M - C^∞ - manifold

$I: TM \to TM, \quad I^2 = -\mathrm{id}$

almost complex structure

$$\boxed{TM \otimes \mathbb{C} = T^{10}M \oplus T^{01}M}$$

integrability:

$$\boxed{[T^{10}M, T^{10}M] \subset T^{10}M}$$

Obstruction to integrability

Nijenhuis Tensor:

$$\boxed{N: T^{10}M \times T^{10}M \to T^{01}M}$$

$$\boxed{N \in \mathrm{Hom}(\Lambda^2 T^{10}M, T^{01}M)}$$

(M, I) almost complex

$\dim_{\mathbb{C}} M = 3$

$$\boxed{\mathcal{N} : \underbrace{\Lambda^2_{\mathbb{C}} T^{1,0} M}_{\dim = 3} \longrightarrow \underbrace{T^{0,1} M}_{\dim = 3}} \quad (*)$$

$\mathcal{N}$: Nijenhuis tensor is "non-degenerate" when $(*)$ is an isomorphism

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$\Rightarrow \det \mathcal{N} \in \left(\Lambda^{3,0} M\right)^{\otimes 2} \otimes \left(\Lambda^{0,3} M\right)^*$

$$\boxed{\det \mathcal{N} \otimes \overline{\det \mathcal{N}} \in \Lambda^{3,0} M \otimes \Lambda^{0,3} M = \Lambda^{3,3} M}$$

A volume form!

"intrinsic volume"

$$\nabla : TM \to TM \otimes \Lambda^1 M$$

a connection

$$T : \Lambda^2 TM \to TM$$

its torsion

$g \in S^2 TM^*$ Riemannian metric

$$\nabla g = 0$$

$$T_g(x, y, z) = g(T(x, y), z)$$

$$T_g \in \Lambda^2 M \otimes \Lambda^1 M$$

<u>Def</u> ∇ - connection with **Totally antisymmetric torsion** on (M, g) if $\boxed{T_g \in \Lambda^3 M}$

<u>Example</u> (G, g) Lie group with Killing form, ∇ - left-invariant connection

Definitions of nearly Kähler manifolds

(M, I, ω) almost complex Hermitian manifold, $\omega \in \Lambda^{11} M$ Hermitian form, g metric

<u>Theorem</u> The following conditions are equivalent (and define N.K.)

- $\nabla^{LC}_X I(X) = 0 \quad \forall\, X \in TM$ (∇^{LC} - Levi-Civita)
- $\nabla^{LC}_X \omega \in \Lambda^3 M$
- M admits a Hermitian connection ∇ w. totally antisymmetric torsion T and
  - $\boxed{\nabla T = 0}$, or, equivalently,
  - $d\omega \in \Lambda^{3,0} M + \Lambda^{0,3} M$, $|d\omega| = \text{const}$
- $d\omega$ is a real part of $(3,0)$-form Ω, and $d\Omega = \frac{2}{3}\sqrt{-1}\,\omega^2$

More definitions of nearly Kähler manifolds

Remark An almost complex structure on NK-manifold is uniquely determined by the metric

Theorem (M, g) - Riemannian 6-manifold. Then (M, g) is nearly Kähler iff one of the following conditions holds

- The Riemannian cone $C(M) = M \times \mathbb{R}^{>0}$, with metric $\lambda t^2 g + dt^2$ has holonomy G_2 (for some λ)
- M admits a real Killing spinor
- M has weak holonomy $SU(3)$ (Gray)

- $\dim \Lambda^3 \mathbb{R}^7 = 35$
- $\dim GL(7,\mathbb{R}) = 49$
- $\rho \in \Lambda^3 \mathbb{R}^7$ generic $\Leftrightarrow$
  $\dim(\mathrm{Stab}_{GL(7)} \rho) = 14$
  (a form of G_2)
- Definition
  ρ stable $\Leftrightarrow$
  $$\mathrm{Stab}_{GL(7)} \rho \cong G_2$$
  (compact form)

Octonions:

$\dim \mathbb{O} = 8$,

$\mathbb{O} = \mathbb{O}_{real} \oplus \mathbb{O}_{im}$

dim = 1, dim = 7

(Cayley)

$\mathbb{O} \times \mathbb{O} \to \mathbb{O}$ octonion product

On $\mathbb{O}_{im}$ it is skew-symmetric

$G_2 = \mathrm{Aut}\, \mathbb{O}$

$x, y, z \to \mathrm{Re}(xyz)$
is a 3-form on $\mathbb{O}_{im}$

G_2 is a stabilizer of this 3-form