Topological D-branes & Noncommutative Geometry

<u>Claim</u>: The tree-level open string dynamics of a system of topological D-branes is described by a cyclic & unital weak A_∞ -category.

$\underline{o}$: By an A_∞-category we mean a $\mathbb{Z}_2$-graded A_∞ category. This grading can be lifted to a $\mathbb{Z}$-grading provided that the TST has a non-anomalous worldsheet $U(1)$ symmetry.

<u>Proof</u>: CIL (2003) Herbst, Lerche & CIL (2004)
The statement follows by a worldsheet analysis of the constraints obeyed by properly regularized integrated boundary amplitudes on the disk (HLL 2004).

- weak A_∞ products r_n : describe $n+1$-point bdry functions on the disk
- (strict) A_∞ units = unit bdry observables for each D-brane
- bilinear pairings = topological metrics

* The category is <u>strong</u> if all products r_0 vanish & minimal if all $r_0, r_1 = 0$.

* A minimal unital & cyclic A_∞ category corresponds to a D-brane background which satisfies the string eqs. of motion

D: A finite A_∞ category is an A_∞ category $\mathcal{A}$ s.t.:

(1) $Ob\,\mathcal{A}$ = finite set

(2) $Hom_{\mathcal{A}}(a,b)$ = finite-dimensional $\mathbb{C}$-supervector space $\forall a,b \in Ob\,\mathcal{A}$

D: A finite topological D-brane system is a D-brane system whose tree-level open string dynamics is described by a finite A_∞-category.

Given a finite top. D-brane system $\mathcal{A}$, let $Q_0 = Ob\,\mathcal{A}$. Consider the finite-dimensional semisimple commutative algebra:

$$\boxed{R = \bigoplus_{u \in Q_0} \mathbb{C}\epsilon_u}$$

with $\epsilon_u \epsilon_v = \epsilon_u \delta_{uv}$ (commuting idempotents)

Let $\begin{cases} E_{uv} := Hom_{\mathcal{A}}(u,v) & \forall u,v \in Q_0 \\ E := \bigoplus_{u,v \in Q_0} E_{uv} & \end{cases}$ (a homogeneous decomposition of the $\mathbb{Z}_2$-graded vector space E)

Claim: The homog. decomposition $E = \bigoplus_{u,v \in Q_0} E_{uv}$ amounts to an R-superbimodule structure on E. Then $E_{uv} = \epsilon_u E \epsilon_v$ $\forall u,v \in Q_0$.

Pf: trivial

P: Giving a finite weak, cyclic & unital A_∞-category $\mathcal{A}$ amounts to giving:

① An R-superbimodule E s.t. $\dim_{\mathbb{C}} E < \infty$

② An R-bilinear & homogeneous symplectic form $\omega : E[1] \times E[1] \to R$ (whose degree we denote by $\tilde{\omega}$)

③ A weak A_∞-structure on E, i.e. a countable sequence of R-multilinear maps:

$$r_n : E^n \to E \qquad (\forall n \geq 0)$$

s.t.:

$$\sum_{1 \leq i \leq j \leq n} (-1)^{\tilde{x}_1 + \dots + \tilde{x}_i} r(x_1 \dots x_i, r(x_{i+1} \dots x_j), x_{j+1} \dots x_n) = 0$$

$\forall n \geq 0$: $\forall x_1 \dots x_n \in E[1]$ homogeneous of degrees $\deg x_i = \tilde{x}_i$

such that:

(I) $(r_n)_{n \geq 0}$ is strictly unital, i.e. $\exists \lambda \in E[1]$ ($\lambda = \Sigma 1$ with $1 \in E$, Σ = suspension operator) s.t.: (a) $r_n(x_1 \dots x_n) = 0$ if $n \neq 2$ and any of $x_j \in E[1]$ coincides with λ

(b) $-r_2(x, \lambda) = (-1)^{\tilde{x}} r_2(\lambda, x) = x \quad \forall x \in E[1]$ homogeneous

(II) $(r_n)_{n \geq 0}$ is ω-cyclic, i.e.:

$$\omega(x_0, r_n(x_1 \dots x_n)) = (-1)^{\tilde{x}_0 + \tilde{x}_1 + \tilde{x}_0(\tilde{x}_1 + \dots + \tilde{x}_n)} \omega(x_1, r_n(x_2 \dots x_n\, x_0))$$

$\forall$ homogeneous $x_0 \dots x_n \in E[1]$

o: Odd unit λ is unique & central
(i.e. $\lambda \in E[1]^R = \{x \in E[1] \mid \alpha x = x\alpha \ \forall \alpha \in R\}$)

Let $A = T_R V = \bigoplus_{n \geq 0} V^{\otimes_R n}$ where

$V = E[1]^\vee = \mathrm{Hom}_{R\text{-Mod}}(E[1]; R) = \underline{\mathrm{Hom}}_{R\text{-Smod}}(E[1], R)$

with "wrong" superbimodule structure:

$(\alpha f \beta)(x) = \alpha f(x\beta) \quad \forall f \in E[1]^\vee$

Let: $(\Omega_R(A), d)$ = relative differential cover of A
($\mathbb{N} \times \mathbb{Z}_2$-graded)

$$(C_R(A), \bar{d}) = \left(\Omega_R[A] \Big/ \underset{\substack{\| \\ \mathrm{im}\,[\cdot;\cdot]}}{[\Omega_R(A), \Omega_R(A)]} ; (\text{induced } d) = \bar{d} \right)$$

($[\cdot;\cdot]$ = supercommutator)

the relative Karoubi complex of A
($\mathbb{N} \times \mathbb{Z}_2$-graded)

P: $H^n_R(A) := H_{\bar{d}}(C^n_R(A)) = \begin{cases} R, & \text{if } n = 0 \\ 0, & \text{if } n > 0 \end{cases}$

(Kontsevich, V. Ginzburg, L. Le Bruyn)

ω induces $\hat{\omega} \in E[1]^\vee \otimes E[1]^\vee = V \otimes V$

and a noncommutative differential form:

$\omega_{\text{form}} \in C^2_R(A)$; $\omega_{\text{form}} = \sum_i \bar{d}x_i \otimes \bar{d}y_i$
if $\hat{\omega} = \sum_i x_i \otimes y_i$

which is symplectic of $\mathbb{Z}_2$-degree $\tilde{\omega}$.

Symplectic means the $\mathbb{C}$-linear map:

$\underbrace{\mathrm{Der}_R(A)}_{\substack{R\text{-linear } \underline{\text{left}} \\ \text{derivatives of A}}} \ni \Theta \longrightarrow i_\Theta \omega \in C^1_R(A)$ is bijective.

Thus (A, ω_{form}) = noncommutative symplectic <u>super</u>-mfol in the sense of Kontsevich.

$\Downarrow$

Have notions of:

- symplectic derivations $\mathrm{Der}_e^{\omega}(A) = \{\theta \in \mathrm{Der}_e(A) \mid \bar{L}_\theta \omega = 0\}$
  where $\bar{L}_\theta$ = Lie superderivative of θ (induced by a superderivation L_θ of $\Omega_R(A)$)

- Hamiltonian derivations:
  $$\theta_f = \Psi_\omega(f) = \text{Ham der with Ham } f \in C_R^0(A)$$
  is given by: $i_{\theta_f}\omega = \bar{d} f$

- $H_R(A)$ is acyclic in positive degrees $\Rightarrow$ any Hamiltonian derivation is symplectic.
  Have exact sequence of super-vector spaces:
  $$0 \rightarrow R \hookrightarrow C_R^0(A) \xrightarrow{\Psi_\omega} \mathrm{Der}_e^{\omega}(A)[\tilde{\omega}] \rightarrow 0$$
  Thus Hamiltonian of $\theta \in \mathrm{Der}_e^{\omega}(A)$ is determined up to addition of els. of R.

- Have <u>Kontsevich bracket</u> $\{\cdot,\cdot\} : C_R^0(A) \times C_R^0(A) \rightarrow C_R^0(A)$
  given by $\{f, g\} = i_{\theta_f} i_{\theta_g} \omega \quad \forall f, g \in C_R^0(A)$
  with properties:
  (1) $\{f, g\} = (-1)^{(\tilde{f}+\tilde{\omega})(\tilde{g}+\tilde{\omega})}\{g, f\}$
  (2) $(-1)^{(\tilde{f}+\tilde{\omega})(\tilde{g}+\tilde{\omega})}\{f,\{g,h\}\} + (-1)^{(\tilde{g}+\tilde{\omega})(\tilde{f}+\tilde{\omega})}\{g,\{h,f\}\} + (-1)^{(\tilde{h}+\tilde{\omega})(\tilde{g}+\tilde{\omega})}\{h,\{f,g\}\} = 0$

i.e. $\boxed{(C_R^0(A)[\tilde{\omega}], \{\cdot,\cdot\}) = \text{Lie superalgebra}}$

P: (1) A weak A_∞-structure on E is the same as as a nilpotent relative derivation $Q \in \mathrm{Der}_e(A)$,
$Q^2 = 0 \Leftrightarrow [Q, Q] = 0$

(2) The weak A_∞ str on E is cyclic iff $Q \in \mathrm{Der}_e^{\omega}(A)$, i.e. $L_Q \omega = 0$

Q: How to describe unitality?

D: The <u>noncommutative generating function</u> $W \in C^0_R(A)$ of the finite D-brane system is the canonical Hamiltonian of the symplectic derivation Q, i.e. that Hamiltonian which <u>vanishes at zero</u>.

O: $\forall f \in C^0_R(A)$, we have:

$f = \sum_{n \geq 0} f_n$ with $f_n \in C^0_R(A)_n$

nth hom. cp. of $C^0_R(A)$ wrt $\mathbb{N}$-grading induced from $A = T_R(V)$.

We say <u>f vanishes at zero if $f_0 = 0$</u>.

P: A cyclic weak A_∞ structure on A is the same as an element $W \in C^0_R(A)$ s.t.:

(1) W vanishes at zero

(2) $\{W, W\} = 0$

P: We have $C^2(A)_{closed} \simeq [A,A]^R$, the central subspace of $[A,A] \subset A$.
The isomorphism can be built explicitly, and the inverse isomorphism is denoted by

$$\kappa : [A,A]^R \to C^2(A)_{closed}$$

D. A two-form is called <u>constant</u> if it belongs to $C^2_R(A)_2$ (degree two component of $C^2_R(A)$ wrt $\mathbb{N}$-grading induced from A).

o: Clearly ω form is constant.

Restriction give isom κ_0: $\underbrace{[A,A]^R_2}_{\substack{\mathbb{N}\text{-degree two cp} \\ \text{of } [A,A]^R}} \xrightarrow{\sim} C^2(A)_2$

Let $\mathrm{Mom}_V = \{ u \in [A,A]^R_2 \mid u \text{ is non-degenerate} \}$
where $u = \sum_a [a, a^*]$ is non-degenerate if a^* is a basis of V as soon as a is a basis of V.

Mom_V is the <u>space of non-commutative moment maps</u>.

Let $C^2(A)_2^{sympl} \subset C^2(A)_2$ be the space of constant non commutative symplectic forms. Restriction of κ gives an isomorphism:

$$\boxed{\mathrm{Mom}_V \xrightarrow[\sim]{\kappa} C^2(A)_2^{sympl}}$$

Consider $A\otimes A$ with the outer A-superbimodule structure:

$$\alpha(a\otimes b)\beta = \alpha a\otimes b\beta \quad \forall \alpha,\beta,a,b\in A.$$

Fix $x\in E[1]$.

Let $D_x \in \mathrm{Der}_e(A; A\otimes A)$ be given by:

$$D_x \eta = \eta(x) \quad \forall \eta \in V = E[1]^\vee$$

Let $m: A\otimes A \to A$ be the $\mathbb{C}$-bilinear map:

$$m(a\otimes b) = (-1)^{\tilde{a}\tilde{b}}\, b a$$

We define the left cyclic derivative with respect to x via:

$$\vec{\partial}_x = m\circ D_x : A\to A$$

This vanishes on $[A,A]$ and thus descends to a map:

$$\vec{\partial}_x : C^0_R(A) \to A \quad \text{(denoted by the same letter)}$$

P: The weak cyclic A_∞ structure determined by W is unital iff $\exists\, \lambda\in E[1]^R$, $\tilde{\lambda}=0$ s.t. $\frac{1}{2}\vec{\partial}_\lambda W = \kappa^{-1}(\omega_{form}) \in \mathrm{Hom}_V$.

Thus $\omega_{form} = \frac{1}{2}\kappa(\vec{\partial}_\lambda W)$ is determined by W and λ.

I: Giving a weak cyclic & unital A_∞ structure on E (i.e. a finite topological D-brane system) amounts to giving an element $W \in C^0_R(A)$ such that:

(1) W vanishes at zero

(2) W is homogeneous of degree $\tilde{\omega}+1$

(3) $\frac{1}{2}\vec{\delta}_\lambda W \in \mathrm{Hom}_V^{\tilde{\omega}}$ (component of degree $\tilde{\omega}$ of Hom_V)

(4) $\{W, W\}=0$ with respect to the Kontsevich bracket determined by the constant noncommutative symplectic form $\omega = \kappa(\frac{1}{2}\vec{\delta}_\lambda W)$.

O: The A_∞ products etc can be reconstructed explicitly.

Symmetries:

P: An autoequivalence of the underlying A_∞ category amounts to a unital ~~superalgebra~~ relative superalgebra ~~endomorphism~~ automorphism $\varphi \in \mathrm{Aut}_R(A)$ [relative means $\varphi|_R = \mathrm{id}_R$].

The autoequivalence is cyclic (preserves the sympl forms) iff $\varphi^*(\omega)=\omega$ where φ^* is the map $C_R(A) \to C_R(A)$ induced by φ.

How to describe unital autoequivalences? Easiest in coords, will see later.

Adapted coords & superquivers

A homogeneous basis e_a of $E[1]$ is adapted if:

(1) $a = (u, v, i)$ is a multi-index with $u, v \in Q_0$, $i \in \overline{1, \dim_{\mathbb{C}} E_{uv}}$

(b) $e_{(u,v,i)} \in E_{uv}$ give a basis of E_{uv} when $i \in \overline{1, \dim_{\mathbb{C}} E_{uv}}$

A homogeneous basis determines an index superquiver Q with vertices $u \in Q_0$ & arrows $\underset{u}{\bullet} \xrightarrow{i} \underset{v}{\bullet}$ for $(u,v,i) = a$ with $i \in \overline{1, \dim_{\mathbb{C}} E_{uv}}$.

Thus we can identify multi-indices with the arrows a. Grading of Q is given by:

$$\deg : \underbrace{Q_1}_{\text{set of all arrow}} \to \mathbb{Z}_2 \quad , \deg a := \tilde{a} := \tilde{e}_a = \deg e_a$$

Clearly path algebra $\mathbb{C}Q$ is an $\mathbb{N} \times \mathbb{Z}_2$-graded algebra ($\mathbb{N}$-grading is by length of paths)

P (obvious): We have $\mathbb{C}Q \simeq T_R V$ as bigraded associative algebras. R identifies with the subalgebra of trivial paths.

For $x \in E[1]$, $x = \sum_{a \in Q_1} x^a e_a$, we have:

$$\vec{\partial}_x = \sum_{a \in Q_1} x^a \vec{\partial}_a \quad \text{where } \vec{\partial}_a := \vec{\partial}_{e_a}.$$

$\vec{\partial}_a$ are "quiver cyclic derivatives".

We let $\lambda_u \in E_{uu}[1]$ be the odd A_∞ units $(\lambda = \sum_u \lambda_u)$ and pick adapted basis s.t. λ_u are basis els. [Let σ_u be the dual basis els and $\sigma := \sum_u \sigma_u$

P: $C^0_R(A)$ identifies with the supervector space generated by all necklaces.

D: A necklace is called <u>null</u> if its root is odd & its period is even.

P: The set of <u>non-null</u> necklaces (including trivial paths) is a basis of $C^0_R(A)$.

O: Can give explicit description of $C^0_R(A)$, $C^1_R(A)$ & $C^2_R(A)$ closed, define cyclic coeffs etc.

An element $\varphi \in \mathrm{Aut}_R(A)$ is determined by $\varphi(a) \in A$ $(a \in Q_1)$.

P: Let $\varphi \in \mathrm{Aut}^\omega_R(A)$ a symplectomorphism (i.e. $\varphi \in \mathrm{Aut}_R(A)$ s.t. $\varphi^*(\omega\text{form}) = \omega\text{form}$). The associated A_∞-autoequivalence is unital iff:

(1) $\varphi(\sigma) = \sigma$

(2) $\varphi(a) =$ indep of σ for all $a \in Q_1 \setminus \{\sigma\}$.

This describes the group of symmetries of the topological D-brane system:

$G = \{\varphi \in \mathrm{Aut}^W_R(A) \mid \varphi$ satisfies (1) & (2)$\} = \{$unital & cyclic A_∞- autoequivalences of the D-brane category$\}$.

D: Let $\mathbb{C}[\mathfrak{X}] = A/\mathcal{J}$ where $\mathcal{J}$ = bilateral ideal ~~[illegible]~~ generated by $(\vec{\partial}_a W)_{a \in Q_1}$

View $\mathbb{C}[\mathfrak{X}]$ as the noncommutative coordinate ring of a "noncommutative superscheme", the noncommutative extended vacuum space of the D-brane system.

D: Let $\mathbb{C}[\mathcal{M}] = \mathbb{C}[\mathfrak{X}]^{\mathcal{G}} = (A/\mathcal{J})^{\mathcal{G}}$.

$\mathcal{M}$ is the noncommutative moduli space of the D-brane system.

One of the generators of $\mathcal{J}$ is:

$$\vec{\partial}_\sigma W = \vec{\partial}_\lambda W \propto \sum_{a \in Q_1} [a, a^*] := \mu$$

where $a^* = \sum_{b \in Q_1} W_{ab}\, b$.

Thus we are imposing the moment map condition $\mu = 0$; therefore, the construction of $\mathcal{M}$ is a noncommutative symplectic reduction.

Some examples of $\mathfrak{X}, \mathcal{M}$ are constructed in CL2005

C.I.L. Generalized complexes and string field theory, JHEP 0106 (2001) 052, hep-th/0102122 (31 pages)

C.I.L. Unitarity, D-brane dynamics and D-brane categories JHEP 0112 (2001) 031, hep-th/0102183 (16 pages)

C.I.L. Graded Lagrangians, exotic topological D-branes and enhanced triangulated categories, JHEP 0106 (2001) 064, hep-th/0105063 (31 pages)

C.I.L., R.Roiban, D.Vaman Graded Chern-Simons field theories and graded topological D-branes, JHEP 0204 (2002) 023, hep-th/0107063 (74 pages)

C.I.L. String field theory and brane superpotentials, JHEP 0110 (2001) 018, hep-th/0107162 (42 pages)

C.I.L., R.Roiban, Holomorphic potentials for graded D-branes, JHEP 0202 (2002) 038, hep-th/0110288 (47 pages)

C.I.L. An analytic torsion for graded D-branes, JHEP 0204 (2002) 023, hep-th/0111239 (28 pages)

C.I.L., R.Roiban Gauge-fixing, semiclassical approximation and potentials for graded Chern-Simons theories, JHEP 0203 (2002) 022, hep-th/0112029 (46 pages)

C.I.L. D-brane categories, Int.J. Mod. Phys A18 (2003) 5299-5335, hep-th/0305095 (37 pages)

M. Herbst, C.I.L., W. Lerche — Superpotentials, A_∞ relations and WDVV equations for open topological strings, JHEP 0502 (2005) 071, hep-th/0402110 (53 pages)

M. Herbst, C.I.L., W. Lerche — D-brane effective action and tachyon condensation in topological minimal models, JHEP 0503 (2005) 078, hep-th/0405138 (36 pages)

C.I.L. — On the non-commutative geometry of topological D-branes, hep-th/0507222 (66 pages)

C.I.L. On the boundary coupling of topological Landau-Ginzburg models, JHEP 0505 (2005) 037, hep-th/0312286

M. Herbst, C.I.L. Localization and traces in open-closed topological Landau-Ginzburg models, JHEP 0505 (2005) 044, hep-th/0404184

D. Orlov Triangulated categories of singularities and D-branes in Landau-Ginzburg models, math.AG/0302304

D. Orlov Triangulated categories of singularities and equivalences between Landau-Ginzburg models, math.AG/0503630

D. Orlov Derived categories of coherent sheaves and triangulated categories of singularities, math.AG/0503632