# Weighted Ergodic Theory

# and

# Dimension free Estimates

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• <u>Hardy - Littlewood maximal function</u>:

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \left| \int_{x+B_r} f(y)\, dy \right|$$

for $f : \mathbb{R}^n \to \mathbb{C}$ loc. intgble. For $1 < p < \infty$, $\exists\, C_p$ s.t.

$$\|Mf\|_p \leq C_p \|f\|_p .$$

• $\mathbb{T}^n$ ; $\mathbb{Z}^n$ with any total (group) order

<u>Conjugate map</u>:

$$H : f \longrightarrow -i \sum_{\substack{\gamma \in \mathbb{Z}^n \\ \gamma > 0}} \hat{f}(\gamma) e^{it.\gamma} + i \sum_{\substack{\gamma \in \mathbb{Z}^n \\ \gamma < 0}} \hat{f}(\gamma) e^{it.\gamma}$$

For $1 < p < \infty$, $\exists\, C_p$ s.t.

$$\|Hf\|_{L^p(\mathbb{T}^n)} \leq C_p \|f\|_{L^p(\mathbb{T}^n)} .$$

$$R_j f = \partial_j (-\Delta)^{-1/2} f \quad , j = 1, \ldots, n.$$

For $1 < p < \infty$, $\exists\, C_p$ s.t.

$$\| (\sum_{j=1}^{n} |R_j f|^2)^{1/2} \|_p \leq C_p \|f\|_p .$$

- <u>Question</u>: $w > 0$ on $(0, \infty)$

For which weights $w$ can we find $C_{p,w}$ $(1 < p < \infty)$ s.t.

$$(*) \quad \| (\sum_{j=1}^{n} |R_j f|^2)^{1/2} \|_{L^p(w(|x|)dx)} \leq C_{p,w} \|f\|_{L^p(w(|x|)dx)}$$

for all $f \in L^p(\mathbb{R}^n, w(|x|)dx)$ ?

<u>$C_{p,w}$</u> <u>independent of $n$</u>.

$m = 1, 2, \ldots$, $\exists\, C_{p,w,m}$ s.t.

$$\Big\| \Big( \sum_{|\ell| = m} |\partial^\ell (-\Delta)^{-m/2} f|^2 \Big)^{1/2} \Big\|_{p,w}$$

$$\leq C_{p,w,m} \, \|f\|_{p,w}$$

for $f \in L^p(\mathbb{R}^n, w(|x|)\,dx)$.

<u>Proof</u> By induction on $m$ and (an extension of) the Marcinkiewicz - Zygmund principle

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<u>Case $m=2$</u>: Get a priori estimates for solutions of

$$\Delta u = f \, .$$

# Weighted Ergodic Theory

Setting: $(X, \mathcal{F}, \mu)$ a $\sigma$-finite measure space

$\mathcal{T} = \{T^t\}_{t\in\mathbb{R}}$ a strongly conts. one parameter group of positive operators on $L^p(\mu)$, $1 \le p < \infty$.

Structure (Kan): For $t \in \mathbb{R}$, there are

- mble $h_t > 0$ on $X$
- isom. of measure algebra $\mathcal{F}/\mu$-null $\Phi^t$ (with natural extension to mble fns.)

such that

$$T^t f = h_t \cdot \Phi^t(f) \qquad (f \in L^p(\mu)).$$

Group structure of $\mathcal{T}$ gives

$$\Phi^{t+s} = \Phi^t \Phi^s \;;\quad h_{s+t} = h_s \Phi^s(h_t)$$

$$M_r f = \frac{1}{2r} \int_{-r}^{r} T^t f \, dt \quad \text{(Bochner integral)}$$

[technical point: can treat $(T^t f)(x)$ as jointly m'ble on $\mathbb{R} \times X$ and so make sense of

$$(M_r f)(x) = \frac{1}{2r} \int_{-r}^{r} (T^t f)(x) \, dt \quad \mu\text{-a.e. in } x \;]$$

Hence form associated <u>ergodic</u> <u>maximal</u> <u>operator</u>

$$(\mathcal{M}_\tau f)(x) = \sup_{r>0} |(M_r f)(x)|$$

for $x \in X$, $\mu$-a.e.

<u>Question</u>: When is $\mathcal{M}_\tau$ bdd, $L^p(\mu) \to L^p(\mu)$?

$$J_t = \frac{d\mu}{d\mu \circ \Phi^t}, \text{ i.e. } \int_X f\,d\mu = \int_X J_t\, \Phi^t(f)\,d\mu.$$

<u>Group property for $\tau$ gives</u>

$$J_{t+s} = J_t\, \Phi^t(J_s).$$

<u>Thm</u> (cf Martin-Reyes, de la Torre for discrete analogue)
Spse $1<p<\infty$. Then TFAE:

(i) $\mathcal{M}_\tau$ is bdd $L^p(\mu) \to L^p(\mu)$.

(ii) $\exists C$ s.t. $\|M_r f\|_p \le C\|f\|_p$, all $r>0$, all $f \in L^p(\mu)$.

(iii) For $\mu$-almost all $x \in X$,

$$t \to h_t^{-p}(x)\, J_t(x)$$

is an $A_p$-weight on $\mathbb{R}$ with an $A_p$-constant independent of $x$.

$v : \mathbb{R} \to (0,\infty)$ is an <u>$A_p$-weight</u>

if $\exists$ C s.t.

$$\left(\frac{1}{|I|}\int_I v\right)\left(\frac{1}{|I|}\int_I v^{-\frac{1}{p-1}}\right)^{p-1} \le C$$

for all intervals $I$ in $\mathbb{R}$. $C$ is then an <u>$A_p$-constant</u> for $v$.

[ $A_1$-weight if $v^\# \le Cv$
$v^\#$ = H-L maximal fn. of $v$. ]

<u>Muckenhoupt</u> (1972): $f \to f^\#$ is bdd

$$L^p(\mathbb{R}, v(x)dx) \to L^p(\mathbb{R}, v(x)dx)$$

iff $v$ is an $A_p$ weight.

<u>Hunt</u> - <u>Muckenhoupt</u> - <u>Wheeden</u>:

Same result for the Hilbert transform.

$\mathcal{T} = \{T_t\}_{t\in\mathbb{R}}$ on $L^p(X,\mu)$.

Consider weight $w : X \to (0,\infty)$. Form $\mathcal{M}_\tau$ as before. Immediate that, for $1<p<\infty$,

$$\mathcal{M}_\tau \text{ is bdd } L^p(w\,d\mu) \to L^p(w\,d\mu)$$

$\Longleftrightarrow$

for $\mu$-almost all $x\in X$,

$$t \to h_t^{-p}(x)\, J_t(x)\, \Phi^t(w)(x)$$

is an $A_p$-weight on $\mathbb{R}$ with an $A_p$-constant indept. of $x$.

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Call such a weight an <u>ergodic</u> $A_p$-<u>wt. for</u> $\mathcal{T}$. Write $w\in E_p(\mathcal{T})$.

[Correct definition of $w\in E_1(\mathcal{T})$ is that

$$t \to h_t(x)\, \Phi^t(w)(x)$$

is an $A_1$-wt. for $\mathbb{R}$ with $A_1$-constant $\mu$-a.e. indept. of $x$. Equivalent to

$$\mathcal{M}_\tau w \le C w \quad \mu\text{-a.e.} ]$$

by Rubio de la Francia (reverse Hölder, factorization ...) have analogues in this ergodic setting. Key result is

<u>Extrapolation</u>: Spse that, for some $p_0$ in the range $1 < p_0 < \infty$, $\mathcal{T}$ acts on $L^{p_0}(\mu)$ and $K$ is a <u>sublinear</u> <u>operator</u> s.t.

$$\|Kf\|_{L^{p_0}(v\,d\mu)} \leq C_v \|f\|_{L^{p_0}(v\,d\mu)}$$

for all $v \in E_{p_0}(\mathcal{T})$, where $C_v$ depends only on any $E_{p_0}(\mathcal{T})$ constant for $v$. <u>Then</u>, for $1 < p < \infty$,

$$\|Kf\|_{L^{p}(w\,d\mu)} \leq C_w \|f\|_{L^{p}(w\,d\mu)}$$

for all $w \in E_p(\mathcal{T})$, where $C_w$ depends only on an $E_p(\mathcal{T})$ constant for $w$.

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In fact, a more general extrapolation thm. holds for families of groups $\{\mathcal{T}_\gamma\}_\gamma$.

groups on $L^{p_0}(\mu)$; $K$ sublinear such that

$$\|Kf\|_{L^{p_0}(v\,d\mu)} \le C_v \|f\|_{L^{p_0}(v\,d\mu)}$$

(*) for all wts. $v$ with an $\eta$-uniform $E_{p_0}(T_\eta)$ constant, where $C_v$ depends only on a bound (w.r.t. $\eta$) of these $E_{p_0}(T_\eta)$ constants.

<u>Then</u>

(*) holds with $p_0$ replaced by $p$, for all $p$ in the range $1 < p < \infty$.

Thm Let $1<p<\infty$ and let $w$ be a weight on $\mathbb{R}^n$ s.t., for all $x \in \mathbb{R}^n$ and all $y \in \Sigma_{n-1}$, the weight

$$t \to w(x+ty)$$

is an $A_p$-weight on $\mathbb{R}$ with an $A_p$ constant $\beta$ (indept. of $x, y$). Then

$$(*) \qquad \left\| \left( \sum_{j=1}^{n} |R_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n, w\,dx)} \leq C_\beta \|f\|_{L^p(\mathbb{R}^n, w\,dx)}$$

for all $f \in L^p(\mathbb{R}^n, w\,dx)$.

In particular, spse $v > 0$ on $(0, \infty)$ s.t. for all $x \in \mathbb{R}^n$, $y \in \Sigma_{n-1}$,

$$t \to v(|x+ty|)$$

is an $A_p$-wt. with an $A_p$-const. $\beta$ indept. of $x$, $y$ and $n$. Then we get the dimension free estimate $(*)$ for $w(x) = v(|x|)$ on $\mathbb{R}^n$.

$-1 < \alpha < p-1$. Then we have (*) with

$C_\beta = C_p$ (and similar higher order) <u>with</u>

<u>no</u> <u>n-dependence</u>.

$$(T_y^t f)(x) = f(x+ty).$$

Gives group $T_y = \{T_y^t\}_{t \in \mathbb{R}}$. Here

$$h_t = J_t = 1 \ ; \ \underline{\Phi_y^t(\sigma) = \sigma + ty}.$$

By <u>extrapolation</u>, only need to prove the theorem in the case $p = 2$.

Set up is

- weight $w$ on $\mathbb{R}^n$
- $\exists\, \beta$ s.t. for all $x \in \mathbb{R}^n$, $y \in \Sigma_{n-1}$

$$t \to w(x+ty) = \Phi_y^t(w)$$

is $A_2$ with $\beta$ an $A_2$ constant.

Need to show

$$\left\| \left( \sum_{j=1}^{\infty} |R_j f|^2 \right)^{1/2} \right\|_{L^2(w\,dx)} \leq C_\beta \|f\|_{L^2(w\,dx)}$$

$$(H_y f)(x) = \int_{\mathbb{R}} \frac{f(x-ty)}{t}\,dt = \int_{\mathbb{R}} \frac{\Phi_y f(x)}{t}\,dt$$

[transfer of the Hilbert transform ; ergodic H.T. ]

<u>Show</u> that, for $x$ fixed,

$$\sum_{j=1}^{n} |R_j f(x)|^2 \le C \int_{\Sigma_{n-1}} |H_y f(x)|^2 \, d\sigma(y)$$

where $C$ indept. of $n$. [Consider orthogonal projⁿ of $y \to (H_y f)(x)$ in $L^2(d\sigma)$ onto span $\{y_1, \dots, y_n\}$. ] <u>Then</u>

$$\int_{\mathbb{R}^n} \sum_{j=1}^{n} |R_j f(x)|^2 \, w \, dx$$

$$\le C \int_{\Sigma_{n-1}} \left\{ \int_{\mathbb{R}^n} |(H_y f)(x)|^2 \, w \, dx \right\} d\sigma(y)$$

uses transference

$$\le C C_\beta \int_{\Sigma_{n-1}} \left\{ \int_{\mathbb{R}^n} |f(x)|^2 \, w \, dx \right\} d\sigma(y)$$

$$= C C_\beta \int_{\mathbb{R}^n} |f(x)|^2 \, w \, dx .$$