CORRELATION FUNCTIONS FOR LATTICE EXACTLY SOLVABLE MODELS.

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A recursion formula for the correlation functions of an inhomogeneous XXX model. (hep-th/0405044)
Reduced qKZ equation and correlation functions of the XXZ model (hep-th/0412191)
Traces on the Sklyanin algebra and correlation functions of the eight-vertex model (hep-th/0504072)
Density matrix of a finite sub-chain of the Heisenberg anti-ferromagnet (hep-th/0506171)

The model.

I shall consider the Heisenberg antiferromagnet with the Hamiltonian

$$H = \sum_{i=-\infty}^{\infty} \left(\sigma_i^1\sigma_{i+1}^1 + \sigma_i^2\sigma_{i+1}^2 + \sigma_i^3\sigma_{i+1}^3\right)$$

$\left(\mathbb{C}^2\right)^{\otimes\infty}$ is bad, we need to extract a separable subspace.

In order to do that we follow the procedure

$$H = \lim_{N\to\infty} H_N$$

$$H = \sum_{i=-N-1}^{N} \left(\sigma_i^1\sigma_{i+1}^1 + \sigma_i^2\sigma_{i+1}^2 + \sigma_i^3\sigma_{i+1}^3\right),$$

$$\sigma_{N+1}^a = \sigma_{-N-1}^a$$

The goal: to find the ground state $|\text{vac}\rangle$ and Fock space over it.

R-matrix:

$$R(\lambda) = \frac{\rho(\lambda)}{\lambda + 1} (\lambda + P) \quad \in \mathrm{End}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$$

where

$$\rho(\lambda) = -\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{1}{2} - \frac{\lambda}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\lambda}{2}\right)}.$$

Considering tensor product of several spaces $V_i \simeq \mathbb{C}^2$ we write $R_{i,j}(\lambda)$ for R-matrix acting non-trivially in $V_i \otimes V_j$.

Yang-Baxter equation:

$$R_{1,2}(\lambda_{1,2})R_{1,3}(\lambda_{1,3})R_{2,3}(\lambda_{2,3}) = R_{2,3}(\lambda_{2,3})R_{1,3}(\lambda_{1,3})R_{1,2}(\lambda_{1,2})$$

We always imply

$$\lambda_{i,j} = \lambda_i - \lambda_j$$

Transfer-matrix.

$$t_N(\lambda) = \mathrm{tr}_{V_a}\left(R_{a,-N-1}(\lambda)R_{a,-N}(\lambda)\cdots R_{a,N-1}(\lambda)R_{a,N}(\lambda)\right)$$

Due to Yang-Baxter

$$[t(\lambda_1), t(\lambda_2)] = 0$$

Moreover

$$t_N(0) = U_N, \qquad U_N\ \sigma_j^a\ U_N^{-1} = \sigma_{j+1}^a$$

$$\frac{d}{d\lambda}\log t_N(\lambda) = \sum_{n=1}^{\infty}\lambda^n I_{N,n}\ , \quad I_{N,1} = H_N$$

Transfer-matrix can be diagonalised by Bethe ansatz.

Spectrum.

Ground state $|\mathrm{vac}\rangle$. Magnon is spin-1/2 particle parametrised by rapidity β,

$$u(\beta) = \tanh\frac{1}{2}\left(\beta + \frac{\pi i}{2}\right), \quad e(\beta) = \frac{1}{\cosh(\beta)}$$

Factorised scattering, two-particle S-matrix:

$$S_{1,2}(\beta_{1,2}) = R_{1,2}\left(-\frac{\beta_{1,2}}{\pi i}\right)$$

Basis:

$$|\mathrm{vac}\rangle, \quad |\beta_1, \cdots, \beta_n\rangle_{\epsilon_1,\cdots,\epsilon_n}$$

with $\beta_1 < \cdots < \beta_n$

$$_{\epsilon'_1,\cdots,\epsilon'_m}\langle\beta'_1, \cdots, \beta'_m|\beta_1, \cdots, \beta_n\rangle_{\epsilon_1,\cdots,\epsilon_n} = \delta_{m,n}\prod_{j=1}^{n}\delta(\beta_j - \beta'_j)\ \delta_{\epsilon_j}^{\epsilon'_j}$$

Form factors.
The matrix elements

$$ {}_{\epsilon'_1,\cdots,\epsilon'_m}\langle\beta'_1,\cdots,\beta'_m|\sigma_k^a|\beta_1,\cdots,\beta_n\rangle_{\epsilon_1,\cdots,\epsilon_n} $$

can be explicitly calculated. The algebra of local spins is represented in the Fock space.

Correlation functions.

Consider an operator

$$ \mathcal{O}=\sigma_{i_1}^{a_1}\sigma_{i_2}^{a_2}\cdots\sigma_{i_k}^{a_k} $$

$$ i_1<i_2<\cdots<i_k $$

The length of operator $l(\mathcal{O})=i_k-i_1+1$. The problem is to find

$$ \langle\mathrm{vac}|\ \mathcal{O}\ |\mathrm{vac}\rangle $$

L-operator.

Let $\{S_a\}_{a=1}^3$ be a basis of $\mathfrak{sl}_2$ satisfying $[S_a, S_b] = 2i\epsilon_{abc}S_c$. Define the L-operator which belongs to $U(\mathfrak{sl}_2) \otimes \mathbb{C}^2$:

$$L(\lambda) = \frac{\rho(\lambda, d)}{\lambda + \frac{d}{2}} L^{(0)}(\lambda),$$

$$L^{(0)}(\lambda) = \lambda + \frac{1}{2} + \frac{1}{2}\sum_{a=1}^{3} S_a \otimes \sigma^a,$$

where d is related to the Casimir operator as

$$C = \sum_{a=1}^{3} S_a S^a = d^2 - 1$$

and

$$\rho(\lambda, d) = -\frac{\Gamma\left(\frac{1}{2} - \frac{d}{4} + \frac{\lambda}{2}\right)\Gamma\left(1 - \frac{d}{4} - \frac{\lambda}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{d}{4} - \frac{\lambda}{2}\right)\Gamma\left(1 - \frac{d}{4} + \frac{\lambda}{2}\right)}.$$

In this normalisation we have the unitarity and crossing symmetry in the form

$$L(\lambda)L(-\lambda) = 1, \quad \sigma^2 \left(L(\lambda)\right)^t \sigma^2 = -L(-\lambda - 1).$$

Yang-Baxter equation:

$$R_{1,2}(\lambda_{1,2})L_1(\lambda_1)L_2(\lambda_2) = L_2(\lambda_2)L_1(\lambda_1)R_{1,2}(\lambda_{1,2})$$

Monodromy matrix.

$$T(\lambda) = \cdots L_1(\lambda)L_2(\lambda)\cdots \in \left(\mathbb{C}^2\right)^{\otimes\infty} \otimes U(\mathfrak{sl}_2)$$

Trace functional.

We define "trace over a space of fractional dimension". By this we mean the unique $\mathbb{C}[d]$ linear map

$$\mathrm{Tr}_d \ : \ U(\mathfrak{sl}_2)\otimes\mathbb{C}[d] \longrightarrow \mathbb{C}[d]$$

such that for any non-negative integer k we have

$$\mathrm{Tr}_{k+1}(A) = \mathrm{tr}_{V^{(k)}}\pi^{(k)}(A) \qquad (A\in U(\mathfrak{sl}_2)).$$

Here tr in the right hand side stands for the usual trace over $(k+1)$-dim irrep $\pi^{(k)}$. We list some properties of the trace function Tr_d.

$$\begin{aligned}
&\mathrm{Tr}_d(AB) = \mathrm{Tr}_d(BA), \quad \mathrm{Tr}_d(1) = d,\\
&\mathrm{Tr}_d(A) = 0 \text{ if } A \text{ has non-zero weight},\\
&\mathrm{Tr}_d(e^{zH}) = \frac{\sinh(dz)}{\sinh z},\\
&\mathrm{Tr}_d\,(CA) = (d^2-1)\mathrm{Tr}_d(A), \quad (A\in U(\mathfrak{sl}_2)\otimes\mathbb{C}[d]).
\end{aligned}$$

By the generating series the traces $\mathrm{Tr}_d(H^a)$ are known, $\mathrm{Tr}_d(H^aE^bF^c)$ is reduced to them inductively for all $a, b, c \geq 0$. We emphasise that $\mathrm{Tr}_d(A)$ is determined by the 'dimension' $\mathrm{Tr}_d(1) = d$ and the value of the Casimir operator; we have

$$\mathrm{Tr}_d(A) = \mathrm{Tr}_d(A') \text{ if } \varpi_d(A) = \varpi_d(A'),$$

where ϖ_d is the projection

$$\varpi_d : U(\mathfrak{sl}_2) \otimes \mathbb{C}[d] \to U(\mathfrak{sl}_2) \otimes \mathbb{C}[d]/I_d$$

and I_d signifies the two-sided ideal of $U(\mathfrak{sl}_2) \otimes \mathbb{C}[d]$ generated by $C - (d^2 - 1)$.

The following are simple consequences of these rules.

$\mathrm{Tr}_{-d}(A) = -\mathrm{Tr}_d(A)$,

$\mathrm{Tr}_d(A) - d\varepsilon(A) \in d(d^2 - 1)\mathbb{C}[d]$,

$\varepsilon : U(\mathfrak{sl}_2) \otimes \mathbb{C}[d] \to \mathbb{C}[d]$ stands for the counit,

The degree of $\mathrm{Tr}_d(H^aE^bF^c)$

is at most $m + 1$ (m even)

or m (m odd) where $m = a + b + c$.

Formally,

$$t(\lambda) = \mathrm{Tr}_2\left(T(\lambda)\right)$$

Consider tensor product of several copies of $U(\mathfrak{sl}_2)$. We define

$$\mathrm{Tr}_{d_1,\cdots,d_k}(A_1 \otimes \cdots \otimes A_k) = \prod_{i=1}^{k} \mathrm{Tr}_{d_i}(A_i)$$

Main formula.
Define

$$\varphi(\lambda) = \frac{\lambda}{\lambda^2 - 1}\left(\frac{d}{d\lambda}\log\rho(\lambda) + \frac{1}{2(\lambda^2-1)}\right)$$
$$= \frac{\lambda}{\lambda^2 - 1}\left(\sum_{k=1}^{\infty}(-1)^k\frac{2k}{\lambda^2 - k^2} + \frac{1}{2(\lambda^2-1)}\right)$$

Ajoint action of monodromy matrix on operators:

$$\mathcal{T}(\lambda)(\mathcal{O}) = T(\lambda)\cdot\mathcal{O}\cdot T(\lambda)^{-1}$$

Notice that in $T(\lambda)$ only finite piece of length $l(\mathcal{O})$ is relevant.

Our result is

$$\langle \text{vac}|\mathcal{O}|\text{vac}\rangle = \frac{1}{2^\infty}\ \text{tr}_{\left(\mathbb{C}^2\right)^{\otimes\infty}}\left(\rho(\mathcal{O})\right), \qquad \rho = e^{\Omega}$$

with

$$\Omega = \frac{1}{2}\iint \frac{d\mu_1}{2\pi i}\frac{d\mu_2}{2\pi i}\ \varphi(\mu_{1,2})$$
$$\times\ \text{Tr}_{2,2,\mu_{1,2}}\Big(\big(P^-\otimes I\big)\left(\mathcal{T}(\mu_1)\otimes\mathcal{T}(\mu_2)\otimes\mathcal{T}\big(\frac{\mu_1+\mu_2}{2}\big)\right)\Big),$$

integrals go around $\mu_1, \mu_2 = 0$. Generally, the singularities are as follows

$\frac{1}{(\mu_1\mu_2)^{l(\mathcal{O})}}$ from $\mathcal{T}\big(\frac{\mu_1+\mu_2}{2}\big)$,
$\frac{1}{(\mu_1^2-1)^{l(\mathcal{O})}}$ from $\mathcal{T}(\mu_1)$
$\frac{1}{(\mu_2^2-1)^{l(\mathcal{O})}}$ from $\mathcal{T}(\mu_2)$

Alternatively,

$$\langle \text{vac}|\mathcal{O}|\text{vac}\rangle = \frac{1}{2^{l(\mathcal{O})}}\ \text{tr}_{\left(\mathbb{C}^2\right)^{\otimes l(\mathcal{O})}}\left(\rho_{l(\mathcal{O})}(\mathcal{O})\right)$$

Density matrix.

Consider a finite sub-chain of length n in infinite environment.

$$\rho_n(\mathcal{O}) = \sum A_i \mathcal{O} B_i$$

then

$$\langle \mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle = \frac{1}{2^n}\mathrm{tr}_{(\mathbb{C}^2)^{\otimes n}}\left(\widehat{\rho}_n \mathcal{O}\right)$$

where

$$\widehat{\rho}_n = \sum B_i A_i$$

$\widehat{\rho}_n$ can be described as follows. It is easy to see that

$$\widehat{\rho}_n = e^{\widehat{\Omega}_n}(I)$$

where

$$\widehat{\Omega}_n = \frac{1}{2}\iint \frac{d\mu_1}{2\pi i}\frac{d\mu_2}{2\pi i}\ \varphi(\mu_{1,2})\mathrm{Tr}_{\mu_{1,2},2,2}\Big(\big(\mathcal{T}_n\big(\frac{\mu_1+\mu_2}{2}\big)^{-1}\otimes \mathcal{T}_n(\mu_2)^{-1}\otimes \mathcal{T}_n(\mu_1)^{-1}\big)\left(P^-\otimes I\right)\Big)\,.$$

Structure of the result.
Taylor expansion of $\varphi(\lambda)$:

$$\varphi(\lambda) = \sum_{l=0}^{\infty} \lambda^{2l+1} \varphi_{2l+1},$$

$$\varphi_{2l+1} = \frac{l+1}{2} - \sum_{p=0}^{l} \zeta_a(2p+1)$$

where the alternating ζ-function is

$$\zeta_a(s) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^s} = \left(1 - 2^{-s+1}\right) \zeta(s)$$

Generating function:

$$U(\alpha_1,\alpha_2) = \frac{1}{2}\iint \frac{d\mu_1}{2\pi i}\frac{d\mu_2}{2\pi i}\ \frac{1}{(\alpha_1-\mu_1)(\alpha_2-\mu_2)}$$
$$\times\,\mathrm{Tr}_{2,2,\mu_{1,2}}\Big(\big(P^-\otimes I\big)\,\big(\mathcal{T}(\mu_1)\otimes\mathcal{T}(\mu_2)\otimes\mathcal{T}\big(\frac{\mu_1+\mu_2}{2}\big)\big)\Big)\,,$$
$$=\sum_{k_1,k_2\geq 0}\alpha_1^{-k_1-1}\alpha_2^{-k_2-1}\ U_{k_1,k_2}$$

where α_i are outside of the contour of integration.
Formally,

$$\Omega = \int\frac{d\alpha_1}{2\pi i}\int\frac{d\alpha_1}{2\pi i}\ \varphi(\alpha_{1,2})\ U(\alpha_1,\alpha_2)$$

Important properties of the generating function.

1. Commutativity:

$$\Big[U(\alpha_1,\alpha_2),U(\alpha_3,\alpha_4)\Big]=0$$

2. Nilpotency:

$$U\big(\alpha_1,\alpha_2\big)U\big(\alpha_3,\alpha_4\big)\cdots U\big(\alpha_{\,2\left[\frac{l(\mathcal{O})}{2}\right]+1},\alpha_{\,2\left[\frac{l(\mathcal{O})}{2}\right]+2}\big)=0$$

The proof will be given later. From these formulae one finds

$$\Omega=\sum_{l=0}^{l(\mathcal{O})-2}\varphi_{2l+1}\Omega_{2l+1}$$

where the commuting family of nilpotent operators Ω_{2l+1} is defined by

$$\Omega_{2l+1}=\sum_{k=0}^{2l+1}\binom{2l+1}{k}U_{k,2l+1-k}$$

So, the functions $\zeta(2l+1)$ are present up to $l=l(\mathcal{O})-2$. Together with nilpotency it means that

$$\langle \mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle = \sum_{k_i \geq 0} \sum_{\sum k_i \leq \left[\frac{l(\mathcal{O})}{2}\right]} r_{k_0,\cdots,k_{l(\mathcal{O})-2}} \prod_{i=0}^{l(\mathcal{O})-2} \zeta_a(2i+1)^{k_i}$$

Inhomogeneous model. Consider the model whose integrals of motion are given by the transfer-matrix:

$$t(\lambda,\lambda_1,\cdots,\lambda_n)=\lim_{N\to\infty}t_N(\lambda,\lambda_1,\cdots,\lambda_n)$$
$$t_N(\lambda,\lambda_1,\cdots,\lambda_n)=\mathrm{tr}_{V_a}\big(R_{a,-N-1}(\lambda)\cdots R_{a,0}(\lambda)$$
$$\times R_{a,1}(\lambda-\lambda_1)\cdots R_{a,n}(\lambda-\lambda_n)$$
$$\times R_{a,n+1}(\lambda)\cdots R_{a,N}(\lambda)\big)$$

Consider the operators localised at the sub-chain $1,\cdots,n$. Corresponding operator $\rho_n(\lambda-1,\cdots,\lambda_n)$ depends oh λ_j, it is given by the same formula as before with

$$\Omega(\lambda_1,\cdots,\lambda_n)=$$
$$=\frac{1}{2}\iint\frac{d\mu_1}{2\pi i}\frac{d\mu_2}{2\pi i}\,\varphi(\mu_{1,2})\mathrm{Tr}_{2,2,\mu_{1,2}}\big((P^-\otimes I)$$
$$\big(\mathcal{T}_n(\mu_1,\lambda_1,\cdots,\lambda_n)\otimes\mathcal{T}_n(\mu_2,\lambda_1,\cdots,\lambda_n)\otimes\mathcal{T}_n\big(\frac{\mu_1+\mu_2}{2},\lambda_1,\cdots,\lambda_n\big)\big)\big),$$

the integrals are taken around $\lambda_1, \cdots, \lambda_n$. Obviously,

$$\Omega(\lambda_1, \cdots, \lambda_n) = \sum_{i<j} \Omega^{(i,j)}(\lambda_1, \cdots, \lambda_n)$$

where

$$\Omega^{(i,j)}(\lambda_1, \cdots, \lambda_n) = \mathrm{res}_{\mu_1=\lambda_i} \mathrm{res}_{\mu_2=\lambda_j} (\text{integrand})$$

Jimbo-Miwa equations.

In inhomogeneous case $\rho_n(\lambda_1, \cdots, \lambda_n)$ must satisfy certain system of equations. Let

$$A_n(\lambda_1, \cdots, \lambda_n) = R_{1,2}(\lambda_{1,2}) \cdots R_{1,n}(\lambda_{1,n})$$

and

$$\begin{aligned} &\mathcal{A}_n(\lambda_1, \cdots, \lambda_n)(\mathcal{O}) \\ &= A_n(\lambda_1, \cdots, \lambda_n)\ \sigma_1^2 (\mathcal{O})^{t_1} \sigma_1^2\ A_n(\lambda_1, \cdots, \lambda_n)^{-1} \end{aligned}$$

Then $\rho_n(\lambda_1, \cdots, \lambda_n)$ obeys three equations:

$$\rho_n(\cdots, \lambda_{j+1}, \lambda_j, \cdots) = \check{\mathcal{R}}_{j,j+1}(\lambda_{j,j+1}) \rho_n(\cdots, \lambda_j, \lambda_{j+1}, \cdots),$$

$$\rho_n(\lambda_1 - 1, \cdots, \lambda_n) = \mathcal{A}_n(\lambda_1, \cdots, \lambda_n) \rho_n(\lambda_1, \cdots, \lambda_n)$$

$$\mathrm{tr}_1 \circ \rho_n(\lambda_1, \lambda_2, \cdots, \lambda_n) = \rho_{n-1}(\lambda_2, \cdots, \lambda_n)$$

where $\check{R} = PR$.

Properties of operators $\Omega^{(i,j)}$.

1. Exchange relation:

$$\check{\mathcal{R}}_{k,k+1}(\lambda_{k,k+1})\Omega^{(i,j)}(\cdots\lambda_k,\lambda_{k+1},\cdots)$$

$$=\Omega^{(i,j)}(\cdots,\lambda_{k+1},\lambda_k,\cdots)$$

2. Commutativity:

$$\left[\Omega^{(i,j)}(\lambda_1,\cdots,\lambda_n),\Omega^{(k,l)}(\lambda_1,\cdots,\lambda_n)\right]=0$$

From here the commutativity of $U(\alpha_1,\alpha_2)$ follows.

3. Nilpotency:

$$\Omega^{(i,j)}(\lambda_1,\cdots,\lambda_n)\Omega^{(k,l)}(\lambda_1,\cdots,\lambda_n)=0 \quad \text{if} \quad \{i,j\}\cap\{k,l\}\neq\emptyset$$

From here nilpotency of Ω follows.

4. Difference equations:

$$\Omega^{(i,j)}(\lambda_1 - 1, \cdots, \lambda_n)$$
$$= \mathcal{A}_n(\lambda_1, \cdots, \lambda_n)\Omega^{(i,j)}(\lambda_1, \cdots, \lambda_n)\mathcal{A}_n(\lambda_1, \cdots, \lambda_n)^{-1} \qquad (i, j \neq 1)$$
$$\Omega^{(1,j)}(\lambda_1 - 1, \cdots, \lambda_n)$$
$$= \mathcal{A}_n(\lambda_1, \cdots, \lambda_n)\left(\Omega^{(1,j)}(\lambda_1, \cdots, \lambda_n) + Y^{(j)}(\lambda_1, \cdots, \lambda_n)\right)$$

where $Y^{(j)}$ satisfy the following

5. Cancellation identity:

$$\sum_{j=2}^{n} Y^{(j)}(\lambda_1, \cdots, \lambda_n) = 1 - \mathcal{A}_n(\lambda_1, \cdots, \lambda_n)^{-1}$$

Proof of JM equations.

We have

$$\rho_n(\lambda_1,\cdots,\lambda_n) = e^{\Omega(\lambda_1,\cdots,\lambda_n)}$$

Due to nilpotency

$$\rho_n(\lambda_1,\cdots,\lambda_n) = \sum_{p=1}^{\left[\frac{n}{2}\right]} \sum_{i_k<j_k,\ i_1<\cdots<i_p} \Omega^{(i_p,j_p)}(\lambda_1,\cdots,\lambda_n)\cdots\Omega^{(i_1,j_1)}(\lambda_1,\cdots,\lambda_n)$$

First of JM equations follows from Exchange relation. To prove the second one rewrites:

$$\rho_n(\lambda_1,\cdots,\lambda_n) =$$
$$\sum_{p=1}^{\left[\frac{n}{2}\right]} \sum_{i_k<j_k,\ 2\leq i_1<\cdots<i_p} \Omega^{(i_p,j_p)}(\lambda_1,\cdots)\cdots\Omega^{(i_1,j_1)}(\lambda_1,\cdots)$$
$$+\sum_{p=1}^{\left[\frac{n}{2}\right]} \sum_{i_k<j_k,\ 1<i_2<\cdots<i_p} \Omega^{(i_p,j_p)}(\lambda_1,\cdots)\cdots\Omega^{(1,j_1)}(\lambda_1,\cdots)$$

Difference equations imply

$$
\begin{aligned}
&\rho_n(\lambda_1 - 1, \cdots, \lambda_n) = \mathcal{A}_n(\lambda_1, \cdots) \\
&\times \sum_{p=1}^{\left[\frac{n}{2}\right]} \Big\{ \sum_{i_k<j_k,\ 2\le i_1<\cdots<i_p} \Omega^{(i_p,j_p)}(\lambda_1, \cdots) \cdots \Omega^{(i_1,j_1)}(\lambda_1, \cdots) \\
&\times \mathcal{A}_n(\lambda_1, \cdots)^{-1} \\
&+ \sum_{i_k<j_k,\ 1<i_2<\cdots<i_p} \Omega^{(i_p,j_p)}(\lambda_1, \cdots) \cdots \Omega^{(1,j_1)}(\lambda_1, \cdots) \\
&+ \sum_{i_k<j_k,\ 2\le i_2<\cdots<i_p} \Omega^{(i_p,j_p)}(\lambda_1, \cdots) \cdots \Omega^{(i_2,j_2)}(\lambda_1, \cdots) \\
&\times \sum_{j=2}^{n} Y^{(j)}(\lambda_1, \cdots) \Big\}
\end{aligned}
$$

The Cancellation Identity

$$\sum_{j=2}^{n} Y^{(j)}(\lambda_1, \cdots) = 1 - \mathcal{A}_n(\lambda_1, \cdots)^{-1}$$

proves the second JM equation.

Conjectures about temperature correlation functions.
Recall that

$$\varphi(\lambda) = \frac{\lambda}{\lambda^2 - 1}\left(\frac{d}{d\lambda}\log\rho(\lambda) + \frac{1}{2(\lambda^2-1)}\right)$$

The function $\omega(\lambda) = \frac{d}{d\lambda}\log\rho(\lambda)$ satisfies the functional equation:

$$\omega(\lambda+1) + \omega(\lambda) = -\frac{1}{\lambda(\lambda+1)}$$

If replace $\varphi(\lambda)$ by

$$\varphi_{c_*}(\lambda) = \frac{\lambda}{\lambda^2 - 1}\left(\frac{d}{d\lambda}\log\rho(\lambda) + \frac{1}{2(\lambda^2-1)} + \sum_{k=\infty}^{\infty} c_{2k+1}e^{(2k+1)\lambda}\right)$$

the JM equations are still valid.
Conjecture. Corresponding solutions describe

$$\langle\beta_1,\cdots,\beta_k|\mathcal{O}|\beta_1,\cdots,\beta_k\rangle_{\text{reg}}$$

Regularisation means the following:

$$\langle\beta_1,\cdots,\beta_k|\mathcal{O}|\beta_1,\cdots,\beta_k\rangle_{\mathrm{reg}} =$$
$$= \int \frac{d\epsilon_1}{\epsilon_1} \int \frac{d\epsilon_k}{\epsilon_k} \langle\beta_1+\epsilon_1,\cdots,\beta_k+\epsilon_k|\mathcal{O}|\beta_1,\cdots,\beta_k\rangle$$

Problem: What is the relation between c_{2i+1} and β_j?
LeClair and Mussardo claim that

$$\langle\mathcal{O}\rangle_T = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} d\beta_1 \cdots \int_{-\infty}^{\infty} d\beta_k \ \langle\beta_1,\cdots,\beta_k|\mathcal{O}|\beta_1,\cdots,\beta_k\rangle_{\mathrm{reg}} \ e^{-\sum \frac{\epsilon(\beta_j)}{T}}$$

All together it leads to
Conjecture.

$$\langle\mathcal{O}\rangle_T = \int Dc_* \ \omega(c_*,T) \ \frac{1}{2^\infty} \ \mathrm{tr}_{(\mathbb{C}^2)^\infty} \left(\rho_{(c_*)}(\mathcal{O})\right)$$

Problem: To find the measure $\omega(c_*,T)$.

Conjectures about VEV in SG theory. The formula

$$\langle \mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle = \mathrm{tr}_{\left(\mathbb{C}^2\right)^\infty}\left(\rho(\mathcal{O})\right)$$

allows generalisation to XYZ model. In that case still

$$\Omega^{\left[\frac{l(\mathcal{O})}{2}\right]+1} = 0$$

By scaling limit we obtain SG-model. The expression including $T(\lambda)$ is quite universal, in continuous limit it becomes an monodromy matrix of Bazhanov-Lukyanov-Zamolodchikov. Spinless local operators belong to

$$\bigoplus_{k\geq 0}\left(W_k \otimes \overline{W}_k\right)$$

It is natural to assume that $l(\mathcal{O}) \sim k$, i.e. in continuous limit

$$\Omega^{\left[\frac{\deg(\mathcal{O})}{2}\right]+1} = 0$$

Problem. Can one describe effectively ρ for every degree and, thus, calculate VEV's?

Problem. To understand better this formula. In XXZ, XYZ cases to consider free fermion point.