The Riemann-Hilbert problem with a vanishing coefficient that arises in nonlinear hydrodynamics

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Motivation

Variational theory of Stokes waves: it is very important to know whether or not every solution w of the equation

$$\mathcal{C}w' = \lambda\{w + w\mathcal{C}w' + \mathcal{C}(ww')\}, \quad \lambda > 0 \qquad (1)$$

satisfies the Bernoulli constant-pressure condition

$$(1 - 2\lambda w)\{w'^2 + (1 + \mathcal{C}w')^2\} = 1 \quad \text{a.e.}$$

Here $\mathcal{C}u$ denotes the periodic Hilbert transform of a 2π-periodic function $u : \mathbb{R} \to \mathbb{R}$:

$$\mathcal{C}u(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(y) \cot \frac{x-y}{2} dy.$$

J.F. Toland (2000):

a solution w of (1) satisfies the Bernoulli constant-pressure condition if and only if $1 - 2\lambda w \geq 0$;

(1) is equivalent to a nonlinear Riemann-Hilbert problem with the coefficient $1 - 2\lambda w$.

<u>Aim:</u> Show that the nonlinear Riemann-Hilbert problem does not have solutions such that $1 - 2\lambda w$ changes sign.

It is sufficient to show that if $1 - 2\lambda w$ *changes sign, then the corresponding* **linear** *Riemann-Hilbert problem does not have nontrivial solutions.*

Hardy classes

Let $\mathbb{D}$ be the unit disc centred at 0 in the complex plane $\mathbb{C}$. For any holomorphic function $f : \mathbb{D} \to \mathbb{C}$, let

$$\|f\|_p := \sup_{r\in(0,1)} \left(\int_0^{2\pi} |f(re^{it})|^p dt\right)^{1/p}, \quad p < \infty,$$

$$\|f\|_\infty := \sup_{|\zeta|<1} |f(\zeta)|.$$

The *Hardy class* $H^p = H^p(\mathbb{D})$ is the set of all such functions f with $\|f\|_p < \infty$.

For any $f \in H^p$, $f^*(t) := \lim_{r\to 1} f(re^{it})$ is well defined for almost all $t \in \mathbb{R}$ and

$$\|f^*\|_{L_p([0,2\pi])} = \|f\|_p.$$

Linear Riemann-Hilbert problem (homogeneous):
Find $\varphi, \psi \in H^p$ such that

$$\varphi^* = a\,\overline{\psi^*}, \tag{2}$$

where $a : \mathbb{R} \to \mathbb{C}$ is a given 2π-periodic continuous function.

(Connection with Stokes waves: $a = 1 - 2\lambda w$.)

Let $\rho(t)$ denote the distance from $t \in \mathbb{R}$ to the set of zeros of a:

$$\rho(t) := \mathrm{dist}(t, \mathcal{N}), \quad \mathcal{N} := \{x \in \mathbb{R} |\ a(x) = 0\}.$$

Theorem. *(ES & J.F. Toland) Suppose* $1 \le p \le \infty$, $0 \le \mu \le 1$, $a : \mathbb{R} \to \mathbb{R}$ *is continuous and*

$$|a(t)| \le \text{const}\,\rho(t)^{\mu} \quad \textit{for all} \quad t \in \mathbb{R}.$$

Then the Riemann-Hilbert problem (2) *has no nontrivial solutions* $\varphi, \psi \in H^p$ *if*

$$p \ge 2/\mu. \tag{3}$$

Suppose additionally that a *changes sign. Then* (2) *has no nontrivial solutions* $\varphi, \psi \in H^p$ *if*

$$p \ge \frac{2}{1+\mu}. \tag{4}$$

Both inequalities (3), (4) are sharp: there are many cases where nontrivial solutions exist for any smaller value of p.

J. Virtanen (2004): If the values of a belong to two rays and the angle between them equals $\gamma \in [0, \pi]$, then the Riemann-Hilbert problem (2) has no nontrivial solutions $\varphi, \psi \in H^p$ if

$$p > \frac{2}{\frac{\gamma}{\pi} + \mu},$$

and the constant in the right-hand side is sharp.

Definition. *We say that* $f : \mathbb{R} \to \mathbb{R}$ *is* non-oscillating *on* $E \subset \mathbb{R}$ *if the limits*

$$\lim_{E \ni x \to t-0} f(x), \quad \lim_{E \ni x \to t+0} f(x)$$

exist, the former for all t *such that* $(t-\varepsilon, t) \cap E \neq \emptyset$ *for any* $\varepsilon > 0$ *and the latter for all* t *such that* $(t, t+\varepsilon) \cap E \neq \emptyset$ *for any* $\varepsilon > 0$*.*

Let

$$S_0 = \{z \in \mathbb{C} \setminus \{0\}| \ -\alpha \le \arg z \le \alpha\},$$

where $\alpha \in [0, \pi)$. Suppose $a : \mathbb{R} \to S_0 \cup \{0\}$ and let

$$E_0 := a^{-1}(S_0) = \{x \in \mathbb{R}| \ a(x) \in S_0\} = \mathbb{R} \setminus \mathcal{N}.$$

E_0 is an open set of full measure.

Theorem. *(ES & J. Virtanen) Let $1 < p < \infty$ and let $a : \mathbb{R} \to S_0 \cup \{0\}$ be a 2π-periodic continuous function such that*

$$|a(t)| \le \text{const}\, \rho(t)^\mu \quad \textit{for all} \quad t \in \mathbb{R}$$

with

$$\frac{2}{p} + \frac{2\alpha}{\pi} < \mu. \tag{5}$$

Then the Riemann-Hilbert problem (2) *has no nontrivial solutions $\varphi, \psi \in H^p$ provided that* $\arg a$ *is non-oscillating on E_0 or $\mu \le 2$.*

The condition (5) is sharp.

Let $S_l = \{z \in \mathbb{C} \setminus \{0\} | \ \alpha_l \leq \arg z \leq \beta_l\}, \quad l = 0, 1$ and $\quad 0 \leq \alpha_0 \leq \beta_0 < \alpha_1 \leq \beta_1 < 2\pi$. Suppose $a : \mathbb{R} \to S_0 \cup S_1 \cup \{0\}$ and let

$$E_0 = \{x \in \mathbb{R} | \ a(x) \in S_0\},$$
$$E_1 = \{x \in \mathbb{R} | \ a(x) \in S_1\}.$$

E_0 and E_1 are open sets.

Theorem. *(ES & J. Virtanen) Let* $1 < p < \infty$, $\mu \geq 0$, $a : \mathbb{R} \to S_0 \cup S_1 \cup \{0\}$ *be a* 2π*-periodic continuous function such that*

$$|a(t)| \leq \text{const}\, \rho(t)^{\mu} \quad \textit{for all} \quad t \in \mathbb{R}$$

and let $0 < |E_l \cap [0, 2\pi]| < 2\pi$*. If*

$$p > \max\left\{\frac{2}{\mu + \frac{\alpha_1 - \beta_0}{\pi}},\ \frac{2}{\mu + \frac{2\pi - \beta_1 - \alpha_0)}{\pi}}\right\} = \frac{2}{\mu + \frac{\min\{\alpha_1 - \beta_0, 2\pi - (\beta_1 - \alpha_0)\}}{\pi}}, \qquad (6)$$

then the Riemann-Hilbert problem (2) *has no nontrivial solutions* $\varphi, \psi \in H^p$ *provided that* $\arg a$ *is non-oscillating on* E_l, $l = 0, 1$ *or*

$$\mu \leq \min\left\{\frac{\alpha_1 - \alpha_0}{\pi},\ \frac{\beta_1 - \beta_0}{\pi}\right\}.$$

The condition (6) is sharp.

Question: can one drop the non-oscillation condition in the above theorems?

Open problem

(which is not really related to my talk)

Let

$$Aw := \mathcal{C}w'.$$

Then

$$A\left(\sum_{k=-\infty}^{\infty} c_k e^{ikt}\right) = \sum_{k=-\infty}^{\infty} |k| c_k e^{ikt};$$

$\sim$ first order ΨDO on the unit circle $\mathbb{T}$ with the symbol $|\xi|$;

$\sim$ $\sqrt{-\triangle}$ on $\mathbb{T}$.

Question: Is there a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(\tau) \to +\infty$ as $\tau \to +\infty$ and

$$\sharp\{\text{negative eigenvalues of } A - qI\} \geq h\left(\|q\|_{L^1(\mathbb{T})}\right),$$

$\forall q \geq 0, \quad q \in L^1(\mathbb{T})$?