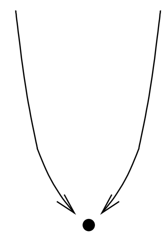
Constant scalar curvature Kähler metrics and stability of algebraic varieties

1. Geometric Invariant Theory
2. Symplectic reduction
3. Balanced varieties, cscK metrics
4. Bundle analogue
5. Stability of algebraic varieties (Joint work with **JULIUS ROSS**)

Geometric Invariant Theory

$$\begin{array}{ccc} G & \curvearrowright & X \\ \cap & & \cap \\ SL(n+1,\mathbb{C}) & & \mathbb{P}^n \end{array} \qquad X/G \ ?$$



G-action not proper.

Quotient not Hausdorff (not separated).

GIT chooses certain "unstable" orbits to remove to give a projective quotient.

Also identifies some "semistable" orbits to compactify quotient.

$$(X, L = \mathcal{O}(1)) \;\longleftrightarrow\; \bigoplus_r H^0(X, \mathcal{O}(r)),$$

$$X/G \;\longleftrightarrow\; \bigoplus_r H^0(X, \mathcal{O}(r))^G.$$

$$(f_1 = 0 = \ldots = f_k) \subset \mathbb{P}^n \quad \longleftrightarrow \quad \frac{\mathbb{C}[x_0, \ldots, x_n]}{(f_1, \ldots, f_k)}.$$

G acts on $\mathbb{C}^{n+1}$ so on $\mathcal{O}(-1) \to \mathbb{P}^n$ so on $\mathcal{O}(r) \to X$.

$H^0(X, \mathcal{O}(r)) = \{$degree r homogeneous polynomials on $\widetilde{X} \subset \mathbb{C}^{n+1}\}$.

$x \in X$ *semistable* iff $\exists f \in H^0(X, \mathcal{O}(r))^G$ such that $f(x) \neq 0$.

So the Kodaira "embedding" of X/G,

$$\begin{array}{rcll} X & \dashrightarrow & \mathbb{P}((H^0(X,\mathcal{O}(r))^G)^*), & \\ x & \mapsto & ev_x & (ev_x(f) := f(x)), \end{array}$$

is well defined at x; i.e. $ev_x \neq 0$.

x is *stable* iff $\bigoplus_r H^0(X, \mathcal{O}(r))^G$ separates orbits at x and the stabiliser of x is finite.

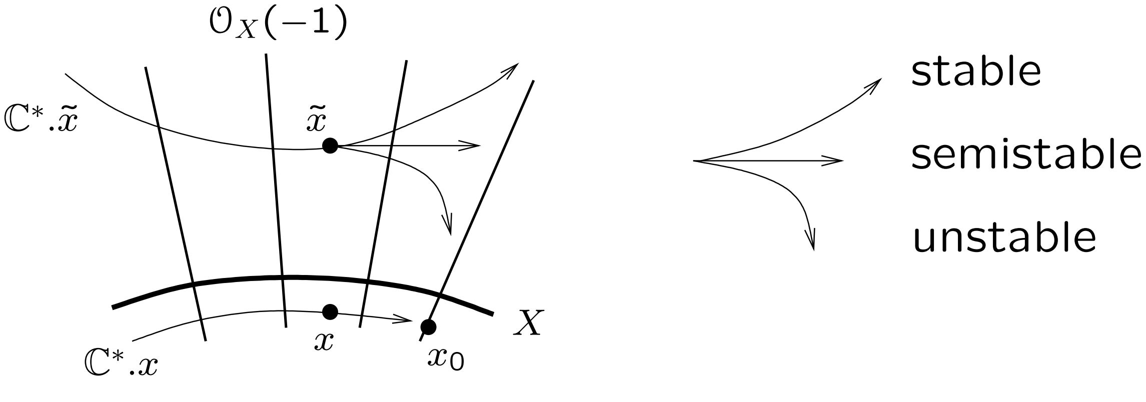
Theorem 1 [Mumford]
x is stable $\iff$ $G.\tilde{x}$ is closed in $\mathbb{C}^{n+1}$ and $\dim G.\tilde{x} = \dim G$.
($G.\tilde{x}$ just closed = polystable.)
x is semistable $\iff$ $0 \notin \overline{G.\tilde{x}}$.

Theorem 2 [Hilbert-Mumford criterion]
The same result is true iff it is true for all one parameter subgroups (1-PS) $\mathbb{C}^* \subset SL(n+1,\mathbb{C})$. So everything reduces to the $\mathbb{C}^*$-action on the line over the limit point $x_0 = \lim_{\lambda\to 0}\lambda.x$.
x_0 *fixed point of $\mathbb{C}^*$-action, so get action on $\mathcal{O}_{x_0}(-1)$.*

Weight $\rho \in \mathbb{Z}$ of action, $\lambda \mapsto \lambda^\rho$,

- $\rho < 0$ *stable*
- $\rho = 0$ *semistable*
- $\rho > 0$ *unstable*

So "just" compute this weight for all $\mathbb{C}^* \subset SL(n+1,\mathbb{C})$; x is stable $\iff$ weight always < 0.



Fundamental example – points in $\mathbb{P}^1$

n points in $\mathbb{P}^1$ $\leftrightarrow$ 0-dim algebraic subvariety!

(Points with multiplicities $\leftrightarrow$ length-n 0-dim subscheme)

$$
\begin{aligned}
& SL(2,\mathbb{C}) \curvearrowright \mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2) \\
\Rightarrow\ & SL(2,\mathbb{C}) \curvearrowright S^n(\mathbb{C}^2)^* \\
& \qquad = \{\deg n \text{ polys on } \mathbb{C}^2\} = H^0(\mathcal{O}_{\mathbb{P}^1}(n)).
\end{aligned}
$$

But $\{n \text{ points}\} = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(n)))$ as roots of the degree n polynomial.

Theorem 3 *n points in $\mathbb{P}^1$.*
Semistable $\iff$ each multiplicity $\leq n/2$.
Stable $\iff$ each multiplicity $< n/2$.

Proof. Diagonalise a given $\mathbb{C}^* \subset SL(2,\mathbb{C})$:

$\begin{pmatrix} \lambda^k & 0 \\ 0 & \lambda^{-k} \end{pmatrix}$ w.r.t. $[x:y]$ coords on $\mathbb{P}^1$. ($k \geq 0$.)

Polynomial $f = \sum_{i=0}^{n} a_i x^i y^{n-i}$.
$\lambda.f$ tends to ∞ iff there are more ys than xs in a nonzero summand.

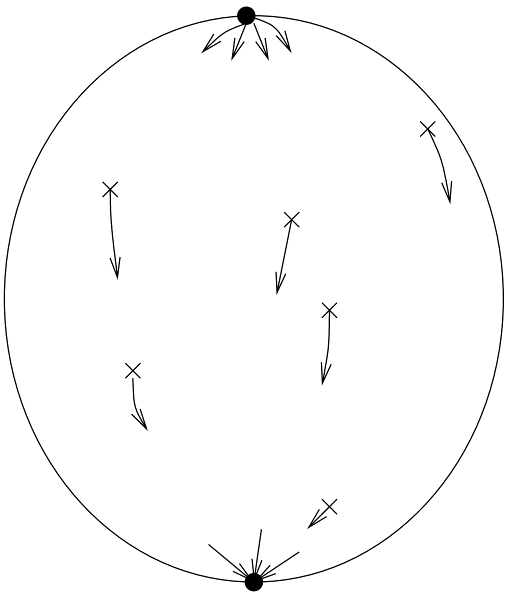
I.e. stable unless $a_i = 0$ for $i < n/2$.
I.e. stable so long as f does not vanish to order $\geq n/2$ at $x = 0$, $\forall \mathbb{C}^* \subset SL(2,\mathbb{C})$. □

Alternatively, use Hilbert-Mumford criterion.

Proof. After rescaling, $\lambda.f \to f_0 = a_j x^j y^{n-j}$, where j is smallest such that $a_j \neq 0$.
$(f = a_j x^j y^{n-j}(1 + \frac{a_{j+1}}{a_j} x y^{-1} + \dots.)$

Weight on $\mathbb{C}.f_0$ is $k(j-(n-j)) = k(2j-n)$.
So stable $\iff k(2j-n) < 0 \iff j < n/2$ as before. □

Subgroup moves all points to the "attractive" fixed point at $x = 0$ (weight $-k$) except those stuck at "repulsive" fixed point $y = 0$ (weight $+k$).



So total weight negative unless $\geq$ half the points are at $y = 0$.

So stability generic; unstable only if "too singular" – destabilised by high multiplicity singularity.

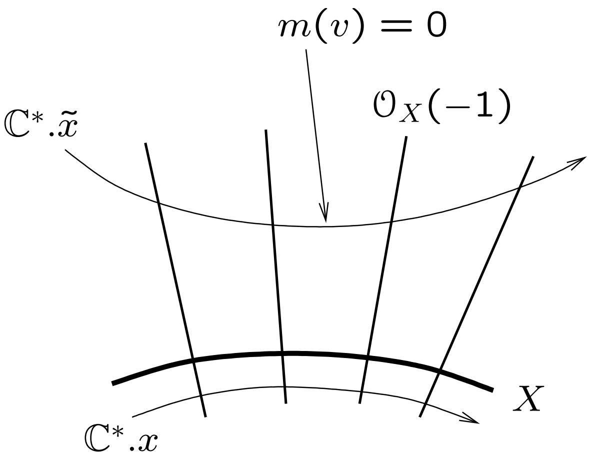
Symplectic reduction

$G \subset SL(N+1,\mathbb{C})$ has compact subgroup $K = G \cap SU(N+1)$. $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$.
K acts on $\mathbb{P}^N$, preserves J *and* g, and so ω too.

So $\forall v \in \mathfrak{k} = LK$ the infinitesimal action X_v is *Hamiltonian*, $X_v \lrcorner\, \omega = dm_v$. i.e. $(X_v = J\nabla m_v)$

Gives **moment map** $m\colon X \to \mathfrak{k}^*$.
(Collection of r hamiltonians m_v, $r = \dim K$.)

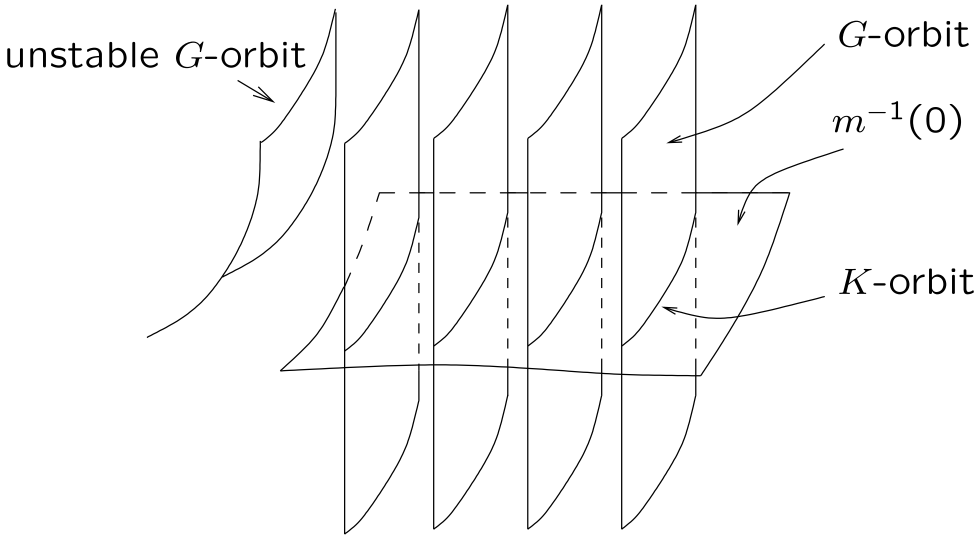
m_v = derivative down $(0,\infty) \subset \mathbb{C}^*$ orbit of $\log ||\lambda\tilde{x}||_{\lambda\in(0,\infty)}$, i.e. down $JX_v = X_{iv}$.



(Poly)Stable $\iff ||\lambda\tilde{x}||$ achieves min on all $\mathbb{C}^*$-orbits $\iff m(v) = 0$ somewhere on orbit $\forall v$.

Theorem 4 [Kempf-Ness]

$$\frac{X}{G} \cong \frac{m^{-1}(0)}{K}.$$



$m^{-1}(0)$ provides slice to $i\mathfrak{k} \subset \mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ part of orbit; K-equivariant.

(Nonlinear generalisation of $V/W \cong W^{\perp}$ for $W \leq V$ vector spaces.)

E.g. $U(1) \subset \mathbb{C}^* \curvearrowright \mathbb{C}^n$, moment map $= |\underline{z}|^2 - a^2$.

$$\frac{\mathbb{C}^n \backslash \{0\}}{\mathbb{C}^*} \cong \frac{S^{2n-1} = \{\underline{z} \,:\, |\underline{z}|^2 = a^2\}}{U(1)} \cong \mathbb{P}^{n-1}.$$

E.g. n points in $\mathbb{P}^1$ again.

$$SL(2,\mathbb{C}) \supset SU(2) \curvearrowright \mathbb{P}^1 \xrightarrow{m} \mathfrak{su}(2)^*$$

is the inclusion $S^2 \subset \mathbb{R}^3$.

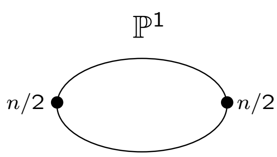
Adding gives, for n points, $m = \sum_{i=1}^n m_i$:

$$S^n\mathbb{P}^1 \longrightarrow \mathbb{R}^3,$$

the sum of n points in $\mathbb{R}^3$ ("centre of mass").

So $m^{-1}(0) = \{$**Balanced configurations**$\}$
(Centre of mass $0 \in \mathbb{R}^3$).

Stable $\iff$ $\exists$ $SL(2,\mathbb{C})$ transformation of $\mathbb{P}^1$ such that points are balanced
$\iff$ mass at each point $< n/2$.

(Note that balanced 

has dim 1 stabiliser.)

Polarised algebraic varieties (X, L)

$$X \hookrightarrow \mathbb{P}(H^0(X, L^r)^*) = \mathbb{P}^N, \quad r \gg 0.$$

Defines a point in Hilb $\subset Gr \subset \mathbb{P}^M$ by the subspace

$$H^0(\mathbb{P}^N, \mathscr{I}_X(k)) \subset H^0(\mathbb{P}^N, \mathcal{O}(k)) = S^k H^0(X, L^r)$$

of deg k polys on $\mathbb{P}^N$ vanishing on X.

I.e. point of $\Lambda^{\dim H^0_{\mathbb{P}^N}(\mathscr{I}_X(k))} S^k H^0(X, L^r), \quad r, k \gg 0.$

Divide by autos $SL(N+1, \mathbb{C})$ of $\mathbb{P}^N$ to get moduli of polarised varieties.

Choice of line bundle on Hilb $\Rightarrow$ notion of stability for (X, L).

Moment map for appropriate ample line bundle / symplectic structure on Hilb.

Fix metric on $\mathbb{C}^{N+1}$ and so g_{FS} on $\mathbb{P}^N$.
Let $m: \mathbb{P}^N \to \mathfrak{su}(N+1)^*$ denote the usual moment map.

Then (Donaldson) moment map takes $X \subset \mathbb{P}^N$ to the centre of mass

$$\int_X m \ \mathrm{vol}_{FS} \ \in \ \mathfrak{su}(N+1)^*.$$

Zeros of moment map = **Balanced** varieties $X \subset \mathbb{P}^N$. (Equivalently, orthonormal basis for $\mathbb{C}^{N+1} \cong H^0(\mathcal{O}_X(1))^*$ is orthonormal in L^2-metric induced by $g_{FS}|_X$.)

Theorem 5 [Zhang/Luo/Paul/Wang] *Balanced + finite automorphism group ⇒ HM stable.*

$s_i \in H^0(\mathcal{O}_X(1)) = H^0(X, L^r)$ L^2-orthonormal basis. Bergman kernel (defines projection of sections of $\mathcal{O}_X(1)$ onto holomorphic sections)

$$B_r(x) = \sum_i s_i(x)^* \otimes s_i(x)$$

is const.id $\iff$ $X \subset \mathbb{P}^N$ is balanced
($\iff$ s_i orthonormal in original metric on $\mathbb{C}^{N+1} \cong H^0(X, L^r)$).

As $r \to \infty$ ($\Rightarrow N \to \infty$) B has an asymptotic expansion (Catlin, Z. Lu, W.-D. Ruan, Tian, Zelditch)

$$B_r(x) \sim r^n + \frac{1}{2\pi} s(g_{FS}) r^{n-1} + O(r^{n-2}),$$

where s is the scalar curvature of g_{FS}.
Roughly, balanced metrics "tend towards" cscK metrics with $[\omega] = [c_1(L)]$.

Theorem 6 [Donaldson] (Aut(X) discrete.)
(X, L) admits cscK metric in $[c_1(L)]$ $\Rightarrow$ (X, L^r) balanced for $r \gg 0$.
(Zhang $\Rightarrow$ HM-stable, Chen-Tian $\Rightarrow$ K-semistable.)

Partial result in converse direction: If $(X, L^r) \subset \mathbb{P}^{N(r)}$ balanced for $r \gg 0$ and resulting $\omega_{FS,r}$ convergent, then limit metric has csc. Also generalisation due to Mabuchi for arbitrary X.

Donaldson and Fujiki also give an infinite dimensional GIT/moment map formulation.
(Think of as $\lim r \to \infty$, where balanced condition has become cscK condition.)

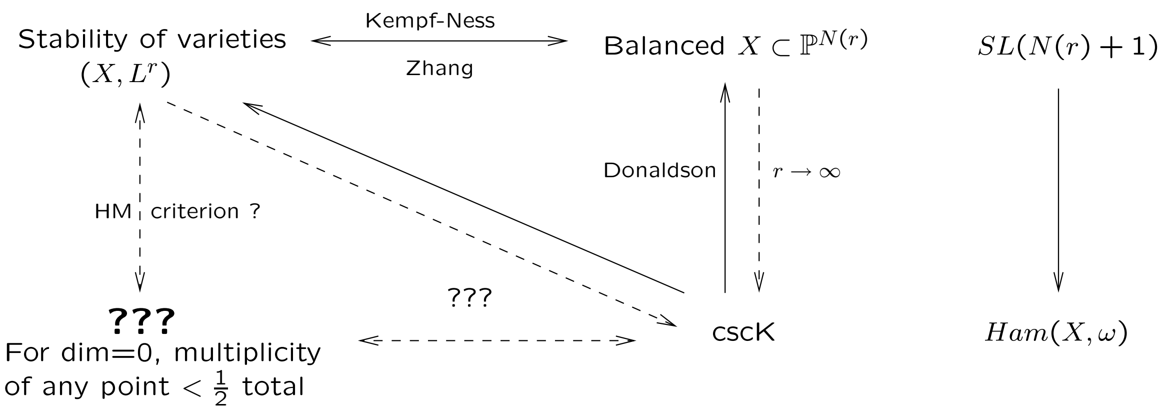
(Hamiltonian diffeomorphisms) $\curvearrowright (X, \omega = c_1(L))$ so $\curvearrowright$ {compatible complex structures on X}.

Moment map = scalar curvature + const.

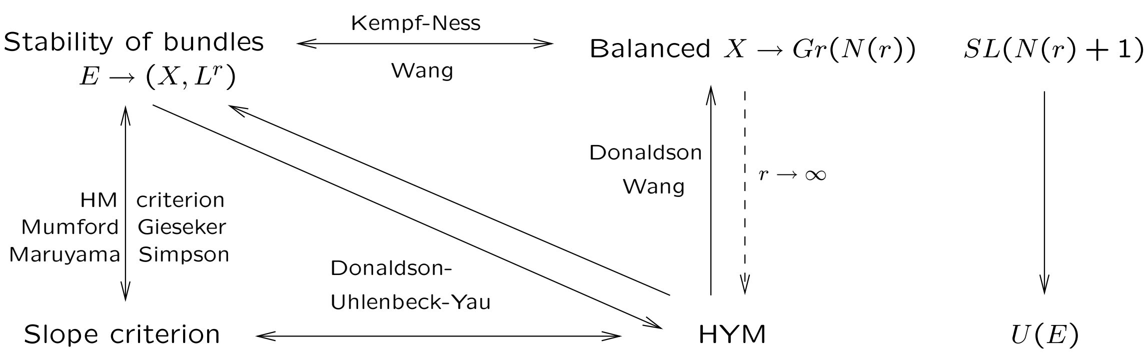
Zeros = cscK metrics.

(When $L = K_X^{\pm n}$, $\omega = \mp n c_1(X)$, cscK=KE. Yau suggested a relationship stability $\leftrightarrow$ KE metrics. Tian proved this for surfaces and suggested the K-stability / cscK relationship.)

So we have the infinite dimensional analogue of the balanced condition for points in $\mathbb{P}^1$ (i.e. cscK metrics) and part of the relationship to stability, but not the algebro-geometric description of stability. I.e. the Hilbert-Mumford criterion, giving the analogue of the multiplicity $< n/2$ condition, is missing.



In the bundle case, all of this is worked out:



Moduli of bundles over (X, L)

Given $E \to X$, form $E(r) := E \otimes L^r$ for $r \gg 0$,

$$H^0(E(r)) \to E(r) \to 0 \qquad \text{on } X.$$

Gives map $X \to Gr$.
$SL(H^0(E(r))) \curvearrowright \text{Maps}(X, Gr)$.
$Gr \subset \mathfrak{su}(N_r + 1)^*$ $\quad (N_r = \dim H^0(E(r)).)$

So can again talk about *balanced* $X \to Gr$ and asymptotics as $r, N_r \to \infty$. (Donaldson)

Gieseker stable bundles admit balanced maps $X \to Gr$. Pulling back the canonical quotient connection on Gr and taking $\lim_{r\to\infty}$, if it exists, gives a HYM connection (X.-W. Wang)

Atiyah-Bott gave an infinite dimensional GIT / moment map formulation.
$U(E) = \{\text{unitary gauge transformations}\}$,
$\mathcal{A} = \{\text{connections } A \text{ with } F_A^{0,2} = 0\}$.
$U(E) \curvearrowright \mathcal{A}$.
Moment map = HYM = $\omega^{n-1} \wedge F_A^{1,1}$.
Donaldson-Uhlenbeck-Yau: E slope polystable $\Rightarrow$ HYM.

In this case HM-criterion can be manipulated (Gieseker, Maruyama, Simpson) to give an algebro-geometric understanding of stability.

Hilbert poly $h^0(E(r)) = a_0 r^n + a_1 r^{n-1} + \dots$

$a_0 = \operatorname{rk} E \int_X \omega^n/n!, \quad a_1 = \int_X c_1(E).\omega^{n-1}/(n-1)! + \varepsilon(X).$

Reduced Hilbert poly $p_E(r) = r^n + \frac{a_1}{a_0} r^{n-1} + \dots.$

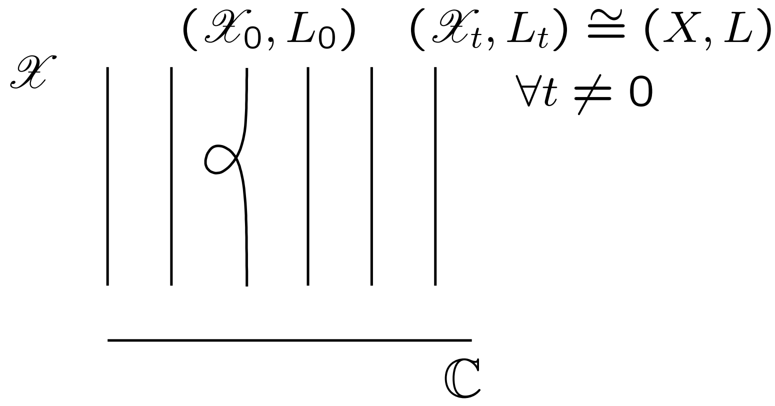
E stable $\iff \forall F \hookrightarrow E,\ p_F(r) < p_E(r) \quad r \gg 0.$

$$E \text{ slope-stable} \iff \frac{a_1(F)}{a_0(F)} < \frac{a_1(E)}{a_0(E)} \iff \mu(F) < \mu(E).$$

($\mu(E) = \int_X c_1(E).\omega^{n-1}/\operatorname{rk}(E)$. Corresponds to a different line bundle on moduli space – Jun Li.)

So bundles/sheaves destabilised by subsheaves $F \subset E$. Can $\mathbb{P}(F) \subset \mathbb{P}(E)$ destabilise as varieties ? Can subschemes $Z \subset (X, L)$ destabilise? (cf. length $\geq n/2$ subschemes of n points in $\mathbb{P}^1$.)

A $\mathbb{C}^* \subset SL(M+1, \mathbb{C})$ orbit of $X \in \mathrm{Hilb} \subset \mathbb{P}^M$ gives a $\mathbb{C}^*$-equivariant flat family (**test configuration**) $\mathscr{X} \to \mathbb{C}$



For the HM-criterion one calculates the weight $w_{r,k}$ of the $\mathbb{C}^*$-action on

$$\Lambda^{\max} H^0(\mathscr{X}_0, L_0^{rk})^* \otimes \Lambda^{\max} S^k H^0(\mathscr{X}_0, L_0^r).$$

$$w_{r,k} = a_{n+1}(r) k^{n+1} + a_n(r) k^n + \dots,$$

where

$$a_i(r) = a_{in} r^n + a_{i,n-1} r^{n-1} + \dots.$$

Definition 7 *The* $\mathbb{C}^* \subset SL(M+1,\mathbb{C})$ *destabilises* (X,L) *if* $w_{r,k} \succ 0$ *in the following sense:*

- *HM(*r*)-unstable:* $w_{r,k} > 0$ *for all* $k \gg 0$*,*

- *Asymptotically HM-unstable: for all* $r \gg 0$*,* $w_{r,k} > 0$ *for all* $k \gg 0$*,*

- *Chow(*r*)-unstable: leading* k^{n+1}*-coefficient* $a_{n+1}(r) > 0$*,*

- *Asymptotically Chow unstable:* $a_{n+1}(r) > 0$ *for* $r \gg 0$*,*

- *K-unstable: leading coefficient* $a_{n+1,n} > 0$*.*

These correspond to different line bundles on Hilb: the standard one, the Chow line, and the Paul-Tian line.

$a_{n+1,n}$ is the Donaldson-Futaki invariant of the $\mathbb{C}^*$-action on $(\mathscr{X}_0, L)$.

Slope for K-stability

$$Z \subset (X, L)$$

$$h^0(\mathcal{O}_X(r)) = a_0 r^n + a_1 r^{n-1} + \dots$$

$$h^0(\mathscr{I}_Z^{xr}(r)) = a_0(x) r^n + a_1(x) r^{n-1} + \dots$$

$a_i(x)$ polynomials in $x \in \mathbb{Q} \cap [0, \epsilon(Z))$ for $r \gg 0$.

(Seshadri constant $\epsilon(Z)$ defined so that $\mathscr{I}_Z^{xr}(r)$ generated by global sections for $x < \epsilon(Z)$ for $r \gg 0$).

$a_0(0) = a_0$, and $a_1(0) = a_1$ for X normal.

$$a_0 = \frac{\int_X \omega^n}{n!}, \ a_1 = \frac{\int_X c_1(X)\omega^{n-1}}{2(n-1)!}.$$

For any $c \leq \epsilon(Z)$, define slope of Z to be

$$\mu_c(\mathscr{I}_Z) = \frac{\int_0^c a_1(x) + \frac{a_0'(x)}{2} dx}{\int_0^c a_0(x) dx}.$$

$Z = \emptyset$ gives

$$\mu(X) = \frac{a_1}{a_0}.$$

Theorem 8
K-(semi)stable $\implies$ slope (semi)stable:
$\mu_c(\mathscr{I}_Z) \leq \mu(X)$ $\forall$ *closed subschemes* $Z \subset X$.
(*Slope stability:* $\mu_c(\mathscr{I}_Z) < \mu(X)$ $\forall c \in (0, \epsilon(Z))$ and $\forall c \in (0, \epsilon(Z)]$ *if* $\epsilon(Z) \in \mathbb{Q}$ *and* $\mathscr{I}_Z^{\epsilon(Z)r}(r)$ *saturated by global sections for* $r \gg 0$.)

Corollary 9
If $\mu_c(\mathscr{I}_Z) > \mu(X)$ *then* X *admits no cscK metric in the class of* $c_1(L)$.
(Donaldson & Chen-Tian: cscK $\implies$ K-semistable.)

Examples.

- $F \subset E$ destabilising subbundle $\implies$ $\mathbb{P}(F) \subset \mathbb{P}(E)$ destabilises, for suitable polarisations $\pi^* L^m \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$, $m \gg 0$.
  (Partial converse (Hong): E stable $\Rightarrow \mathbb{P}(E)$ cscK for $m \gg 0$.)
  And for **all** polarisations if the base is a curve, so in this case (modulo automorphisms)

$$\mathbb{P}(E) \text{ cscK} \iff E \text{ stable} \iff E \text{ HYM}.$$

  ($\Leftarrow$ by Narasimhan-Seshadri, projectively flat connection.)

- -1-curves on del Pezzo surfaces for appropriate L. So Aut(X) reductive (or trivial) does not imply cscK (unless $L \neq K^{-1}$, by Tian).

- $\mathbb{P}^2$ blown up in one point. Aut(X) not reductive $\Longrightarrow$ not stable. Destabilised by the -1-curve for all polarisations.

- Generically stable varieties can specialise to unstable ones. Move two -1-curves together on a del Pezzo to give a limit -2-curve.
  (Blow up 2 "infinitely near" points: blow up one, then another on the exceptional curve.)
  The -2-curve destabilises for suitable L.

- Calabi-Yau manifolds, and varieties with canonical singularities and $mK_X \sim 0$ are slope stable.

- Canonically polarised varieties with canonical singularities (i.e. the canonical models of Mori theory) are slope stable.

- Partial results towards converse (i.e. slope stability $\Rightarrow$ K-stability) complete for curves. Gives geometric (rather than analytic) proof that curves are K-stable ($\mathbb{P}^1$ is K-polystable).

  Similarly Chow-slope results (below) and converse give geometric (rather than combinatorial) proof that curves are Chow stable ($\mathbb{P}^1$ is Chow polystable).

Slope for Chow stability

$Z \subset (X, \mathcal{O}_X(1)) \subset (\mathbb{P}^N, \mathcal{O}(1))$ embedded by sections of $\mathcal{O}_X(1)$.

$$h^0(\mathcal{O}_X(r)) = a_0 r^n + a_1 r^{n-1} + \dots$$

$$h^0(\mathscr{I}_Z^{xr}(r)) = a_0(x) r^n + a_1(x) r^{n-1} + \dots$$

$\forall c \leq \epsilon(Z)$ define Chow slope of Z:

$$Ch_c(\mathscr{I}_Z) = \frac{\sum_{i=1}^{c} h^0(\mathscr{I}_Z^i(1))}{\int_0^c a_0(x)dx}\,.$$

$Z = \emptyset$ gives

$$Ch(X) = \frac{h^0(\mathcal{O}_X(1))}{a_0} = \frac{N+1}{a_0}\,.$$

Theorem 10
Chow (semi)stable $\implies$ slope (semi)stable:

$$Ch_c(\mathscr{I}_Z) \underset{(\leq)}{<} Ch(X) \quad \forall Z \subset X.$$

Review of Hilbert-Mumford criterion for bundles or sheaves over (X, L)

Given $E \to X$, form $E(r) := E \otimes L^r$ for $r \gg 0$, making $E(r)$ a quotient of a trivial bundle:

$$H^0(E(r)) \to E(r) \to 0 \qquad \text{on } X. \qquad (11)$$

Fix isomorphism $H^0(E(r)) \cong \mathbb{C}^{P(r)}$
$\implies [E] \in \mathrm{Quot}\,(\underline{\mathbb{C}}^{P(r)})$.

(Quot subset of a Grassmannian: quotient (11) classified by induced vector space quotient $H^0(E(r)) \otimes H^0(L^R) \twoheadrightarrow H^0(E(r+R))$.)

Divide by $SL(P(r)) \implies$ moduli of sheaves.

HM-criterion gives (Gieseker, Maruyama, Simpson) algebro-geometric criterion for stability (dependent on choice of line bundle on Quot).

A 1-PS $\mathbb{C}^* \subset SL(P(r))$ gives a filtration of E

$$F_0 \subset F_1 \subset \ldots \subset F_p \subset E,$$

($F_i \subset E$ image of ith piece of weight filtration of $H^0(E(r))$ under map (11)) and a degeneration of E to

$$E_0 := F_0 \;\oplus\; F_1/F_0 \;\oplus\; \ldots \;\oplus\; F_p/F_{p-1} \;\oplus\; E/F_p.$$

Different 1-PSs can give the same filtration. But to every filtration there are canonical 1-PSs with the most unstable (largest) weights (for filtration $F_i \subset E$ choose weight filtration $H^0(F_i(r)) \subset H^0(E(r))$).

So need only consider these 1-PSs. Weight = positive linear combination of weights of the canonical 1-PSs associated to the splittings

$$F_i \;\oplus\; E/F_i.$$

So need only control the weights of these simpler splittings. So stability controlled by single subsheaves $F \subset E$.

Weight calculations give the following.

Hilbert poly $h^0(E(r)) = a_0 r^n + a_1 r^{n-1} + \dots$

$a_0 = \mathrm{rk}\, E \int_X \omega^n/n!, \quad a_1 = \int_X c_1(E).\omega^{n-1}/(n-1)! + \varepsilon(X).$

Reduced Hilbert poly $p_E(r) = r^n + \frac{a_1}{a_0} r^{n-1} + \dots.$

E stable $\iff \forall F \hookrightarrow E,\ p_F(r) < p_E(r) \quad r \gg 0.$

$$E \text{ slope-stable} \quad \begin{array}{ll} \iff & \frac{a_1(F)}{a_0(F)} < \frac{a_1(E)}{a_0(E)} \\ \iff & \mu(F) < \mu(E). \end{array}$$

$\mu(E) = \int_X c_1(E).\omega^{n-1}/\,\mathrm{rk}\,(E)$. Corresponds to a different line bundle on moduli space – Jun Li.

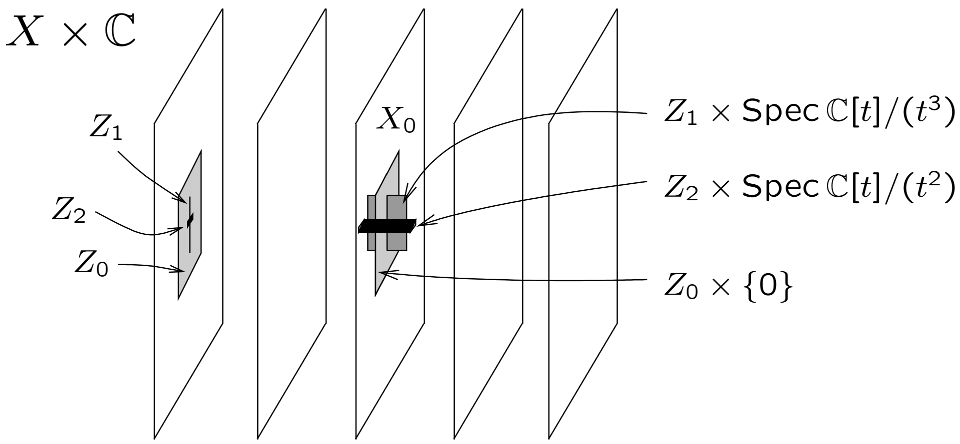
The proofs and converse for varieties

Any test configuration $(\mathscr{X}, \mathcal{L})$ is $\mathbb{C}^*$-birational to $(X \times \mathbb{C}, L)$, so is (a contraction p of) the blow up of $X \times \mathbb{C}$ in a $\mathbb{C}^*$-invariant ideal I supported on the central fibre. $p^*\mathcal{L} = L(-cE)$.

$$\begin{array}{ccc} & \mathrm{Bl}_I(X \times \mathbb{C}) & \\ \swarrow & & \searrow^{p} \\ X \times \mathbb{C} & & \mathscr{X} \end{array}$$

$\exists\ Z_{p-1} \subseteq \ldots \subseteq Z_1 \subseteq Z_0 \subseteq X$, ideal sheaves $\mathscr{I}_{p-1} \supseteq \ldots \supseteq \mathscr{I}_1 \supseteq \mathscr{I}_0$ such that

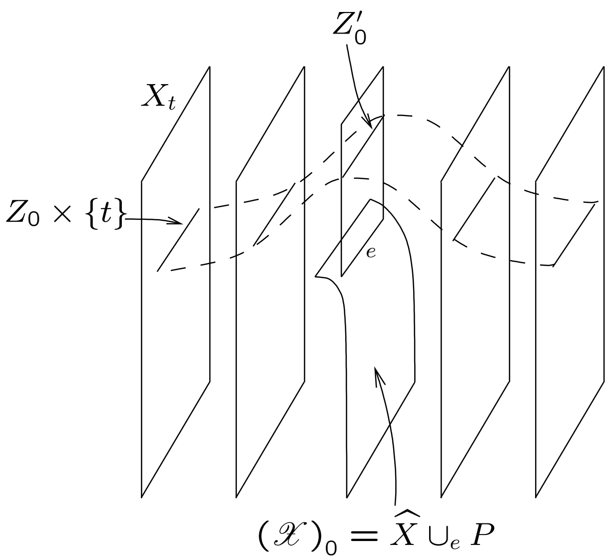
$$I = \mathscr{I}_0 + t\mathscr{I}_1 + t^2\mathscr{I}_2 + \ldots + t^{p-1}\mathscr{I}_{p-1} + t^p.$$



Show weights more stable than normalisation of blow up of $X \times \mathbb{C}$ in I, so consider only these.

$p = 1 \implies I = \mathscr{I}_0 + t \implies$ blow up in $Z_0 \times \{0\} \subset X \times \mathbb{C}$ = *deformation to normal cone of* Z_0.

Exceptional divisor P (= normal cone of $Z_0 = \mathbb{P}(\nu_{Z_0} \oplus \underline{\mathbb{C}}) \to Z_0$ if $Z_0 \subset X$ smooth).



$\mathbb{C}^* \ni \lambda$ acts on blow up (as $[1 : \lambda] = [\lambda^{-1} : 1]$ on $\mathbb{P}(\nu_{Z_0} \oplus \underline{\mathbb{C}})$ in smooth case); equivariant line bundle $\pi^* L(-cP)$.

Deformation to normal cone of Z replaces $H^0_X(L^r)$ (filtered by $H^0(L^r \otimes \mathscr{I}_Z^j)$) by, on central fibre, associated graded of filtration:

$$H^0_X(\mathscr{I}_Z^{cr}(r)) \oplus H^0_X\left(\mathscr{I}_Z^{cr-1}(r)/\mathscr{I}_Z^{cr}(r)\right) \oplus \dots$$
$$\oplus\; H^0_X\left(\mathscr{I}_Z(r)/\mathscr{I}_Z^2(r)\right) \oplus H^0_X(\mathcal{O}_Z(r)).$$

This is the weight space decomposition $\implies$ weight on top exterior power is

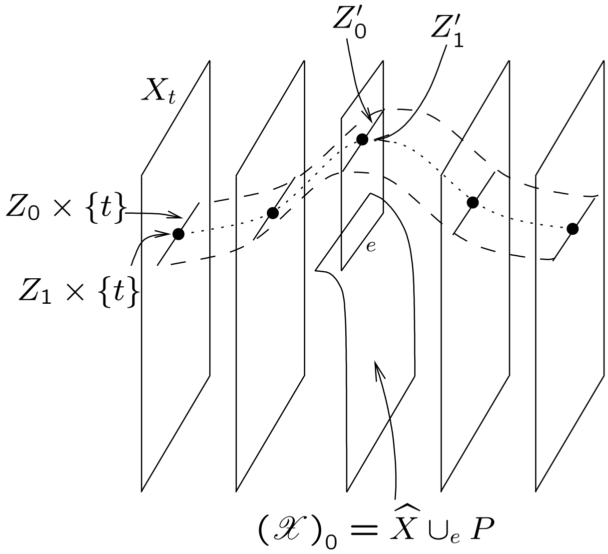
$$\begin{aligned} w_r &= -\sum_{j=1}^{cr} j h^0_X\left(\mathscr{I}_Z^{cr-j}(r)/\mathscr{I}_Z^{cr-j+1}(r)\right) \\ &= -\sum_{j=1}^{cr} h^0(\mathscr{I}_Z^j(r)) - crh^0(\mathcal{O}_X(r)). \end{aligned}$$

Trapezium rule $\implies$ to $O(r^{n-1})$,

$$-\left(\int_0^c a_0(x)dx\right)r^{n+1} + \int_0^c\left(a_1(x) + \frac{a_0'(x)}{2}\right)dx\ r^n.$$

Normalising (to make 1-PS lie in $SL(H^0(\mathcal{O}_X(r))^*)$ instead of GL) gives slope criterion.

Proper transform $\overline{Z_0 \times \mathbb{C}}$ of $Z_0 \times \mathbb{C}$:



Gives $Z'_{p-1} \subseteq \ldots \subseteq Z'_1 \subset Z'_0$. Now blow up in Z'_1, giving $Z''_{p-1} \subseteq \ldots \subseteq Z''_1$; next blow up Z''_2 etc.

Theorem 12 *The blow up of* $X \times \mathbb{C}$ *in* $I = \mathscr{I}_0 + t\mathscr{I}_1 + \ldots + t^{p-1}\mathscr{I}_{p-1} + t^p$ *is a contraction of this iterated blow up.*

Theorem 13 *At ith stage, blow up $Z_i^{(i)}$. If all thickenings of $\overline{(Z_i \times \mathbb{C})}$ are flat over $\mathbb{C}$ then this adds $w(Z_i)$ to the weight, to $O(r^n)$.*
($w(Z_i)$ is weight on deformation to normal cone of Z_i.)

So if this flatness holds, total weight is $w(Z_0)+\ldots+w(Z_{p-1})$. Stability iff

$$w(Z_0)+\ldots+w(Z_{p-1}) \prec 0 \iff w(Z) \prec 0 \ \forall Z. \tag{14}$$

Holds for Z_i smooth, or simple normal crossing (snc) divisors.

In general, resolution of singularities:

$$(X \supset Z_i) \xleftarrow{\pi} (\widehat{X} \supset m_i D_i), \quad D_i \text{ snc divisors.}$$

Work with $(\widehat{X}, \pi^* L)$ to give (14) for X *normal* (to equate $H^0(X,L)$ with $H^0(\widehat{X}, \pi^* L)$) so long as $m_i = 1 \ \forall i$.

D_i nonreduced ? E.g. $\mathscr{I}_0 = (x^2)$ so deformation to normal cone is blow up in (x^2, t).

Square $\mathbb{C}^*$-action (then halve the weight) $\Longrightarrow$ blow up in (x^2, t^2).

Take integral closure (normalise blow up) $\Longrightarrow$ get more unstable test configuration by blowing up in $(x^2, xt, t^2) = (x, t)^2$. I.e. just blow up in (x, t) with different line bundle ($E \mapsto 2E$ or $c \mapsto 2c$).

So can deal with D_i with multiplicities m_i when they *all have the same support*.

So can show slope stability = stability for curves (K- and Chow).

Would like to combine two approaches to deal with, for instance, $D_0 = (x^2y = 0)$, $D_1 = (x = 0)$. ??