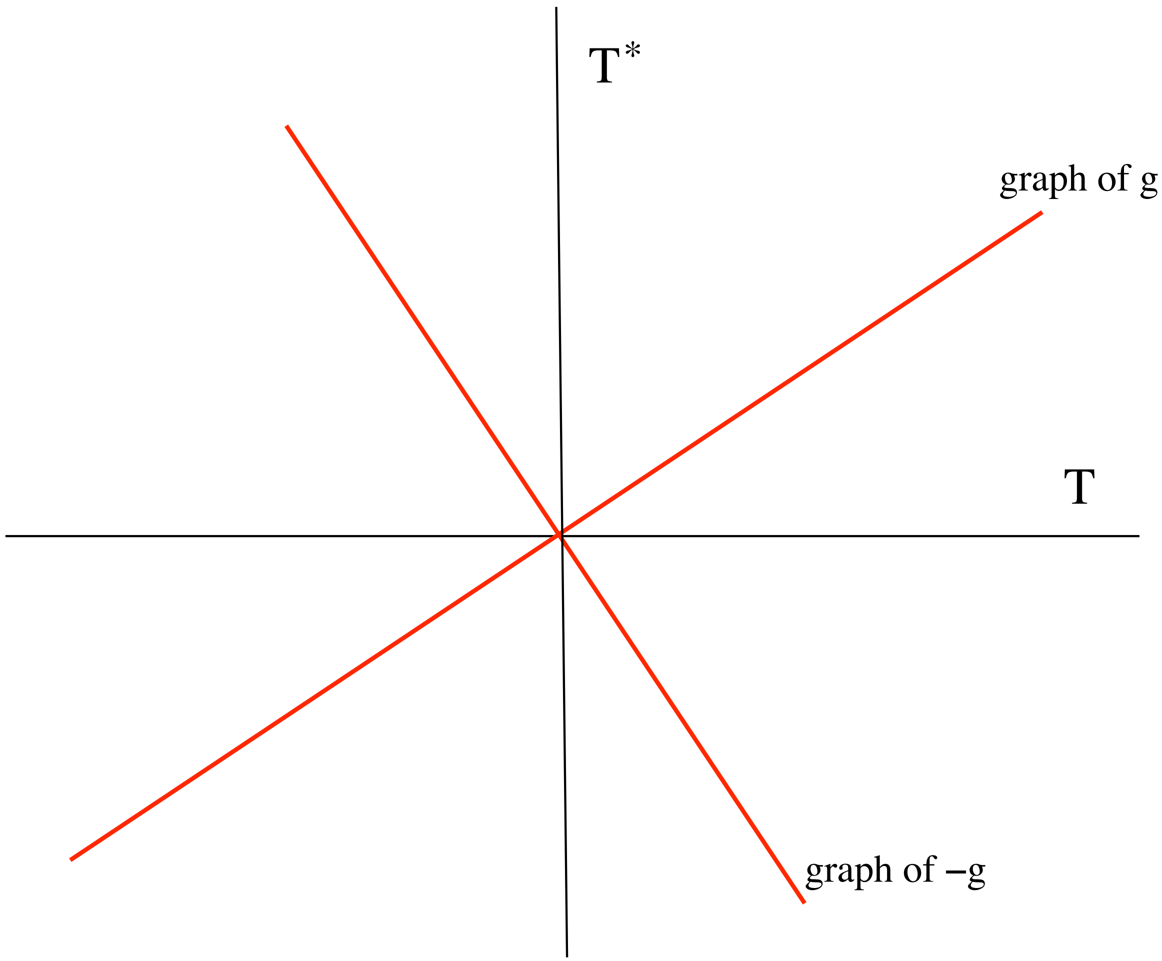
- $SO(n,n)$ compatibility

- integrability $\sim d$ or Courant bracket

- transform by Diff$(M) \ltimes \Omega^2_{closed}(M)$

## RIEMANNIAN METRIC

- Riemannian metric $g_{ij}$
- $X \mapsto g(X, -) : g : T \to T^*$
- *graph* of $g = V \subset T \oplus T^*$
- $T \oplus T^* = V \oplus V^\perp$


T*
graph of g
T
graph of −g

# GENERALIZED RIEMANNIAN METRIC

- $V \subset T \oplus T^*$ positive definite rank $n$ subbundle

- $=$ graph of $g + B : T \to T^*$

- $g + B \in T^* \otimes T^*$: $g$ symmetric, $B$ skew

# GERBES

## GERBES: ČECH 2-COCYCLES

- $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to S^1$

- $(g_{\alpha\beta\gamma} = g^{-1}_{\beta\alpha\gamma} = \ldots)$

- $\delta g = g_{\beta\gamma\delta} g^{-1}_{\alpha\gamma\delta} g_{\alpha\beta\delta} g^{-1}_{\alpha\beta\gamma} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

This *defines* a gerbe.

# CONNECTIONS ON GERBES

Connective structure:

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}$$

Curving:

$$B_\beta - B_\alpha = dA_{\alpha\beta}$$

$\Rightarrow dB_\beta = dB_\alpha = H|_{U_\alpha}$ global three-form $H$

J.-L. Brylinski, *Characteristic classes and geometric quantization*, Progr. in Mathematics **107**, Birkhäuser, Boston (1993)

# TWISTING $T \oplus T^*$

$$dA_{\alpha\beta} + dA_{\beta\gamma} + dA_{\gamma\alpha} = d[g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}] = 0$$

- identify $T \oplus T^*$ on $U_\alpha$ with $T \oplus T^*$ on $U_\beta$ by

  $X + \xi \mapsto X + \xi + i_X dA_{\alpha\beta}$

- defines a vector bundle $E$

  $0 \to T^* \to E \to T \to 0$

- with ... an inner product and a Courant bracket.

# TWISTED COHOMOLOGY

- identify $\Lambda^* T^*$ on $U_\alpha$ with $\Lambda^* T^*$ on $U_\beta$ by

  $\varphi \mapsto e^{dA_{\alpha\beta}} \varphi$

- defines a vector bundle $S$ = spinor bundle for $E$

- $d : C^\infty(S) \to C^\infty(S)$ well-defined

- $\ker d / \operatorname{im} d$ = twisted cohomology.

# ... WITH A CURVING

- $\varphi_\alpha = e^{dA_{\alpha\beta}}\varphi_\beta$
- $d\varphi_\alpha = 0$
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- $d\varphi_\alpha = 0$
- Curving: $B_\beta - B_\alpha = dA_{\alpha\beta}$
- $e^{B_\alpha} \varphi_\alpha = e^{B_\beta} \varphi_\beta = \psi$
- $d\psi + H \wedge \psi = 0$

Definition: A **generalized metric** is a subbundle $V \subset E$ such that $\operatorname{rk} V = \dim M$ and the inner product is positive definite on $V$.

- $V \cap T^* = 0 \Rightarrow$ splitting of the sequence

  $0 \to T^* \to E \to T \to 0$

- $V^\perp \subset E$ another splitting

- difference $\in \operatorname{Hom}(T, T^*) = T^* \otimes T^* =$ Riemannian metric

# SPLITTINGS IN LOCAL TERMS

- splitting: $C_\alpha \in C^\infty(U_\alpha, T^* \otimes T^*)$ : $C_\beta - C_\alpha = dA_{\alpha\beta}$
- $Sym(C_\alpha) = Sym(C_\beta) =$ metric
- $Alt(C_\alpha) = B_\alpha =$ curving of the gerbe
- $H = dB_\alpha$ closed 3-form

- *two* splittings $V$ and $V^{\perp}$ of $0 \to T^* \to E \to T \to 0$
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- $X$ vector field, lift to $X^+ \in C^\infty(V)$ and $X^- \in C^\infty(V^\perp)$ in $E$

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- $[X^+, Y^-] - [X, Y]^+$ is a one-form

- $[X^+, Y^-] - [X, Y]^+ = 2g(\nabla_X^+ Y)$
- 
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- $[X^+, Y^-] - [X, Y]^+ = 2g(\nabla^+_X Y)$

- $\nabla^+$ Riemannian connection with skew torsion $H$

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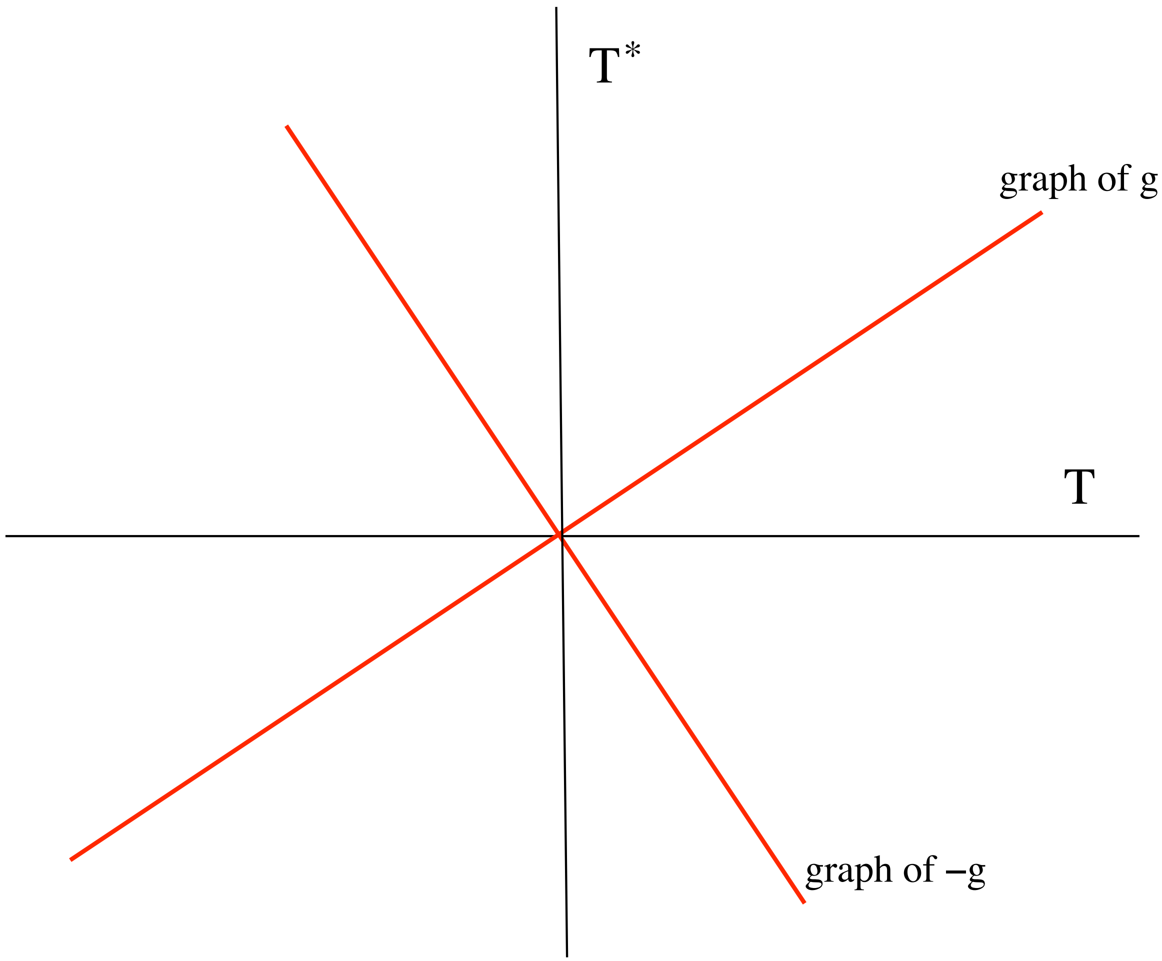
- $[X^+, Y^-] - [X, Y]^+ = 2g(\nabla^+_X Y)$

- $\nabla^+$ Riemannian connection with skew torsion $H$

- $[X^-, Y^+] - [X, Y]^- = -2g(\nabla^-_X Y)$ has skew torsion $-H$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# EXAMPLE: the Levi-Civita connection

$$\left[\frac{\partial}{\partial x_i} - g_{ik}dx_k, \frac{\partial}{\partial x_j} + g_{jk}dx_k\right] - \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]^+ =$$

$$= \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right) dx_k = 2g_{\ell k}\Gamma^{\ell}_{ij}$$

# GENERALIZED COMPLEX STRUCTURES

A *generalized complex structure* is:

- $J : T \oplus T^* \to T \oplus T^*, J^2 = -1$

- $(JA, B) + (A, JB) = 0$

- if $JA = iA, JB = iB$ then $J[A, B] = i[A, B]$ (*Courant bracket*)

- $U(m, m) \subset SO(2m, 2m)$ structure on $T \oplus T^*$

M. Gualtieri: *Generalized complex geometry* math.DG/0401221

## EXAMPLES

- complex manifold $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$$J = i : [\ldots \partial/\partial z_i \ldots, \ldots d\bar{z}_i \ldots]$$

- symplectic manifold $J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$

$$J = i : [\ldots, \partial/\partial x_j + i \sum \omega_{jk} dx_k, \ldots]$$

# EXAMPLE: HOLOMORPHIC POISSON MANIFOLDS

$$\sigma = \sum \sigma^{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

$$[\sigma, \sigma] = 0$$

$$J = i : \left[ \ldots, \frac{\partial}{\partial z_j}, \ldots, d\bar{z}_k + \sum_{\ell} \bar{\sigma}^{k\ell} \frac{\partial}{\partial \bar{z}_\ell}, \ldots \right]$$

# GENERALIZED KÄHLER MANIFOLDS

Kähler $\Rightarrow$ complex structure + symplectic structure

- complex structure $\Rightarrow J_1$ on $T \oplus T^*$
- symplectic structure $\Rightarrow J_2$ on $T \oplus T^*$
- compatibility ($\omega \in \Omega^{1,1}$): $J_1 J_2 = J_2 J_1$

A **generalized Kähler structure**: two commuting generalized complex structures $J_1, J_2$ such that $(J_1 J_2(X+\xi), X+\xi)$ is definite.

GUALTIERI'S THEOREM A generalized Kähler manifold is:

- two integrable complex structures $I_+, I_-$ on $M$
- a metric $g$, hermitian with respect to $I_+, I_-$
- a 2-form $B$
- $U(m)$ connections $\nabla^+, \nabla^-$ with skew torsion $\pm H = \pm dB$

- $(J_1 J_2)^2 = 1$

- eigenspaces $V, V^{\perp}$, metric on $V$ positive definite

- connections $\Rightarrow \nabla^{+}, \nabla^{-}$ torsion $\pm dB$

- ... also define on $0 \to T^* \to E \to T \to 0$

- $J_1 = -J_2$ on $V \Rightarrow \pm I_+$

- $J_1 = J_2$ on $V^\perp \Rightarrow \pm I_-$

- $\{J_1 = i\} \cap \{J_2 = -i\} = V^{1,0}$ is Courant integrable

- $[X^{1,0} + \xi, Y^{1,0} + \eta] = Z^{1,0} + \zeta \Rightarrow I_+$ integrable

V Apostolov, P Gauduchon, G Grantcharov, *Bihermitian structures on complex surfaces*, Proc. London Math. Soc. **79** (1999), 414–428

S. J. Gates, C. M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear $\sigma$-models*. Nuclear Phys. B **248** (1984), 157–186.

- $[J_1, J_2] = 0$, $[I_+, I_-] \neq 0$ in general
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- $[J_1, J_2] = 0$, $[I_+, I_-] \neq 0$ in general
- $g([I_+, I_-]X, Y) = \Phi(X, Y)$ 2-form
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- $\Rightarrow \sigma \in \Lambda^{0,2} \cong \Lambda^2 T$

- $\sigma$ is holomorphic

- $\sigma$ is a holomorphic Poisson structure

# EXAMPLES

- $T^4$ and $K3$, $\sigma =$ holomorphic 2-form (D. Joyce)

- $\mathbf{C}P^2$, $\sigma = \partial/\partial z_1 \wedge \partial/\partial z_2$

- $\mathbf{C}P^1 \times \mathbf{C}P^1 =$ projective quadric.

  $\sigma = \partial Q/\partial z_3[\partial/\partial z_1 \wedge \partial/\partial z_2]$

- moduli space of ASD connections on above.

J. Bogaerts, A. Sevrin, S. van der Loo, S. Van Gils, *Properties of Semi-Chiral Superfields*, Nucl.Phys. B562 (1999) 277-290

V Apostolov, P Gauduchon, G Grantcharov, *Bihermitian structures on complex surfaces*, Proc. London Math. Soc. **79** (1999), 414–428

NJH, *Instantons, Poisson structures and generalized Kaehler geometry*, math.DG/0503432

C. Bartocci and E. Macrì, *Classification of Poisson surfaces*, Communications in Contemporary Mathematics, **7** (2005), 1-7