Permutation branes and linear matrix factorisations

on a joint project together with
Håkon Enger and Andreas Recknagel

1. Introduction

__Setup:__ • Landau-Ginzburg models on $\mathbb{C}^n$ with superpotentials $W(x_1,\dots,x_n) = x_1^{d_1} + \dots + x_n^{d_n}$

• at criticality: CFT realization in terms of tensorproducts of A-series $\mathcal{N}=2$-minimal models $\mathcal{M}_{d_1-2} \otimes \dots \otimes \mathcal{M}_{d_n-2}$

• a construction of certain class of B-type conformal boundary conditions in these models is available: permutation boundary conditions

__Goal:__ Construct all matrix factorisations associated to this class of boundary conditions.

Motivation:

- Relation between boundary conditions and matrix factorisations rather indirect
- Better understanding:
  - → new CFT constructions
  - → deformations
  - → stability

Why are LG-models interesting?

Orbifolds of Landau-Ginzburg models with quasi homogeneous superpotential W are via GLSM connected to non-linear σ-models on $\{W=0\}$ in weighted projective space.

2. Tensor products of minimal models

A-series-($\mathcal{N}=2$) minimal model $\mathcal{M}_k$

Hilbertspace $\mathcal{H}_k \cong \bigoplus_{[l,m,s] \in I_k} V_{[l,m,s]} \otimes \left(\bar{V}_{[l,m,s]} \oplus \bar{V}_{[l,m,s+2]} \right)$

$\mathcal{N}=2$-Super-Virasoro algebra $\mathcal{A}_k$ at central charge $c_k = \frac{3k}{k+2}$.

Bosonic part is coset $\mathcal{W}$ algebra $\frac{\widehat{su}(2)_k \oplus \hat{u}(1)_2}{\hat{u}(1)_{k+2}}$.

$I_k = \{ (l,m,s) \mid 0 \le l \le k;\ m \in \mathbb{Z}_{2(k+2)};\ s \in \mathbb{Z}_4;\ l+m+s \text{ even} \} / \sim$

$(l,m,s) \sim (k-l, m+k+2, s+2)$

$\mathbb{Z}_{k+2}$-action on $\mathcal{H}_k$:

$$g\big|_{V_{(l,m,s)} \otimes \bar{V}_{(l,\bar{m},\bar{s})}} = id \cdot e^{\frac{\pi i}{k+2}(m+\bar{m})}$$

$\to$ corresponds to action $X \mapsto \zeta X$, $\zeta^d = 1$ in LG-model

Tensor products

$$\mathcal{H}_{k_1,\dots,k_n} \cong \bigoplus_{\substack{[l_i;m_i;s_i]\in I_{k_i} \\ s_i - s_j \in 2\mathbb{Z}}} \bigotimes_{i=1}^{n} V_{[l_i;m_i;s_i]} \otimes \left(\overline{V}_{[l_i;m_i;s_i]} \oplus \overline{V}_{[l_i;m_i;s_i+2]} \right)$$

carries representation of $\mathcal{A}_{k_1,\dots,k_n} = \underbrace{\mathcal{A}_{k_1} \oplus \dots \oplus \mathcal{A}_{k_n}}_{\cup\!\mathsf{I} \; (N=2)_{\mathrm{diag}}}$.

3. Boundary Conditions

... can be (partly) encoded in

boundary states $\|B\rangle\rangle \in \mathcal{H}$,

they have to satisfy gluing conditions:

if boundary preserves subalgebra $\mathcal{A} \subseteq \mathcal{W}$ then there is $\Omega \in \mathrm{Aut}(\mathcal{A})$ such that $\|B\rangle\rangle$ intertwines the actions of $\mathcal{A} \subseteq \mathcal{W}$ and $\overline{\Omega(\mathcal{A})} \subseteq \overline{\mathcal{W}}$:

$$\forall \omega \in \mathcal{A}: \ (\omega - \overline{\Omega(\omega)}^{*})\,\|B\rangle\rangle = 0 .$$

$\longrightarrow$ Ishibashi states

irreducible representations

if $\mathcal{H} = \bigoplus_{(\lambda,\bar\lambda)\in I} V_\lambda \otimes \overline{V}_{\bar\lambda}$

then there is one Ishibashi state $|\lambda\rangle\rangle_\Omega$

for each $(\lambda, \overline{\Omega(\lambda)^{*}}) \in I$.

$$\|B\rangle\rangle = \sum_{(\lambda,\overline{\Omega(\lambda)^{*}})\in I}{}' \; B_\lambda \, |\lambda\rangle\rangle_\Omega$$

E.g. B-type boundary states in $\mathcal{M}_k$:

$$\Omega_B\left([l,m,\bar{s}]^+\right) = [l,-m,-\bar{s}] \doteq (l,m,s]$$

Ishibashi states: $\quad |[l,0,s]\rangle\rangle_B$

and there is a standard construction for the boundary states:

$\forall\, [\hat{L},\hat{M},\hat{S}] \in I_k$:

$$\|(\hat{L},\hat{M},\hat{S}]\rangle\rangle_B = \sqrt{k+2}\, \sum_{[l,0,s]\in I_k}{}' \frac{S_{(\hat{L}\hat{M}\hat{S})(l0s)}}{\sqrt{S_{(000)(l0s)}}}\, |[l,0,s]\rangle\rangle_B$$

To construct boundary state in this case is rather easy, because there are only finitely many Ishibashi states.

In tensor products $\mathcal{M}_{k_1,\dots,k_n} = \mathcal{M}_{k_1} \otimes \dots \otimes \mathcal{M}_{k_n}$ of minimal models, one is also interested in constructing boundary conditions, which only preserve the diagonal $N=2$ superVirasoro algebra $\subseteq \mathcal{A}_{k_1} \oplus \dots \oplus \mathcal{A}_{k_n}$.

$\longrightarrow$ ∞-tly many Ishibashi states

Construct boundary conditions preserving bigger algebra, e.g. the whole $\mathcal{A}_{k_1} \oplus \dots \oplus \mathcal{A}_{k_n}$. $\longrightarrow$ only finitely many Ishibashi states.

If all $\mathcal{A}_{k_i}$ are preserved separately, we arrive at the tensor product of boundary states in single $\mathcal{M}_{k_i}$. (Up to sector alignment.)

If some of the k_i are equal then one can also consider permutation gluing automorphisms $\Omega = \tau_B \sigma$, where σ permutes factors with equal k_i.

$\longrightarrow$ still finitely many Ishibashi states

$$\mathcal{M} = \underbrace{\mathcal{M}_4 \otimes \dots \otimes \mathcal{M}_4}_{n \text{ times}}$$

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ 2 & 3 & \dots & 1 \end{pmatrix}$$

Ishibashi states:

$$[l_i, m_i, s_i] = \tau_B \left(l_{\sigma(i)}, m_{\sigma(i)}, \bar{s}_{\sigma(i)} \right) \qquad i < n$$

$$= [l_{i+1}, -m_{i+1}, -\bar{s}_{i+1}]$$

$$[l_n, m_n, s_n] = [l_1, -m_1, -\bar{s}_1]$$

$$\Rightarrow l_i = l$$

$m_{2i+1} = m = -m_{2i}$, $m = 0$ for odd n.
$\bar{s}_{i+1} = -s_i$

n odd: $|[l, 0, s_1, \dots, s_n]\rangle\rangle_B^\sigma$ $\quad l + s_i$ even

n even: $|[l, m, s_1, \dots, s_n]\rangle\rangle_B^\sigma$ $\quad l + m + s_i$ even.

odd n:

$$\| \hat{L}, \hat{M}, \hat{S}_1, \ldots, \hat{S}_n \rangle\rangle^{G}_{B}$$

$$= \frac{\sqrt{k+2}}{2^{\frac{n-1}{2}}} \sum_{\substack{[\ell,0,s_1] \in I_k \\ s_i - s_1 \in 2\mathbb{Z}}}' \frac{S_{[\hat{L}\hat{M}\hat{S}_1][\ell,0,s_1]}}{\left(S_{[000][\ell 0 s_1]} \right)^{\frac{n}{2}}} e^{i\frac{\pi}{2} \sum_{i=1}^{n}{}' \hat{S}_i s_i} \,|[\ell,0,s_1,\ldots,s_n]\rangle\rangle^{G}_{B}$$

even n:

$$\| \hat{L}, \hat{M}, \hat{T}, \hat{S}_1, \ldots, \hat{S}_n \rangle\rangle^{G}_{B}$$

$$= \frac{1}{2^{\frac{n-1}{2}}} \sum_{\substack{[\ell,m,s_1] \in I_k \\ s_i - s_1 \in 2\mathbb{Z}}}' \frac{S_{[\hat{L},\hat{M}-2\hat{T},\hat{S}_1][\ell m s_1]}}{\left(S_{[000][\ell 0 s_1]} \right)^{\frac{n}{2}}} e^{i\frac{\pi}{2} \sum_{i=1}^{n}{}' \hat{S}_i \cdot s_i} \,|[\ell,m,s_1,\ldots,s_n]\rangle\rangle^{G}_{B}$$

Extract topological partition functions from boundary states:

n odd $\quad \alpha = \|L M S=0\rangle\rangle^{\mathcal{B}}_{B} \quad \alpha' = \|L' M' S'=0\rangle\rangle^{\mathcal{B}}_{B}$

$$tr_{\mathcal{H}_0^t(\alpha,\alpha')}(g^t) = \sum_{l_i \in \{0,\dots,k\}}{}' \, N_{LL'}^{l_1 * \dots * l_n} \, e^{\frac{2\pi i t}{(k+2)} \frac{1}{2}(\sum_i l_i - M + M')}$$

$$tr_{\mathcal{H}_1^t(\alpha,\alpha')}(g^t) = \sum_{l_i \in \{0,\dots,k\}}{}' \, N_{LL'}^{(k-l_1) * \dots * l_n} \, e^{\frac{2\pi i t}{(k+2)} \frac{1}{2}(\sum l_i - M + M' + k + 2)}$$

n even $\quad \alpha = \|L M T S=0\rangle\rangle^{\mathcal{B}}_{B} \quad \alpha' = \|L' M' T' S'=0\rangle\rangle^{\mathcal{B}}_{B}$

$$tr_{\mathcal{H}_0^t(\alpha,\alpha')}(g^t) = \sum_{l_i}{}' \, N_{LL'}^{l_1 * \dots * l_n} \, e^{\frac{2\pi i t}{k+2} \frac{1}{2}(\sum l_i - M + M')} \times \delta^{(2k+4)}_{M - M' - 2(T - T'), \sum_i (-1)^i l_i}$$

$$tr_{\mathcal{H}_1^t(\alpha,\alpha')}(g^t) = \sum_{l_i}{}' \, N_{LL'}^{(k-l_1) * \dots * l_n} \, e^{\frac{2\pi i t}{k+2} \frac{1}{2}(\sum l_i - M + M' + k + 2)} \times \delta^{(2k+4)}_{M - M' - 2(T - T'), \sum_i (-1)^i l_i - k - 2}$$

n even/odd: $\alpha = \|\lambda_1, \dots, \lambda_n, M, S=0\rangle\rangle^{id}_{B}$

$$\alpha' = \|\lambda'_1 M'_1 (T'), S'=0\rangle\rangle^{B}_{B}$$

$$tr_{\mathcal{H}^t_0(\alpha\alpha')}(g^t) = (k+2)^{\left[\frac{n-1}{2}\right]} \sum_{l \in \{0 \dots k\}}{}' \mathcal{N}_{\lambda_1 + \dots + \lambda_n, 2}{}^{l}\, e^{\frac{2\pi i t}{k+2} \frac{1}{2}(l - M + M')}$$

$$tr_{\mathcal{H}^t_1(\alpha\alpha')}(g^t) = (k+2)^{\left[\frac{n-1}{2}\right]} \sum_{l \in \{0 \dots k\}}{}' \mathcal{N}_{\lambda_1 + \dots + \lambda_n, 2}{}^{(k-l)}\, e^{\frac{2\pi i t}{k+2} \frac{1}{2}(l - M + M' + k + 2)}$$

$$\mathcal{N}_{rs}{}^{t} = \begin{cases} 1, & |r-s| \le t \le \min\{r+s, 2k-r-s\} \wedge r+s+t \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

are the $\widehat{su}(2)_k$ - fusion rules

$$r * s = \sum_t \mathcal{N}_{rs}{}^{t}\, t$$

4. Topological LG models (B-twisted)

Target space $\mathbb{C}^n$, superpotential $W \in A = \mathbb{C}[x_1, \dots, x_n]$

bulk chiral ring: $\mathcal{H}^t \cong A/(\partial W)$

boundary conditions: matrix factorisations

P_0, P_1 : free A-modules (graded)

$$P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0$$

$$p_1 p_0 = W \, id_{P_0} \qquad p_0 p_1 = W \, id_{P_1}$$

boundary chiral ring:

$$\tilde{H}(P,Q) = \mathrm{Hom}(P_1 \oplus P_0, Q_1 \oplus Q_0) \,, \quad d : \tilde{H} \to \tilde{H}$$

$$d\varphi = \begin{pmatrix} 0 & q_1 \\ q_0 & 0 \end{pmatrix} \varphi - \sigma(\varphi) \begin{pmatrix} 0 & p_1 \\ p_0 & 0 \end{pmatrix}$$

$\sigma = \pm\, id$ on even/odd part.

$$\mathcal{H}^t = H^*(\tilde{H}, d)$$

E.g. in even degree:

$$\begin{tikzcd} P_1 \arrow[r, shift left, "p_1"] \arrow[d, "\varphi_1"'] & P_0 \arrow[l, shift left, "p_0"] \arrow[d, "\varphi_0"] \\ Q_1 \arrow[r, shift left, "q_1"] & Q_0 \arrow[l, shift left, "q_0"] \end{tikzcd}$$

$(\varphi_1, \varphi_0) \in \ker(d)$

$\Longleftrightarrow$ diagram commutative

$$\begin{tikzcd} P_1 \arrow[r, shift left, "p_1"] \arrow[dr, "\psi_1"' near start] & P_0 \arrow[l, shift left, "p_0"] \arrow[dl, "\psi_0" near start] \\ Q_1 \arrow[r, shift left, "q_1"] & Q_0 \arrow[l, shift left, "q_0"] \end{tikzcd}$$

$$d(\psi_1, \psi_0) = (q_0 \psi_1 + \psi_0 p_1, q_1 \psi_0 + \psi_1 p_0)$$

$R = A/(w)$, $\hat{P}_i := P_i \otimes R \quad \hat{Q}_i := Q_i \otimes R$

$\longrightarrow$ 2 periodic chain complexes

$$\cdots \to \hat{P}_0 \xrightarrow{\hat{p}_0} \hat{P}_1 \xrightarrow{\hat{p}_1} \hat{P}_0 \xrightarrow{\hat{p}_0} \hat{P}_1 \xrightarrow{\hat{p}_1} \hat{P}_0 \to \operatorname{coker}(\hat{p}_1) \to 0$$

$$\mathcal{H}^t_{even}(P, Q) \cong \operatorname{Ext}^{2i}_R(\operatorname{coker}(\hat{p}_1), \operatorname{coker}(\hat{q}_1))$$

$i > 0$

$$\mathcal{H}^t_{odd}(P, Q) \cong \operatorname{Ext}^{2i-1}_R(\operatorname{coker}(\hat{p}_1), \operatorname{coker}(\hat{q}_1))$$

Example: $W(x) = x^d$; $A = \mathbb{C}(x)$

$$\mathcal{M}_\ell^\alpha = \left(P_1 = A(\alpha+\ell+1) \underset{x^{\ell+1}}{\overset{x^{d-\ell-1}}{\rightleftarrows}} A(\alpha) = P_0 \right)$$

$$\mathcal{H}_0^t(\mathcal{M}_\ell^\alpha, \mathcal{M}_{\ell'}^{\alpha'}) \cong \left(x^{\max\{0,\ell-\ell'\}}\mathbb{C}(x) \big/ x^{\min\{\ell+1,d-\ell'-1\}}\mathbb{C}(x) \right)(\alpha'-\alpha)$$

$$\mathcal{H}_1^t(\mathcal{M}_\ell^\alpha, \mathcal{M}_{\ell'}^{\alpha'}) \cong \left(x^{\max\{0,d-2-\ell-\ell'\}}\mathbb{C}(x) \big/ x^{d-1-\max\{\ell,\ell'\}}\mathbb{C}(x) \right)(\alpha'-\alpha-\ell-1)$$

boundary state $\| \ell, \ell-2\alpha, 0 \rangle\rangle_B \longmapsto \mathcal{M}_\ell^\alpha$

Tensor product of matrix factorisations

$$P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \qquad\qquad Q_1 \underset{q_0}{\overset{q_1}{\rightleftarrows}} Q_0$$

$$p_1 p_0 = W_1(x_1, \dots, x_m) \qquad\qquad q_1 q_0 = W_2(x_{m+1}, \dots, x_n)$$

$R = P \otimes Q$: is factorisation of $W_1 + W_2$

$$R_1 = P_1 \otimes Q_0 \oplus P_0 \otimes Q_1$$

$$R_0 = P_0 \otimes Q_0 \oplus P_1 \otimes Q_1$$

$$r_1 = \begin{pmatrix} p_1 \otimes id & -id \otimes q_1 \\ id \otimes q_0 & p_0 \otimes id \end{pmatrix}$$

$$r_0 = \begin{pmatrix} p_0 \otimes id & id \otimes q_1 \\ -id \otimes q_0 & p_1 \otimes id \end{pmatrix}$$

$M_{L_1}^{\alpha_1}(x_1) \otimes \dots \otimes M_{L_n}^{\alpha_n}(x_n)$ is matrix factorisation of $W(x_1, \dots, x_n) = x_1^d + \dots + x_n^d$.

→ It describes the twisted tensorproduct boundary conditions.

5. Linear Matrix Factorisations

$A = \mathbb{C}[x_1, \dots, x_n]$

$W \in A$ homogeneous of degree d

A linear matrix factorisation of size k of W is given by d matrices $\alpha_0, \dots, \alpha_{d-1} \in \mathrm{Mat}(k, k, A)$ linear in the x_i such that

$$\alpha_0 \cdot \ldots \cdot \alpha_{d-1} = W \mathbb{1}_k .$$

Out of these one obtains 2-term-factorisations

$1 \leq \ell \leq d-1$, π cyclic permutation of $\{0, \dots, d-1\}$:

$$p_0 = \alpha_{\pi(0)} \cdot \ldots \cdot \alpha_{\pi(\ell-1)} ,$$

$$p_1 = \alpha_{\pi(\ell)} \cdot \ldots \cdot \alpha_{\pi(d-1)}$$

Linear factorisations of $W(x_1,\dots,x_n) = x_1^d + \dots + x_n^d$

Examples: $n=1$ $\quad W(x_1) = x_1^d \qquad \alpha_i = X_1$

$n=2$ $\quad W(x_1,x_2) = x_1^d + x_2^d \quad \alpha_i = (X_1 - \eta \xi^i X_2)$

$\xi^d = 1,\ \eta^d = -1,$

Generalisation:

$$\left.\begin{aligned}(\varepsilon_1)_{i,j} &:= \xi^{i-1}\delta^{(d)}_{i+1,j}\\ (\varepsilon_2)_{i,j} &:= \xi^{i-1}\delta^{(d)}_{i,j}\\ (\varepsilon_3)_{i,j} &:= \delta^{(d)}_{i+1,j}\end{aligned}\right\} \quad \varepsilon_i\varepsilon_j = \begin{cases}\xi\,\varepsilon_j\varepsilon_i, & i<j\\ \xi^{-1}\varepsilon_j\varepsilon_i, & i>j\end{cases} \qquad \varepsilon_i^d = \mathbb{1}$$

$$\Delta_{d,1}(x_i) := 0, \qquad \Delta_{d,2}(x_i) := x_2,$$

$$\Delta_{d,n+2}(x_i) := \varepsilon_2 \otimes \Delta_{d,n}(x_i) + \varepsilon_3 \otimes \mathbb{1}\cdot x_{n+1} + \varepsilon_1 \otimes \mathbb{1} x_{n+2}$$

Thm: $\Delta_{d,n+2}^d = \mathbb{1} \otimes \Delta_{d,n}^d + \mathbb{1} \otimes \mathbb{1}\, x_{n+1}^d + \mathbb{1} \otimes \mathbb{1}\, x_{n+2}^d$

For general n, $W(x_1, \dots, x_n) = x_1^d + \dots + x_n^d$

$$\alpha_i = \left(x_1 \mathbb{1} - \eta\, \xi^i \beta_{d,n}(x_i) \right)$$

is size $d^{\left[\frac{n-1}{2}\right]}$ linear matrix factorisation of W.

→ Have been constructed in

[Backelin-Herzog-Sanders 1988]

with the help of generalised Clifford algebras

E.g. $n = 3$

$$\alpha_i = \begin{pmatrix} x_1 & \xi^i(x_2+x_3) & & & \\ & x_1 & \xi^i(x_2+\xi x_3) & & \\ & & x_1 & \ddots & \\ & & & \ddots & \ddots \\ \xi^i(x_2+\xi^{d-1}x_3) & & & & x_1 \end{pmatrix}$$

Note

$$[\alpha_i, \alpha_j] = 0 .$$

For $I \subset \{0,\dots,d-1\}$, $I^* := \{0,\dots,d-1\} \setminus I$,

$\alpha \in \mathbb{Z}_d$ define

$$M_I^\alpha := \left(p_0 = \prod_{i \in I} \alpha_i \,,\; p_1 = \prod_{i \in I^*} \alpha_i \right)$$

$$A^\gamma(|I|+\alpha) =: P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 := A^\gamma(\alpha)$$

where $\gamma = d^{[\frac{n-1}{2}]}$.

Equivalences:

n even: $\quad M_I^\alpha \not\cong M_{I'}^{\alpha'}$ for $I \neq I' \vee \alpha \neq \alpha'$

n odd: $\quad M_I^\alpha \cong M_{I'}^{\alpha'}$ iff $I = I' + a \wedge \alpha = \alpha'$

6. Matrix factorisations for permutation branes

$W = x_1^d + \dots + x_n^d\ , \quad \sigma = \begin{pmatrix} 1 & 2 & \dots & d \\ 2 & 3 & \dots & d & 1 \end{pmatrix}$

n odd:

$$\| L, M = L - 2\alpha, S_i = 0 \rangle\rangle^{\sigma}_{B} \longmapsto \mathcal{M}^{\alpha}_{\{0,\dots,L\}}$$

n even:

$$\| L, M = L - 2\alpha, T = m - \alpha, S_i = 0 \rangle\rangle^{\sigma}_{B} \longmapsto \mathcal{M}^{\alpha}_{\{0,\dots,L\}+m}$$

For $n=1$: Single variable case,

$n=2$: discussed also in [Brunner - Gaberdiel 2005]

Note: More matrix factorisations than boundary states!

Checks:

- Transformations under $x_i \mapsto \zeta^{\delta_{ij}} x_j$
- Transformations under $(x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$
- Topological "spectra".
- Bulk-1-point functions.

Tensor product factorisations and the Koszul complex

$$(q_0, q_1) = \mathcal{M}_{L_1}(x_1) \otimes \cdots \otimes \mathcal{M}_{L_n}(x_n)$$

$$A = \mathbb{C}[x_1, \ldots, x_n]$$

$$N = \mathbb{C}[x_1, \ldots, x_n] \big/ (x_1^{L_1+1}, \ldots, x_n^{L_n+1})$$

Koszul complex: $K_m := \Lambda^m V$,

$V = A^n$ with basis $(e_1, \ldots, e_n)$

$\delta : K_m \to K_{m-1}$

$$\delta(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} x_{i_j}^{L_{i_j}+1} e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_p}$$

$$0 \to K_n \xrightarrow{\delta} \cdots \xrightarrow{\delta} K_1 \xrightarrow{\delta} K_0 \longrightarrow N \longrightarrow 0$$

A-free resolution of N.

Want to construct $R = A/(w)$ - free resolution

$\sigma : K_m \to K_{m+1}$

$\omega \mapsto \left(\sum_1^l x_i^{d-l_i-1} e_i\right) \wedge \omega$

$\sigma\delta + \delta\sigma = W$

$T_i := A \cdot t^i$ $\qquad$ $\tau : T_i \to T_{i-1}$
$\qquad\qquad\qquad t^i \mapsto t^{i-1}$

$$\tilde{F}_i = \bigoplus_{j=0}^{[\frac{i}{2}]} K_{i-2j} \otimes T_j$$

$\tilde{\delta} := \delta \otimes id + \sigma \otimes \tau$ $\qquad$ $\tilde{\delta}^2 = W \otimes \tau$

$\hat{F}_i := \tilde{F}_i \otimes R$

$$\ldots \to \hat{F}_e \to \hat{F}_{e-1} \to \ldots \to \hat{F}_2 \to \hat{F}_1 \to \hat{F}_0 \to N \to 0$$

R- free resolution of N.

2-periodic from position $(n-1)$!

Claim: 2-periodic part is equivalent to (q_0, q_1)

2-periodic part:
$$\Phi_0 \cong \bigoplus_j \Lambda^{n-2j} V$$
$$\Phi_1 \cong \bigoplus_j \Lambda^{n-1-2j} V$$
$$\varphi_0 = \delta + \delta : \Phi_0 \to \Phi_1$$
$$\varphi_1 = \delta + \delta : \Phi_1 \to \Phi_0.$$

has the structure of a tensor product.

For only one variable $n=1$:

$$\Phi_0 \cong \Lambda^1 A \cong \mathbb{C}[x] \qquad \varphi_0 = x^{l+1}$$
$$\Phi_1 \cong \Lambda^0 A \cong \mathbb{C}[x] \qquad \varphi_1 = x^{d-l-1}.$$

This can be used to calculate

$$Ext^i_R(\operatorname{coker}(q_0) \otimes R, \cdot) \cong Ext^{i+n}_R(N, \cdot)$$

Permutation	M	Charges				Chern characters			
		$D6$	$D4$	$D2$	$D0$	rk	ch_1	ch_2	ch_3
(1)(2)(3)(4)(5)	0	1	0	0	0	1	0	0	0
	2	−4	−1	−8	5	−4	1	5/2	5/6
	4	6	3	19	−10	6	−3	−5/2	5/2
	6	−4	−3	−14	10	−4	3	−5/2	−5/2
	8	1	1	3	−5	1	−1	5/2	−5/6
(1)(2)(3)(45)	0	3	2	11	−6	3	−2	0	7/3
	2	−1	−1	−3	4	−1	1	−5/2	−1/6
	4	0	0	0	−1	0	0	0	−1
	6	1	0	0	−1	1	0	0	−1
	8	−3	−1	−8	4	−3	1	5/2	−1/6
(1)(2)(345)	0	0	0	−5	0	0	0	5	0
	2	−5	0	−5	5	−5	0	5	5
	4	10	5	35	−15	10	−5	−15/2	35/6
	6	−5	−5	−20	15	−5	5	−15/2	−35/6
	8	0	0	−5	−5	0	0	5	−5
(1)(23)(45)	0	0	0	−1	0	0	0	1	0
	2	−1	0	−1	1	−1	0	1	1
	4	2	1	7	−3	2	−1	−3/2	7/6
	6	−1	−1	−4	3	−1	1	−3/2	−7/6
	8	0	0	−1	−1	0	0	1	−1
(1)(2345)	0	5	4	22	−10	5	−4	0	20/3
	2	0	−1	2	5	0	1	−15/2	5/6
	4	0	−1	−3	0	0	1	−5/2	−25/6
	6	0	−1	−8	0	0	1	5/2	−25/6
	8	−5	−1	−13	5	−5	1	15/2	5/6
(12)(345)	0	5	4	22	−10	5	−4	0	20/3
	2	0	−1	2	5	0	1	−15/2	5/6
	4	0	−1	−3	0	0	1	−5/2	−25/6
	6	0	−1	−8	0	0	1	5/2	−25/6
	8	−5	−1	−13	5	−5	1	15/2	5/6
(12345)	0	−5	0	−25	0	−5	0	25	0
	2	−5	5	15	0	−5	−5	25/2	125/6
	4	20	10	80	−25	20	−10	−25	50/3
	6	−5	−10	−30	25	−5	10	−25	−50/3
	8	−5	−5	−40	0	−5	5	25/2	−125/6

Table 1: Charges and Chern characters of $L = 0$ permutation branes for the quintic