The resolvent test for admissibility of semigroups and Volterra equations

Jonathan R. Partington

University of Leeds

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Joint work with Birgit Jacob (Dortmund/Berlin)

Linear systems associated with semigroups

H a complex Hilbert space, $(T_t)_{t\geq 0}$ a strongly continuous semigroup of bounded operators,

i.e., $T_{t+u} = T_t T_u$ and $t \mapsto T_t x$ is continuous.

A the infinitesimal generator, defined on domain $\mathcal{D}(A) \subseteq H$.

$$Ax = \lim_{t\to 0} \frac{1}{t}(T_t - I)x.$$

A continuous-time linear system in state form:

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$

$$y(t) = Cx(t) + Du(t),$$

with $x(0) = x_0$, say.

Here u is the **input**, x the **state**, and y the **output**.

Often we take $D = 0$. In general B and C (the control and observation operators) are unbounded.

Note that

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

is said to have mild solution

$$x(t) = T_t x_0.$$

(Infinite-time) admissibility

There is a **duality** here between control and observation. We discuss just observation operators.

Admissibility of observation operators.

Consider
$$\frac{dx(t)}{dt} = Ax(t),$$

$$y(t) = Cx(t),$$

with $x(0) = x_0$, say.

Let $C : \mathcal{D}(A) \to \mathcal{Y}$, Hilbert, be an A-bounded 'observation operator', i.e.,

$$\|Cz\| \leq m_1\|z\| + m_2\|Az\|$$

for some m_1, $m_2 > 0$.

C is **admissible**, if $\exists m_0 > 0$ such that $y(t) = CT_t x_0$ satisfies $y \in L^2(0,\infty;\mathcal{Y})$ and

$$\|y\|_2 \leq m_0\|x_0\|.$$

The Weiss conjecture

Suppose C admissible, take Laplace transforms,

$$\hat{y}(s) = \int_0^\infty e^{-st} y(t)\, dt,$$

$$= C(sI - A)^{-1} x_0.$$

Now if $y \in L^2(0,\infty;\mathcal{Y})$, then $\hat{y} \in H^2(\mathbb{C}_+,\mathcal{Y})$,

Hardy space on RHP (Paley–Wiener), and

$$\|\hat{y}(s)\| = \left\| \int_0^\infty e^{-st} y(t)\, dt \right\| \le \frac{\|y\|_2}{\sqrt{2\,\mathrm{Re}\, s}},$$

by Cauchy–Schwarz.

Thus admissibility, i.e.,

$$\|CT_t x_0\|_{L^2(0,\infty;\mathcal{Y})} \le m_0 \|x_0\|,$$

implies the **resolvent condition**: $\exists m_1 > 0$ such that

$$\|C(sI-A)^{-1}\| \leq \frac{m_1}{\sqrt{\operatorname{Re} s}}, \qquad \forall s \in \mathbb{C}_+.$$

George Weiss (1991) conjectured that the two conditions are equivalent.

This would imply several big theorems in function theory in an elementary way.

1. The case $\dim \mathcal{Y} < \infty$.

Weiss proved it for normal semigroups and right-invertible semigroups.

A decade later, other special cases were considered.

Jacob–JRP (2001). Contraction semigroups.

Le Merdy (2003). Bounded analytic semigroups.

Jacob–Zwart (2004). Not true for all semigroups.

Example 1

$$H = L^2(\mathbb{C}_+, \mu),$$

$$(T_t(x))(\lambda) = e^{-\lambda t} x(\lambda),$$

$$(Ax)(\lambda) = -\lambda x(\lambda).$$

For a Borel measure μ on $\mathbb{C}_+$ take C defined by

$$Cf = \int_{\mathbb{C}_+} f(\lambda)\, d\mu(\lambda).$$

Easily checked that the Weiss conjecture for the above A and C is equivalent to the **Carleson–Vinogradov embedding theorem**:

Let $k_\lambda(s) = 1/(s+\lambda)$. If

$$\|k_\lambda\|_{L^2(\mathbb{C}_+,\mu)} \le M \, \|k_\lambda\|_{H^2} \, ,$$

for each $\lambda \in \mathbb{C}_+$, then a similar inequality holds for all H^2 functions.

Example 2

Take the right shift semigroup on $H = H^2(\mathbb{C}_+)$:

$$(T_t(x))(\lambda) = e^{-\lambda t} x(\lambda),$$

$$(Ax)(\lambda) = -\lambda x(\lambda).$$

Now $C : \mathcal{D}(A) \to \mathbb{C}$ is A-bounded iff it has the form

$$Cx = \int_{-\infty}^{\infty} \overline{c(i\omega)} x(i\omega)\, d\omega,$$

where $c(z)/(1+z) \in H^2(\mathbb{C}_+)$ (easy).

Consider the Hankel operator:

$$\Gamma_c : H^2(\mathbb{C}_-) \to H^2(\mathbb{C}_+), \qquad \Gamma_c u = \Pi_+(c.u),$$

where Π_+ is the orthogonal projection from

$$L^2(i\mathbb{R}) = H^2(\mathbb{C}_+) \oplus H^2(\mathbb{C}_-)$$

onto $H^2(\mathbb{C}_+)$.

Now the Weiss conjecture for this semigroup is equivalent to a theorem given by **Bonsall (1984):**

the Hankel operator Γ_c is bounded if and only if it's bounded on normalized rationals of degree 1 (reproducing kernel thesis).

2. The case $\dim \mathcal{Y} = \infty$.

Weiss conjecture fails even for the shift semigroup on $L^2(0,\infty)$ – Jacob–JRP–Pott (2002).

No straightforward analogue of Bonsall's theorem.

* Some positive results on this case are known.

* Several open questions remain in this area.

Volterra systems

$$\dot{x}(t) = Ax(t) + \int_0^t k(t-s)Ax(s)\,ds, \qquad t \geq 0,$$

$$y(t) = Cx(t), \qquad \text{with} \quad x(0) = x_0,$$

where A generates a C_0 semigroup and

$k \in W^{1,2}(0,\infty)$.

Note that for the choice $k(t) \equiv 0$ we obtain the Cauchy system

$$\dot{x}(t) = Ax(t), \qquad t \geq 0,$$

$$y(t) = Cx(t), \qquad \text{with} \quad x(0) = x_0.$$

For Volterra systems we write

$$x(t) = S_t x_0, \quad t \geq 0.$$

Now $S_{\cdot}$ is not a semigroup, but still turns out to be exponentially bounded in this case, i.e.,

$$\|S_t\| \leq Me^{\omega t}, \quad (t \geq 0)$$

for some constants M and ω.

Can use Laplace transform methods again. Let

$$\begin{aligned} H(s)x_0 &= \hat{S}(s)x_0 \\ &= (sI - (1 + \hat{k}(s))A)^{-1}x_0 \end{aligned}$$

for $\operatorname{Re} s > \omega$.

A larger Cauchy system

Idea of Engel and Nagel. Consider

$$\dot{z}(t) = \mathcal{A}z(t), \qquad t \geq 0,$$

$$w(t) = \mathcal{C}z(t).$$

State space $\mathcal{H} = H \times L^2(\mathbb{R}_+, H)$, and

$$\mathcal{A}\begin{pmatrix} x_0 \\ \\ f_0 \end{pmatrix} = \begin{pmatrix} A & \delta_0 \\ \\ \phi & \frac{d}{d\tau} \end{pmatrix}\begin{pmatrix} x_0 \\ \\ f_0 \end{pmatrix},$$

where $(\phi x)(\tau) = k(\tau)Ax$, $x \in D(A)$, $\tau > 0$, and

$$\mathcal{C}\begin{pmatrix} x_0 \\ \\ f_0 \end{pmatrix} = Cx_0.$$

Then $\mathcal{A}$ generates a C_0 semigroup $\mathcal{T}_.$, say.

Indeed $\mathcal{S}_. = \mathcal{T}_.^{(1,1)}$ in a natural way.

Finite-time admissibility is easier to handle here.

This means for some K, γ we have

$$\|CS_{\cdot}x_0\|_{L^2(0,t;Y)} \le Ke^{\gamma t}\|x_0\|$$

for every $x_0 \in H$ and $t > 0$.

Theorem (Jacob–JRP, 2005) Suppose $\alpha > 0$ and A generates a semigroup $T_{\cdot}$ with $\|T_t\| \le e^{\alpha t}$ for each $t \ge 0$. Then TFAE:

1. C finite-time admissible for Volterra system $S_{\cdot}$.

2. $\mathcal{C}$ finite-time admissible for Cauchy system $\mathcal{T}_{\cdot}$.

3. There are constants $M > 0$ and $\beta \in \mathbb{R}$ with

$\|CH(s)\| \le \frac{M}{\sqrt{\operatorname{Re} s - \beta}}$ for $\operatorname{Re} s > \beta$.

Proof uses the fact that finite-time admissibility is equivalent to infinite-time admissibility for exponentially stable semigroup systems* plus fact that the Weiss conjecture holds for contraction semigroups.

*Not true for Volterra systems. Hence all our woe.

Natural Weiss-type conjecture: for infinite-time admissibility, is it enough to check the usual resolvent-type condition (e.g. if A generates a contraction semigroup)?

NO!

Example with $H = Y = \mathbb{C}$, $A = -I$, $C = I$

and k defined by

$$\hat{k}(s) = -1 + \sqrt{\frac{s}{s+1}} \qquad (s \in \mathbb{C}_+).$$

Exercise for audience: $k \in W^{1,2}(0, \infty)$.

Then

$$CH(s) = \frac{1}{s + \sqrt{\frac{s}{s+1}}},$$

a function not in $H^2(\mathbb{C}_+)$ that still satisfies

$$\|CH(s)\| \leq \frac{M}{\sqrt{\operatorname{Re} s}} \qquad (s \in \mathbb{C}_+)$$

for some $M > 0$.

Summary and conclusions

* Weiss conjecture for semigroup systems subsumes classical results on Hankel operators and Carleson embeddings.

* For Volterra systems, a natural generalization, finite-time admissibility can be analysed by embedding in a bigger semigroup.

* Infinite-time admissibility does not generalize as expected.