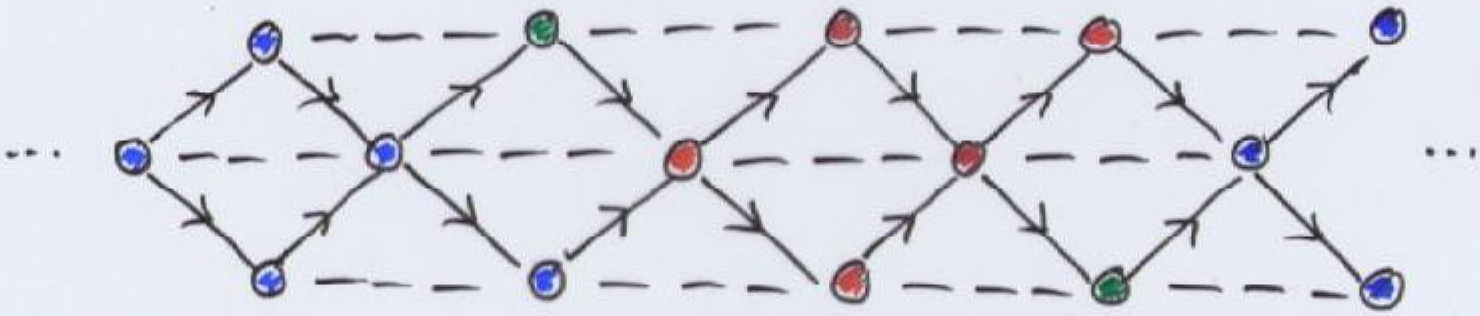
How to 'knit' a 'scarf'



Alastair King (Durham, July 2005)

[to absent friends]

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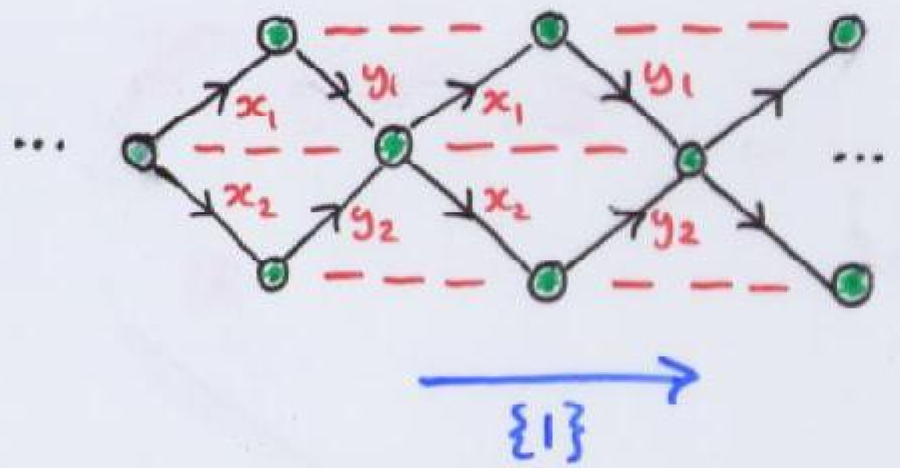
'<u>Definition</u>' To any bipartite graph Δ, e.g. A_3 

can associate a '<u>scarf</u>' $\mathbb{Z}(\Delta)$, which is an infinite '<u>translation</u>' quiver with '<u>mesh</u>' relations

e.g. $\mathbb{Z}(A_3)$



$$\forall v \quad \sum_{\overset{\bullet \xrightarrow{e}}{v}} x_e y_e = 0$$

$$\forall v \quad \sum_{\underset{v}{\xrightarrow{e}\bullet}} y_e x_e = 0$$

c.f. 'Bratteli diagram' (without relations)

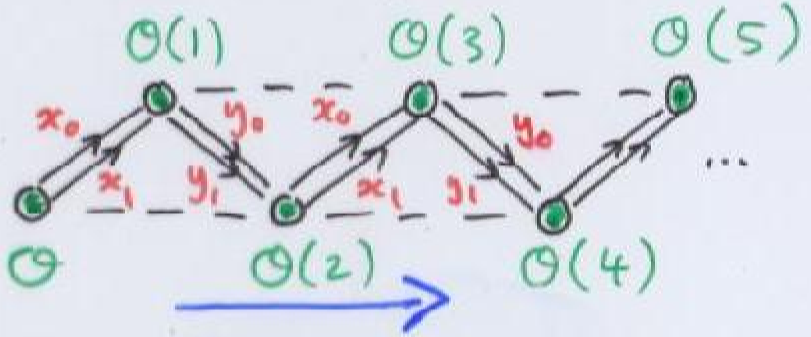
$\mathbb{Z}(\Delta)$ is a picture of a category

- $\circledcirc$: complete set of indecomposable objects
- $\rightarrow$: basis of irreducible maps

Example 1 [Grothendieck / Kronecker]

Vec $(\mathbb{P}^1)$: finite rank vector bundles on $\mathbb{P}^1$

$K =$ 

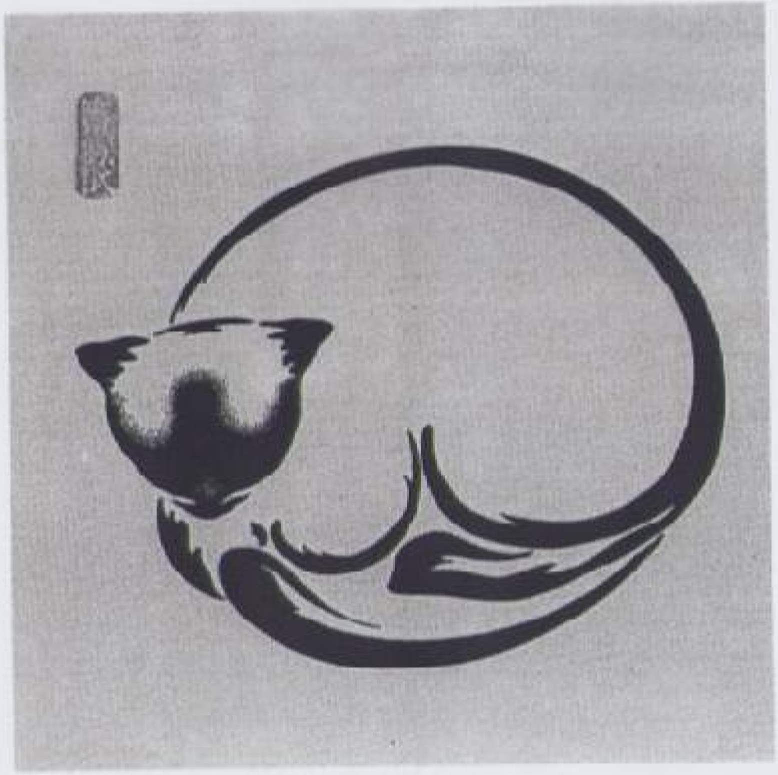
$\mathbb{Z}(K) = \ldots$  $\ldots$

with mesh relation

$$x_0 y_0 + x_1 y_1 = 0 \iff \begin{cases} y_1/y_0 = -x_0/x_1 & \text{in affine coords} \\ (y_0 : y_1) = (-x_1 : x_0) & \text{in proj. coords} \end{cases}$$

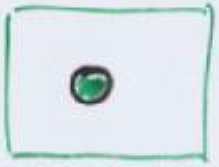
Translation $\{1\}$ = anti-canonical 'twist' $(2) : E \mapsto E \otimes \mathcal{O}(2)$,

so $\exists$ 'square root' $\boxed{\{1/2\} = (1)}$



3-ε

A simple picture of a category:

 $= \underline{\mathrm{Vec}}(\mathbb{C})$: fin. dim'l vector spaces over $\mathbb{C}$ with linear maps

the 'skeleton' is $\begin{cases} \underline{\text{obj}}: & \text{finite } \oplus \text{ of } \bullet, \text{ i.e. one for each } n \in \mathbb{N}_0 \\ \underline{\text{mor}}: & \text{matrices with values in } \mathrm{End}(\bullet) = \mathbb{C} \end{cases}$

+ symmetries : $GL(n, \mathbb{C})$ for $n \in \mathbb{N}_0$

eg. definitions of det

UG: for $A \in \mathrm{End}(\mathbb{C}^n)$, $\det A$ is invariant under $GL(n, \mathbb{C})$

so can define $\det \alpha = \det A$ for $\alpha \in \mathrm{End}(V)$

PG: if $\dim V = \dim W = n$, then

$$\det(\phi : V \to W) = \Lambda^n \phi : \Lambda^n V \longrightarrow \Lambda^n W$$

for natural functors $\Lambda^n : \underline{\mathrm{Vec}}(\mathbb{C}) \circlearrowleft$

'Quantisation' of translation

ambiguity in $\{\tfrac{1}{2}\}$ $\quad \{\tfrac{1}{2}\}_t : \; x_i \mapsto t\, y_i, \quad y_i \mapsto t^{-1} x_i$

deformation of $\{1\}$ $\quad \{1\}_q : \; x_i \mapsto q\, x_i, \quad y_i \mapsto q\, y_i$ $\qquad \{\tfrac{1}{2}\}_q : \; x_i \mapsto q^{1/2} y_i, \quad y_i \mapsto q^{1/2} x_i$

combination of both $\quad \{1\}_w : \; x_i \mapsto w\, x_i, \quad y_i \mapsto w\, y_i$ $\qquad \{\tfrac{1}{2}\}_{uv} : \; x_i \mapsto u\, y_i, \quad y_i \mapsto v\, x_i$

where $\boxed{uv = w}$

Example 2 Matrix factorisation of x^d [c.f. lectures of M. Gaberdiel & D. Roggenkamp]

e.g. $d=4$

$$M_a = x \cdot x^3$$

$$M_b = x^2 \cdot x^2$$

$$M_c = x^3 \cdot x$$

$$\boxed{\text{N.B. } 1 \cdot x^4 \simeq 0 \simeq x^4 \cdot 1}$$

$$\begin{tikzcd}
 & {} & & & \\
M_a \arrow[ur, "\beta=0"] \arrow[rr, dashed, no head] \arrow[dr, "\alpha"] & & M_a\{1\} \arrow[dr, "\alpha"] & & \\
\cdots & M_b \arrow[ur, "\beta"'] \arrow[rr, dashed, no head] \arrow[dr, "\alpha"] & & M_b\{1\} \arrow[dr, "\alpha"] & \cdots \\
 & & M_c \arrow[ur, "\beta"'] \arrow[rr, dashed, no head] & & M_c\{1\} \\
 & & {} \arrow[rr, "\{1\}"'] & & {}
\end{tikzcd}$$

The natural relation is $\boxed{\alpha\beta = \beta\alpha}$,

so must take $\boxed{\begin{array}{l} \{1\} : \alpha \mapsto -\alpha \\ \qquad \beta \mapsto -\beta \end{array}}$ to obtain the scarf $\mathbb{Z}(A_{d-1})$

Also $\exists$ 'shift' $[1]$: e.g. $M_a \mapsto M_c\{1\}$ with $\boxed{[2] = \{d\}}$

Sample Calculations $\quad P_i = \mathbb{C}[x]\{i\}$, i.e. graded $\mathbb{C}[x]$ module with 1 in deg $-i$

$$
\begin{array}{cccccccccc}
M_a : & \cdots & P_0 & \xrightarrow{x} & P_1 & \xrightarrow{x^3} & P_4 & \cdots \\
\alpha \downarrow & & \downarrow 1 & & \downarrow x & & \downarrow 1 & \\
M_b : & \cdots & P_0 & \xrightarrow{x^2} & P_2 & \xrightarrow{x^2} & P_4 & \cdots \\
\beta \downarrow & & \downarrow x & & \downarrow 1 & & \downarrow x & \\
M_a\{1\} : & \cdots & P_1 & \xrightarrow{x} & P_2 & \xrightarrow{x^3} & P_5 & \cdots
\end{array}
$$

$$
\begin{array}{cccccccccc}
M_a & \cdots & P_0 & \xrightarrow{x} & P_1 & \xrightarrow{x^3} & P_4 & \cdots \\
\alpha\cdot\beta = \gamma \downarrow & & \downarrow x & & \downarrow x & & \downarrow x & \\
M_a\{1\} & \cdots & P_1 & \xrightarrow{x} & P_2 & \xrightarrow{x^3} & P_5 & \cdots
\end{array}
$$

Clearly $\boxed{\alpha\cdot\beta = \gamma = \beta\cdot\alpha}$ as reqd. $\boxed{\text{N.B. In this case } \beta\cdot\alpha = 0}$.

Also $M_a[1] = \cdots P_1 \xrightarrow{x^3} P_4 \xrightarrow{x} P_5 \cdots = M_c\{1\}$

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Why are A, D, E special?

Only Dynkin scarves have a shift $\boxed{[1] = \{h/2\} \cdot \nu}$,

defined by Coxeter number and Nakayama automorphism,

$$h = \begin{cases} n+1 & A_n \\ 2n-2 & D_n \\ 12, 18, 30 & E_{6,7,8} \end{cases} \qquad \nu = \begin{cases} \text{id} & D_{\text{even}} \quad E_{7,8} \\ \text{obvious involution} & A_n \quad D_{\text{odd}} \quad E_6 \end{cases}$$

such that $\boxed{\omega = \{-1\} \cdot [1]}$ is the Serre functor,

$$\mathrm{Hom}(A, B)^* \underset{\text{nat}}{\cong} \mathrm{Hom}(B, \omega A)$$

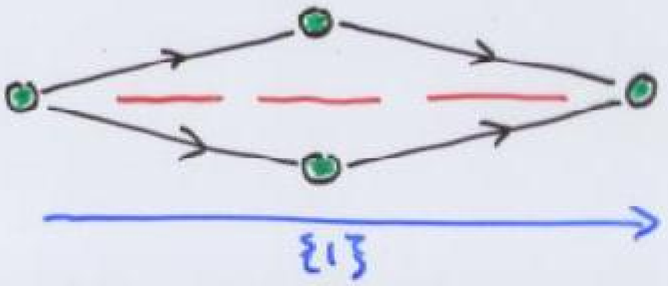
i.e. $\boxed{\tau = \{-1\}}$ is the Auslander-Reiten functor.

$$\mathrm{Hom}(A, B)^* \underset{\text{nat}}{\cong} \mathrm{Hom}(B, \tau A[1])$$

It is A-R duality that provides the meshes of the scarf:

If A is indecomposable,
then $\mathrm{Hom}(A,A)$ is a local algebra (unique max. ideal)
so $\mathrm{Hom}(A, \tau A[1]) \cong \mathrm{Hom}(A,A)^*$
has a unique element Z (up to normalisation)
which gives the mesh 'triangle':

$$\tau A \xrightarrow{x} \bigoplus_i B_i \xrightarrow{y} A \xrightarrow{Z} \tau A[1]$$

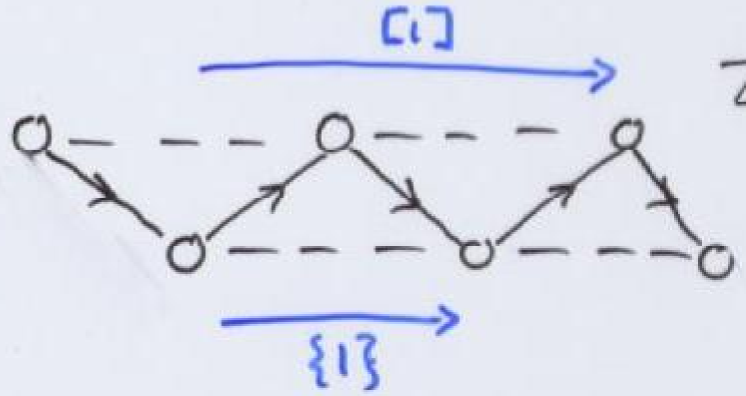


Triangles determine K-theory 'charge lattice'

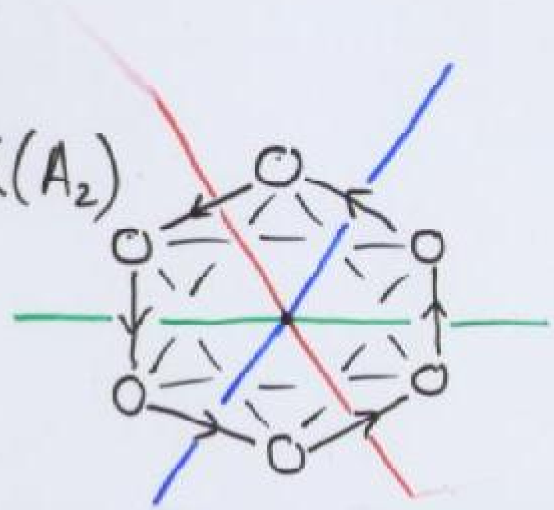
$$\mathbb{Z}(\Delta) \xrightarrow{\kappa} K(\Delta)$$

If $a \to b \to c \to a[1]$ is a triangle in $\mathbb{Z}(\Delta)$,
then $a + c = b$ in $K(\Delta)$

e.g. $\Delta = A_2$



$\mathbb{Z}(A_2) \longrightarrow K(A_2)$

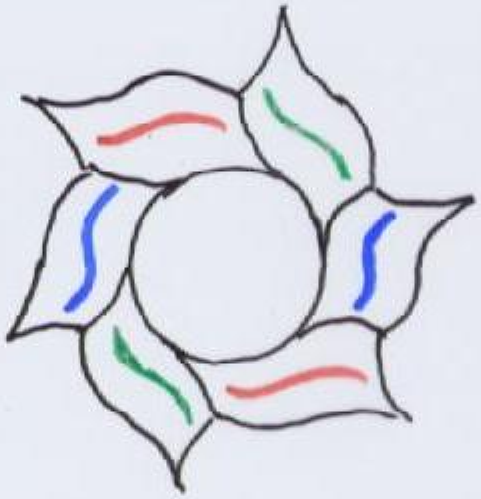


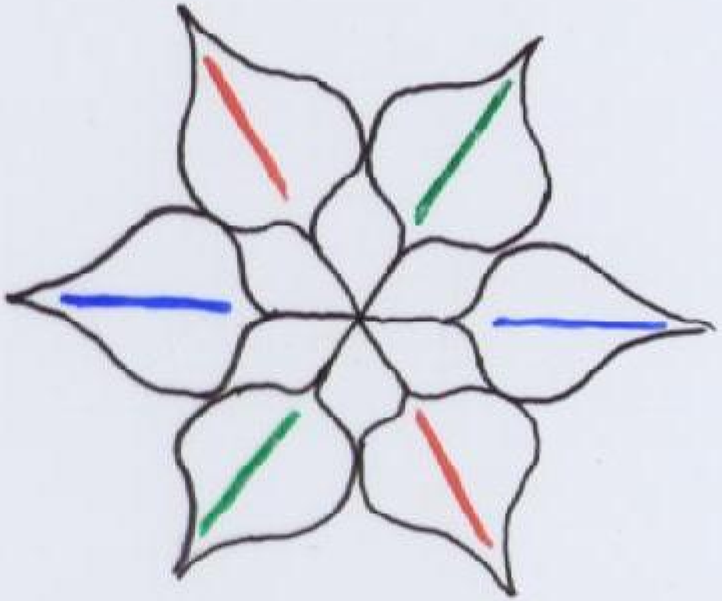
Always $\quad [1] \longmapsto -1 \quad$ i.e. π rotation

$\boxed{\{3\} = [2]} \Rightarrow \{1\} \longmapsto (-1)^{2/3}$ i.e. $\frac{2\pi}{3}$ rotation

N.B.
$\mathrm{Gal} = C_6$
$\Rightarrow K(A_2) \underset{\text{nat}}{\not\cong} K(SU_3)$

Designs on C13 Persian bowls in the Oriental Museum, Durham





'water weed'

C_6

minai-lajvardina

D_{12}

9+ε

"A mathematician is like
a blind man searching in a dark room
for a black cat that isn't there."

Charles Darwin

"A category theorist is like a beaver."

Terry Gannon

"A physicist is like
a woman searching in the real world
for a colourful cat."

"A category theorist is a 'naturalist'."

If symmetry comes from 'breaking naturality', then 'breaking symmetry' is 'restoring naturality'.

Can 'knit' the positive roots of Δ

as dimension vectors (K theory classes)

of indecomposable quiver representations

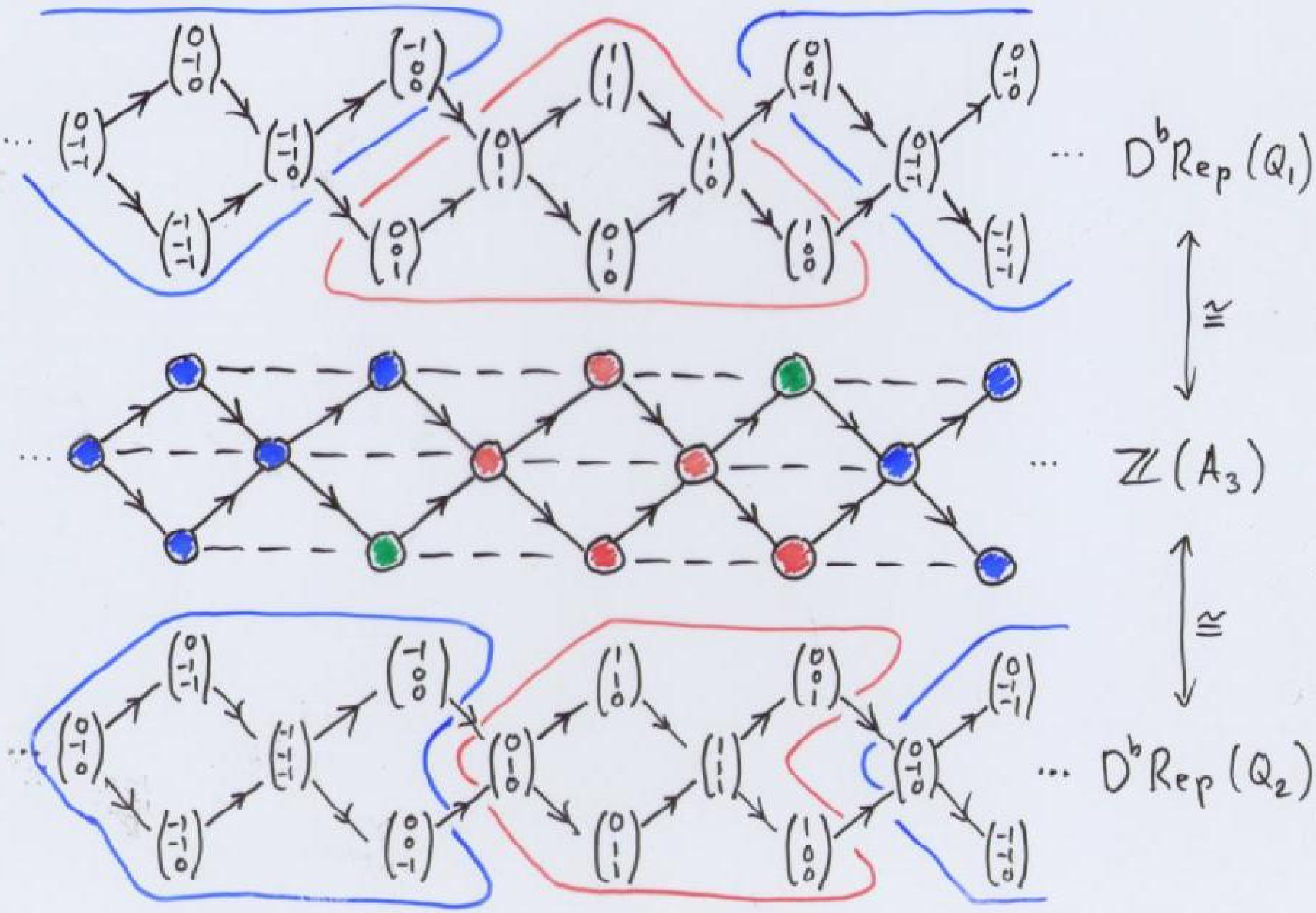
starting from the projectives.

$$Q_1 \qquad \begin{tikzcd} \bullet\, c \arrow[d] \\ \bullet\, b \arrow[d] \\ \bullet\, a \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & & P_c = \begin{pmatrix}1\\1\\1\end{pmatrix} \arrow[dr] & & \\ & P_b = \begin{pmatrix}0\\1\\1\end{pmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dashed, mapsto] & & \begin{pmatrix}1\\1\\0\end{pmatrix} \arrow[dr] & \\ P_a = \begin{pmatrix}0\\0\\1\end{pmatrix} \arrow[ur] \arrow[rr, dashed, mapsto] & & \begin{pmatrix}0\\1\\0\end{pmatrix} \arrow[ur] \arrow[rr, dashed, mapsto] & & \begin{pmatrix}0\\0\\1\end{pmatrix} \end{tikzcd} \qquad \mathrm{Rep}(Q_1)$$

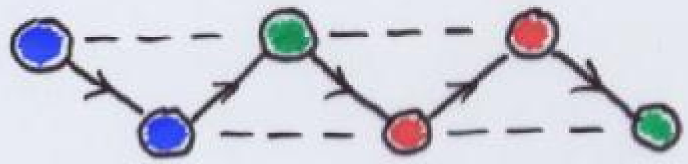
$$Q_2 \qquad \begin{tikzcd} \bullet\, c \arrow[d] \\ \bullet\, b \\ \bullet\, a \arrow[u] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & P_c = \begin{pmatrix}1\\1\\0\end{pmatrix} \arrow[dr] \arrow[rr, dashed, mapsto] & & \begin{pmatrix}0\\0\\1\end{pmatrix} \\ P_b = \begin{pmatrix}0\\1\\0\end{pmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dashed, mapsto] & & \begin{pmatrix}1\\1\\1\end{pmatrix} \arrow[ur] \arrow[dr] & \\ & P_a = \begin{pmatrix}0\\1\\1\end{pmatrix} \arrow[ur] \arrow[rr, dashed, mapsto] & & \begin{pmatrix}1\\0\\0\end{pmatrix} \end{tikzcd} \qquad \mathrm{Rep}(Q_2)$$

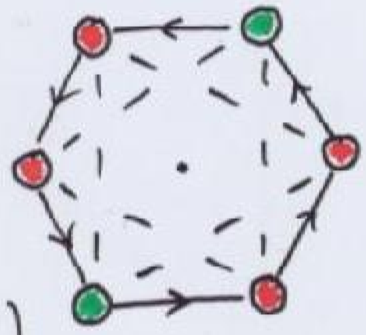
'Black magic' (you can keep knitting, both ways)

becomes 'tilting' (an elementary equivalence of derived categories).

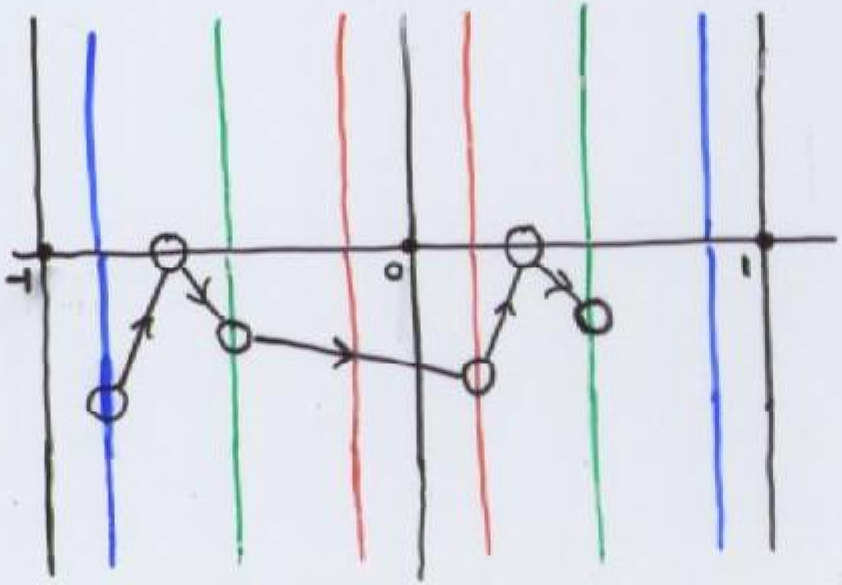


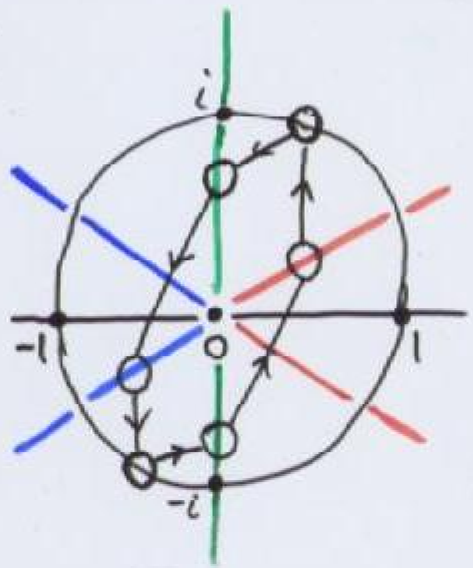
Tilting via π-stability [Douglas / Bridgeland]





$$\begin{tikzcd} \mathbb{Z}(A_2) \arrow[r] \arrow[d, "\Phi"'] & K(A_2) \arrow[d, "Z"] \\ \mathbb{C} \arrow[r] & \mathbb{C} \end{tikzcd}$$





$$Z = (-1)^{\Phi} = e^{i\pi\Phi}$$