RESOLVENT CONDITIONS FOR PERTURBATIONS

Well-posed Cauchy problem in a Banach space X

$$u'(t) = Au(t) \quad (t \geq 0), \qquad u(0) = x$$

A generates a C_0-semigroup $\{T(t) : t \geq 0\}$, where

$$u(t) = T(t)x, \quad T(t) \text{ “}=\text{” } e^{tA},$$

$$R(\lambda, A) := (\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} T(t)\, dt$$

Given $B : D(A) \to X$ (always bounded for the graph norm), when does $A + B$ generate a C_0-semigroup?

Dyson-Phillips series

$$S(t) = T(t) + \sum_{n=1}^{\infty} (V^n T)(t)$$

$$(VF)(t) = \int_0^t T(t-s) B F(s)\, ds, \qquad F : [0, \infty) \to \mathcal{B}(X)$$

This works for:

- $B \in \mathcal{B}(X)$ (Phillips)
- Miyadera-Voigt conditions (Schrödinger operators, delay equations):

  $$\int_0^t \|BT(s)x\|\, ds \leq q\|x\| \quad (x \in D(A))$$

  where $q < 1$.
- Desch-Schappacher conditions (population dynamics)

Resolvent estimates

General Hille-Yosida conditions are not amenable to perturbations.

First-order resolvent conditions are amenable:

$$R(\lambda, A + B) = R(\lambda, A)(I - BR(\lambda, A))^{-1}$$

- T contractive, A dissipative, $\|\lambda R(\lambda, A)\| \leq 1$ $(\lambda > 0)$.

  If B is dissipative with A-bound less than 1, then $A + B$ generates. (Lumer-Phillips)

- T holomorphic, $\|\lambda R(\lambda, A)\| \leq c$ $(\operatorname{Re} \lambda > \omega)$.

  If B has A-bound 0, then $A + B$ generates. (Hille)

  If B is A-compact, then $A + B$ generates. (Desch-Schappacher)

A second-order integral condition

Second-order resolvent conditions are reasonably amenable to perturbations.

Theorem (Gomilko, Shi-Feng). *Suppose that A is closed and densely defined with $\sigma(A) \subseteq \{\lambda : \operatorname{Re}\lambda \le 0\}$, and suppose that for all $x \in X$ and $y \in X^*$,*

$$\sup_{a>0} a \int_{-\infty}^{\infty} \left|\langle R(a+is,A)^2 x, y\rangle\right| \, ds < \infty.$$

Then A generates a bounded C_0-semigroup on X.

Corollary. *Suppose that A is closed and densely defined on a Hilbert space H. Then A generates a C_0-semigroup if and only if there exists ω such that, for all $x \in H$,*

$$\sup_{a>\omega} a \int_{-\infty}^{\infty} \|R(a+is,A)x\|^2 \, ds < \infty,$$

$$\sup_{a>\omega} a \int_{-\infty}^{\infty} \|R(a+is,A)^* x\|^2 \, ds < \infty.$$

Theorem (Kaiser-Weis; B.). *Suppose that A generates a C_0-semigroup on a Hilbert space H, and $B : D(A) \to H$. Suppose that there exist $q < 1$ and ω such that $\sigma(A) \subset \{\operatorname{Re}\lambda \le \omega\}$ and*

$$\|BR(\lambda,A)\| \le q, \qquad \|R(\lambda,A)By\| \le q\|y\| \quad (y \in D(A))$$

whenever $\operatorname{Re}\lambda > \omega$. Then $A+B$ generates a C_0-semigroup on H.

A converse result

Desch and Schappacher showed that their theorem for relatively compact perturbations of holomorphic semigroups does not apply to any other semigroups:

Theorem. *Suppose that $A+B$ generates a C_0-semigroup T for every rank-1 operator $B : D(A) \to X$ of arbitrarily small A-norm. Then T is holomorphic.*

Sketch of proof. For each B, $R(\lambda, A+B)$ is bounded on a right half-plane (depending on B). A Baire category argument implies that $\lambda R(\lambda, A)$ is bounded on a right half-plane.

The argument can be abstracted. Suppose that

- A is densely defined,
- $C : D(A) \to X$ is A-bounded,
- $CR(\lambda, A)x$ is bounded in some region for sufficiently many x,
- for each B of the form $Bx = \langle Cx, b^* \rangle a$ with $\|a\| \, \|b^*\|$ arbitrarily small, $A+B$ satisfies one of a countable family of more or less arbitrary resolvent growth conditions in suitable regions.

Then $CR(\lambda, A)$ is bounded in one of the regions.

Theorem above remains valid if $A+B$ generates a "distribution semigroup" in the sense of Lions.

Cosine functions

Cosine functions are to second-order Cauchy problems as C_0-semigroups are to first-order problems. Thus A generates a cosine function $\{C(t) : t \geq 0\}$ if and only if

$$\begin{aligned} u''(t) &= Au(t) \quad (t \geq 0), \\ u(0) &= x, \\ u'(0) &= 0 \end{aligned}$$

is well-posed. The solutions are given by $u(t) = C(t)x$, and

$$R(\lambda^2, A) = \lambda \int_0^\infty e^{-\lambda t} C(t)\, dt \qquad (\mathrm{Re}\, \lambda > \omega).$$

Example. Let A_0 generate a C_0-group $\{U(t) : t \geq 0\}$ and $A = A_0^2$. Then A generates a cosine function given by

$$C(t) = \frac{1}{2}(U(t) + U(-t)).$$

If A generates a cosine function, then there is a unique “phase space” W. If $B : W \to X$ is bounded, then $A + B$ generates a cosine function.

If X is a UMD-space, then $W = D((\omega I - A)^{1/2})$ for suitable ω. (Fattorini)

Theorem. *Suppose that A generates a cosine function, and let $\gamma > \frac{1}{2}$. Suppose that, for each $B : D((\omega I - A)^\gamma) \to X$ of rank-1 and arbitrarily small norm, $A + B$ generates an (integrated) cosine function. Then A is bounded.*

Semigroups and fractional powers

Suppose that A generates a semigroup. Fix $\gamma \in (0,1)$ and assume that, for each $B : D((\omega I - A)^\gamma) \to X$ (of rank-1 and small norm), $A + B$ generates a semigroup. Then
(CP)
$$\|R(a+is, A)\| = O(|s|^{-\alpha}) \text{ as } |s| \to \infty \text{ for some/all } a;$$

equivalently,

$$T(t)(X) \subseteq D(A) \text{ and } \|AT(t)\| = O(t^{-\beta}) \text{ as } t \downarrow 0.$$

Here α is approximately equal to γ and β is approximately its reciprocal.

Conversely, suppose that A generates a C_0-semigroup and satisfies (CP). Let $B : D((\omega I - A)^\gamma) \to X$ be bounded, where $0 < \gamma < \alpha$. Then $A + B$ generates a C_0-semigroup (via Phillips-Miyadera-Voigt) and also satisfies (CP).

This is also true if X is a Hilbert space, $\alpha = \gamma$ and B is finite rank (via Gomilko-Shi-Feng).

Perturbations of differentiable semigroups

A C_0-semigroup T is *eventually differentiable* if it is norm-differentiable on (t_0, ∞) for some $t_0 \geq 0$; equivalently, $T(t)$ maps X into $D(A)$ for $t > 0$; i.e., mild solutions of the homogeneous Cauchy problem become classical solutions.

T is *immediately differentiable* if $t_0 = 0$.

Phillips asked: If A generates an immediately differentiable semigroup and $B \in \mathcal{B}(X)$ is the semigroup generated by $A + B$ eventually differentiable?

Pazy: T is eventually/immediately differentiable if and only if $\|R(\lambda, A)\| \leq C|\lambda|^m$ in an exponential region $|y| \geq ce^{-bx}$, for some/all $b > 0$.

Hence, Phillips's question has a positive answer when $\|R(\lambda, A)\| \to 0$ as $|\operatorname{Im}\lambda| \to \infty$ in an exponential region.

Renardy showed that the answer to Phillips's question is negative.

In fact,

> $A + B$ generates an eventually differentiable semigroup for every $B \in \mathcal{B}(X)$ in a uniform way

if and only if

> $\|R(\lambda, A)\| \to 0$ as $|\operatorname{Im}\lambda| \to \infty$ in an exponential region.

Delay equations

Consider the delay differential equation:

$$\text{(DDE)} \qquad u'(t) = Au(t) + \Phi u_t \quad (t \geq 0), \qquad u_0 = f.$$

Here,

$$\begin{aligned} u_t(\theta) &= u(t+\theta) \qquad (t \geq 0, \theta \in [-1,0]), \\ \Phi : \mathcal{C} &:= C([-1,0],X),X) \to X \quad \text{(bounded)} \end{aligned}$$

There is an associated semigroup $V\Phi$ on $\mathcal{C}$ generated by B_Φ:

$$\begin{aligned} D(B_\Phi) &= \left\{ f \in C^1 : f(0) \in D(A) \text{ and } f'(0) = Af(0) + \Phi f \right\} \\ B_\Phi f &= f'. \end{aligned}$$

Solutions of (DDE) are given by $u(t) = (V_\Phi(t)f)(0)$.

Question: When is V_Φ eventually differentiable, i.e., when do all mild solutions of (DDE) become classical solutions after some fixed time?

Theorem. *Assume that the semigroup generated by A is immediately differentiable. The following are equivalent:*

(1) *V_Φ is eventually differentiable for every Φ;*
(2) *V_Φ is eventually differentiable when $\Phi(f) = f(-1)$;*
(3) *A satisfies* (CP).