Enclosure Methods for Elliptic Partial Differential Equations

Michael Plum, Karlsruhe, Germany

Consider boundary value problem

$$\begin{aligned} -\Delta u + F(x, u, \nabla u) &= 0 \quad \text{on} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

or

$$\begin{aligned} \Delta\Delta u + F(x, u, \nabla u, \dots) &= 0 \quad \text{on} \quad \Omega \\ u = \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

$\Omega \subset \mathbb{R}^n$ domain with some regularity, F given nonline
smoothness

AIM: Derive conditions for existence of a solutio
"close" and explicit neighborhood of some approx

"Conditions": either of general type, to be verified
more special, to be verified automatically on a compu

General concept:

Transformation into *fixed-point equation*

$$u = Tu$$

and computation of appropriate set U such that

$$\boxed{TU \subset U}$$

and moreover, T has certain properties (e.g. contract
or compactness)

Application of some Fixed-Point Theorem (Banach, S

$\rightsquigarrow$ *Existence* of a solution $u^* \in U$

The set U provides *enclosure*

Abstract formulation

Let $(X, \langle\cdot,\cdot\rangle_X), (Y, \langle\cdot,\cdot\rangle_Y)$ Hilbert spaces

Let $\mathcal{F} : X \to Y$ continuously (Fréchet) differentiable r

problem : $\boxed{u \in X,\ \mathcal{F}(u) = 0}$

Aim now (first): Existence and bounds for this abstra

Let $\omega \in X$ *approximate* solution,

$$L := \mathcal{F}'(\omega) : X \to Y \;\; (linear, bounded)$$

Suppose that constants δ and K, and a nondecreasin $g : [0, \infty) \to [0, \infty)$ have been computed such that

a) $\|\mathcal{F}(\omega)\|_Y \leq \delta$,

b) $\|u\|_X \leq K \|Lu\|_Y$ for all $u \in X$,

c1) $\|\mathcal{F}'(\omega + u) - \mathcal{F}'(\omega)\|_{\mathcal{B}(X,Y)} \leq g(\|u\|_X)$ for all $u \in X$

c2) $g(t) \to 0$ as $t \to 0+$

Need in addition (note that L is one-to-one by b))

d) $L : X \to Y$ *onto*

Here, two ways for obtaining d):

1) $\widehat{X} \supset X$ Banach space, embedding $E_X^{\widehat{X}} : X \hookrightarrow \widehat{X}$ *co*

$$\begin{aligned} \mathcal{F} = L_0 + \mathcal{G}\,,\; L_0 \;&:\; X \to Y \; linear, bounded, bijecti \\ \mathcal{G} \;&:\; \widehat{X} \to Y \; continuously \;\; differenti \end{aligned}$$

Then $Lu = r \Leftrightarrow u = \underbrace{-L_0^{-1}\mathcal{G}'(\omega)E_X^{\widehat{X}}}_{\text{compact!}} u + L_0^{-1} r \;\rightsquigarrow$ *Fr*

2) $Y = X'$ *dual* space, $\Phi : X \to X'$ canonical isometri

i.e. $(\Phi u)[v] = \langle u, v \rangle_X$ for

Assume that $\Phi^{-1}L : X \to X$ is *symmetric* (i.e. $(Lu)[v$

for al

Then $\Phi^{-1}L$ selfadjoint, one-to-one $\Rightarrow$ range $(\Phi^{-1}L)$

$\Rightarrow$ range (L) *dens*

Moreover,

$\left.\begin{array}{l} D(L) = X \text{ closed} \\ L \text{ bounded} \end{array}\right\} \Rightarrow \left.\begin{array}{l} L \text{ closed} \\ L \text{ one-to-one} \end{array}\right\} \Rightarrow L$

$\left.\begin{array}{l} L^{-1} \text{ closed} \\ L^{-1} \text{ bounded by b)} \end{array}\right\} \Rightarrow D(L^{-1}) \text{ closed} \Rightarrow$ ran

Transformation of $\mathcal{F}(u) = 0$ into fixed-point problem.

$$\mathcal{F}(u) = 0 \quad \Leftrightarrow \quad \mathcal{F}'(\omega)[u - \omega] = -\mathcal{F}(\omega) - \Big[\mathcal{F}(u) - \mathcal{F}(\omega)$$

$$\Leftrightarrow \underbrace{\mathcal{F}'(\omega)}_{=L}[v] = -\mathcal{F}(\omega) - \Big[\mathcal{F}(\omega + v) - \mathcal{F}(\omega) - \mathcal{F}'(\omega)$$

$$\Leftrightarrow v = -L^{-1}\Big\{\mathcal{F}(\omega) + \Big[\mathcal{F}(\omega + v) - \mathcal{F}(\omega) - \mathcal{F}'(\omega)[$$

Let $V := \{v \in X : \|v\|_X \leq \alpha\}$, $\alpha > 0$ to be chosen

Then $T(V) \subset V$ if $\boxed{\delta \leq \frac{\alpha}{K} - G(\alpha)}$, $G(t) := \int_0^t g(s)ds$

Need either i) T *compact* ($\rightsquigarrow$ Schauder's Fixed-Point Theorem

or ii) T *contractive* ($\rightsquigarrow$ Banach's Fixed-Point Theorem)

ad i) $\widehat{X} \supset X$, $E_X^{\widehat{X}}$ compact, $\mathcal{F} = L_0 + \mathcal{G}$ as before

ad ii) additional contraction condition

$$\boxed{Kg(\alpha) < 1}$$

Theorem: *For some* $\alpha \geq 0$, *let* $\boxed{\delta \leq \frac{\alpha}{K} - G(\alpha)}$ *, and let i) or ii)*

Then, there exists a solution $u \in X$ *of* $\mathcal{F}(u) = 0$ *satisfying*

$$\|u - \omega\|_X \leq \alpha$$

Applications to second-order boundary value problem

$$-\Delta u + F(x,u) = 0 \text{ on } \Omega,\ u = 0 \text{ on } \partial\Omega$$

A) **strong solutions:** $\Omega \subset \mathbb{R}^n$ $(n \leq 3)$ *bounded* and

H^2*-regular* (Poisson's problem uniquely solvable), F g

$$X = H^2(\Omega) \cap \overset{\circ}{H}{}^1(\Omega), \quad Y = L^2(\Omega),$$

$$L_0 = -\Delta, \quad \mathcal{G}(u)(x) := F(x,u(x)) \quad (\widehat{X} = C(\overline{\Omega}))$$

a) $\| -\Delta\omega + F(\cdot,\omega)\|_{L^2} \leq \delta$ *explicitly or by verified q*

b) $\|u\|_{H^2} \leq K\|Lu\|_{L^2} \quad (u \in X)$:

eigenvalue bounds, Sobolev embeddings, a priori e

c) $\left|\frac{\partial F}{\partial u}\Big(x,\omega(x)+y\Big) - \frac{\partial F}{\partial u}\Big(x,\omega(x)\Big)\right| \leq \tilde{g}(|y|)$

$$-\Delta u + F(x,u) = 0 \text{ on } \Omega,\ u = 0 \text{ on } \partial\Omega$$

B) **weak solutions:** $\Omega \subset \mathbb{R}^n$ *Lipschitz*

$X = \overset{\circ}{H}{}^1(\Omega)$, $\langle u,v\rangle_X := \langle \nabla u, \nabla v\rangle_{L^2} + \sigma\langle u,v\rangle_{L^2}$, $Y =$

Fréchet differentiability requires *growth conditions*
however exponential growth if $n \leq 2$.

a) $\|-\Delta\omega + F(\cdot,\omega)\|_{H^{-1}} \leq \|-div(\nabla\omega - \rho)\|_{H^{-1}} + \|di$

$\leq \|\nabla\omega - \rho\|_{L^2} + \widehat{c}\|div\rho - F(\cdot,\omega)\|_{L^2}$,

$\rho \in H(div;\Omega)$ approximation to $\nabla\omega$, $\|u\|_{L^2} \leq \widehat{c}\|$

b) $Lu = -\Delta u + cu$, $c(x) = \dfrac{\partial F}{\partial u}\Big(x, \omega(x)\Big)$

Let $\Phi : X \to Y$, $\Phi u := -\Delta u + \sigma u$ canonical isomet

$\Phi^{-1} L$ is *symmetric*, so

$\|u\|_X \leq K \|Lu\|_Y = K \|\Phi^{-1} Lu\|_X$ for $u \in X$

$$\Longleftrightarrow \boxed{K \geq \Big[\min\Big\{|\lambda| : \lambda \in \text{ spectrum of } \Phi^{-1} L\Big\}\Big]^{-1}}$$

$\rightsquigarrow$ need bounds for essential spectrum (analyticall

eigenvalue bounds:

$$\Phi^{-1} Lu = \lambda u \Longleftrightarrow \boxed{-\Delta u + \sigma u = \frac{1}{1-\lambda}\Big(\sigma - c(x)\Big)u} \;,$$

choose $\sigma > c(x) \quad (x \in \Omega)$

$$\boxed{\Delta\Delta u + \mu\Delta u + F(x,u) = 0 \text{ on } \Omega,\ u = \frac{\partial u}{\partial \nu} = 0 \text{ on }}$$

$\Omega \subset \mathbb{R}^n$ Lipschitz, F given C^1-function, $\mu \geq 0$

$X := \overset{\circ}{H}{}^2(\Omega),\ \langle u, v\rangle_X := \langle \Delta u, \Delta v\rangle_{L^2} + \sigma\langle u, v\rangle_{L^2},\ Y$

a) $\|\Delta\Delta\omega + \mu\Delta\omega + F(\cdot,\omega)\|_{H^{-2}} \leq \|\Delta(\Delta\omega + \mu\omega - \rho)\|_{}$

$$+\|\Delta\rho +$$

$\leq \|\Delta\omega + \mu\omega - \rho\|_{L^2} + \hat{\hat{c}}\|\Delta\rho + F(\cdot,\omega)\|_{L^2},$

$\rho \in L^2(\Omega)$ s.t. $\Delta\rho \in L^2(\Omega)$, ρ approximation to Δ

$\|u\|_{L^2} \leq \hat{\hat{c}}\|u\|_X$ for $u \in X$.

b) $Lu = \Delta\Delta u + \mu\Delta u + cu,\ c(x) = \frac{\partial F}{\partial u}\Big(x, \omega(x)\Big)$

Let $\Phi : X \to Y,\ \Phi u := \Delta\Delta u + \sigma u$ canonical isomet

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and *eigenvalue bounds*:

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choose $\sigma > c(x) \quad (x \in \Omega)$

Eigenvalue bounds

weak EVP $\boxed{\langle u, v\rangle_X = \lambda b(u, v) \text{ for all } v \in X}$

where b bounded, Hermitian, positive bilinear form on

Upper eigenvalue bounds: Rayleigh-Ritz

Let $\tilde{u}_1, \ldots, \tilde{u}_N \in X$ linearly independent. Define $N \times N$

$$A_0 := (\langle \tilde{u}_i, \tilde{u}_j \rangle_X),\ A_1 := (b(\tilde{u}_i, \tilde{u}_j))$$

$\Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_N$ eigenvalues of the matrix EVP

$$A_0 x = \Lambda A_1 x.$$

Then, if $\Lambda_N < \underline{\sigma}_{\text{ess}} := \inf\{$ essential spectrum $\}$, there at least N eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ below $\underline{\sigma}_{\text{ess}}$, and

$$\boxed{\lambda_i \leq \Lambda_i \quad (i = 1, \ldots, N)}$$

Lower eigenvalue bounds: Temple-Lehmann

Let $\tilde{u}_1, \ldots, \tilde{u}_N$ and $\Lambda_1, \ldots, \Lambda_N < \underline{\sigma}_{\text{ess}}$ as before.

Let $w_1, \ldots, w_N \in X$ satisfy

$$\langle w_i, v\rangle_X = b(\tilde{u}_i, v) \text{ for all } v \in X \qquad (*)$$

and let $\rho \in \mathbb{R}$ be such that

$$\Lambda_N < \rho \leq \left\{ \begin{array}{ll} \lambda_{N+1} & \text{, if } \lambda_{N+1} < \underline{\sigma}_{\text{ess}} \text{ exists} \\ \underline{\sigma}_{\text{ess}} & \text{, otherwise} \end{array} \right\} \quad (*$$

Define, besides A_0 and A_1,

$$A_2 := (\langle w_i, w_j\rangle_X),$$

and let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N < 0$ be the eigenvalues of

$$(A_0 - \rho A_1)x = \mu(A_0 - 2\rho A_1 + \rho^2 A_2)x.$$

Then, $\boxed{\lambda_i \geq \rho\left(1 - \frac{1}{1-\mu_{N+1-i}}\right) \quad (i = 1, \ldots, N)}$

$(*)$ often difficult in practice; considerable improvement by *Goe*

$(**)$ *homotopy method*

homotopy method for obtaining ρ such that

$$\boxed{\Lambda_N < \rho \leq \lambda_{N+1}}\ .$$

Let $(b_t)_{t\in[t_0,t_1]}$ family of bilinear forms on X such that

i) for $s \leq t:\ b_s(u,u) \geq b_t(u,u)\ (u \in X)$

ii) for each t: The eigenvalue problem $\langle u,v\rangle_X = \lambda b_t(u,v)$ for all
has at least $N+1$ eigenvalues $\lambda_1^{(t)} \leq \cdots \leq \lambda_{N+1}^{(t)}$ below its es

iii) for $t = t_0$, the eigenvalues of (EVP_t), or at least bounds to

iv) for $t = t_1$, problem (EVP_t) is the *given* one

Consequences: By i), ii), and the min-max-principle $\lambda_k^{(t)}$ *increasi*
fixed $k \in \{1,\ldots,N+1\}$.

In particular, $\lambda_{N+1}^{(t_0)} \leq \lambda_{N+1}^{(t_1)} = \lambda_{N+1}$.

Thus, $\boxed{\rho := \lambda_{N+1}^{(t_0)}}$ can be chosen $\boxed{\text{if } \Lambda_N < \lambda_{N+1}^{(t_0)}.}$

The last condition requires that problem (EVP_{t_0}) (solvable in clo
given one are sufficiently close.