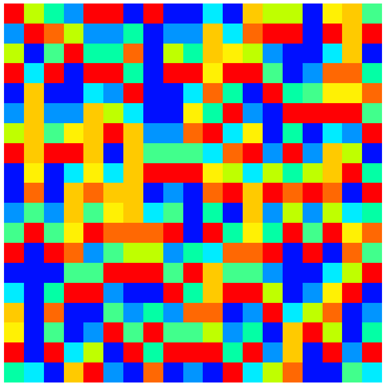
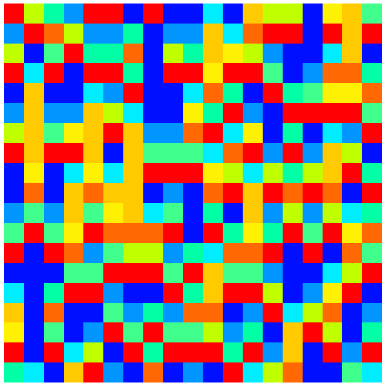
Statistical Mechanics of Nonhyperbolic Coupled Map lattices



Christian Beck
Queen Mary, University of London

Statistical Mechanics of Nonhyperbolic Coupled Map lattices



Christian Beck
Queen Mary,
University of
London

Content

1 Introduction

large 1-dim lattices, lattice sites i. Dynamics given by

$$\Phi_{n+1}^i = (1-a)T(\Phi_n^i) + \frac{a}{2}(T(\Phi_n^{i-1}) + T(\Phi_n^{i+1}))$$

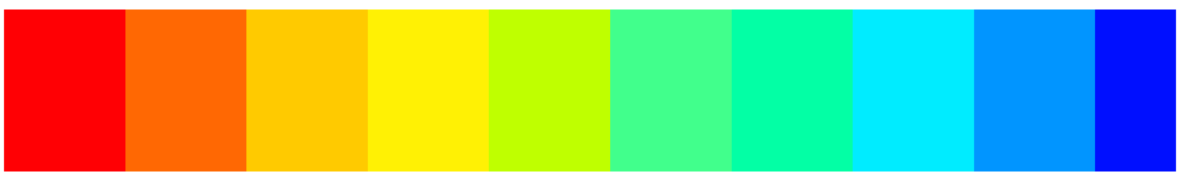
i: discrete spatial coordinate (periodic boundary conditions)
n: discrete time
a: coupling constant
T: local map, e.g. $T(\Phi) = 2\Phi^2 - 1$ (negative Ulam map)

1 Introduction

large 1-dim lattices, lattice sites i. Dynamics given by

$$\Phi_{n+1}^{i} = (1-a)T(\Phi_n^i) + \frac{a}{2}(T(\Phi_n^{i-1}) + T(\Phi_n^{i+1}))$$

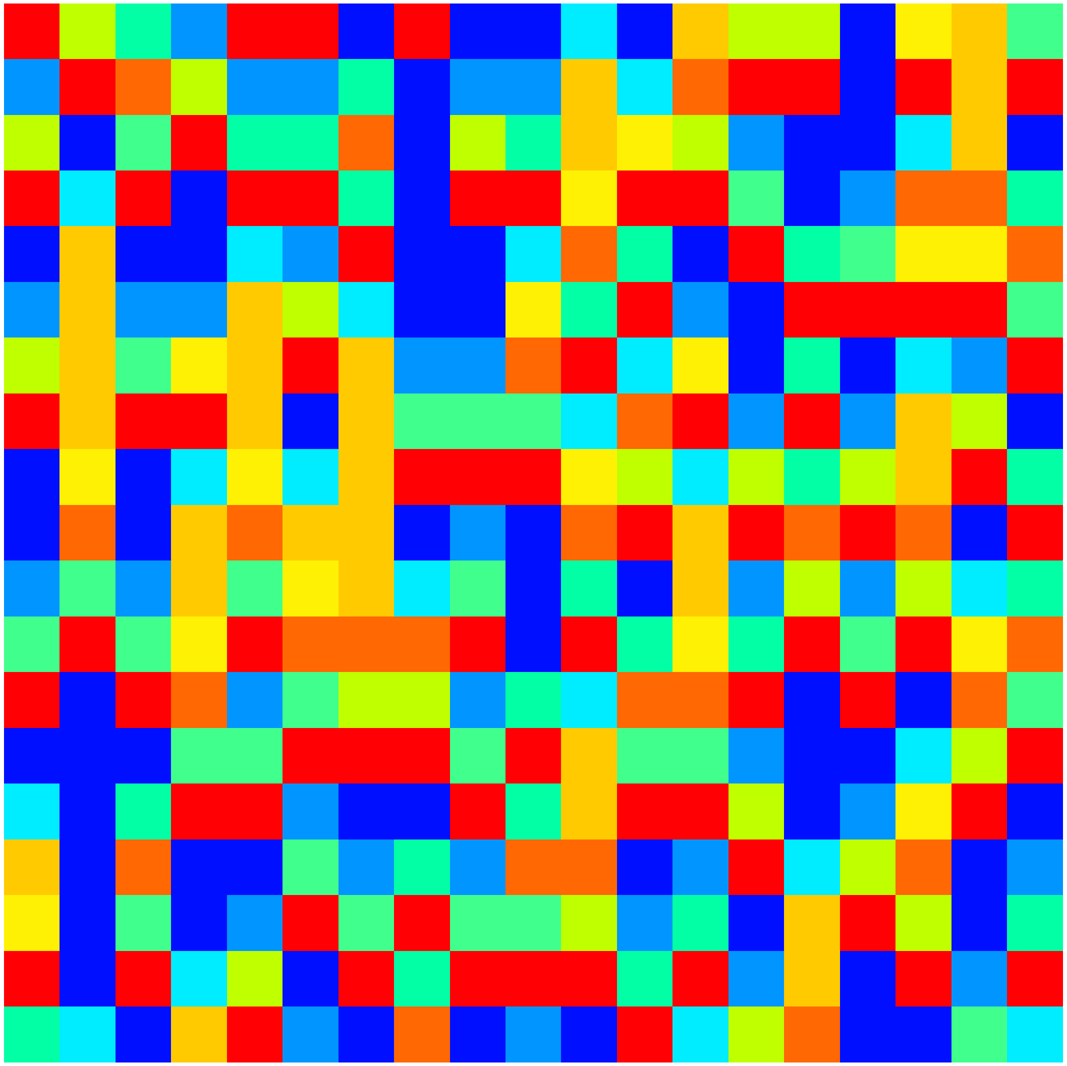
i: discrete spatial coordinate (periodic boundary conditions)
n: discrete time
a: coupling constant
T: local map, e.g. $T(\Phi) = 2\Phi^2 - 1$ (negative Ulam map)
Colour coding:



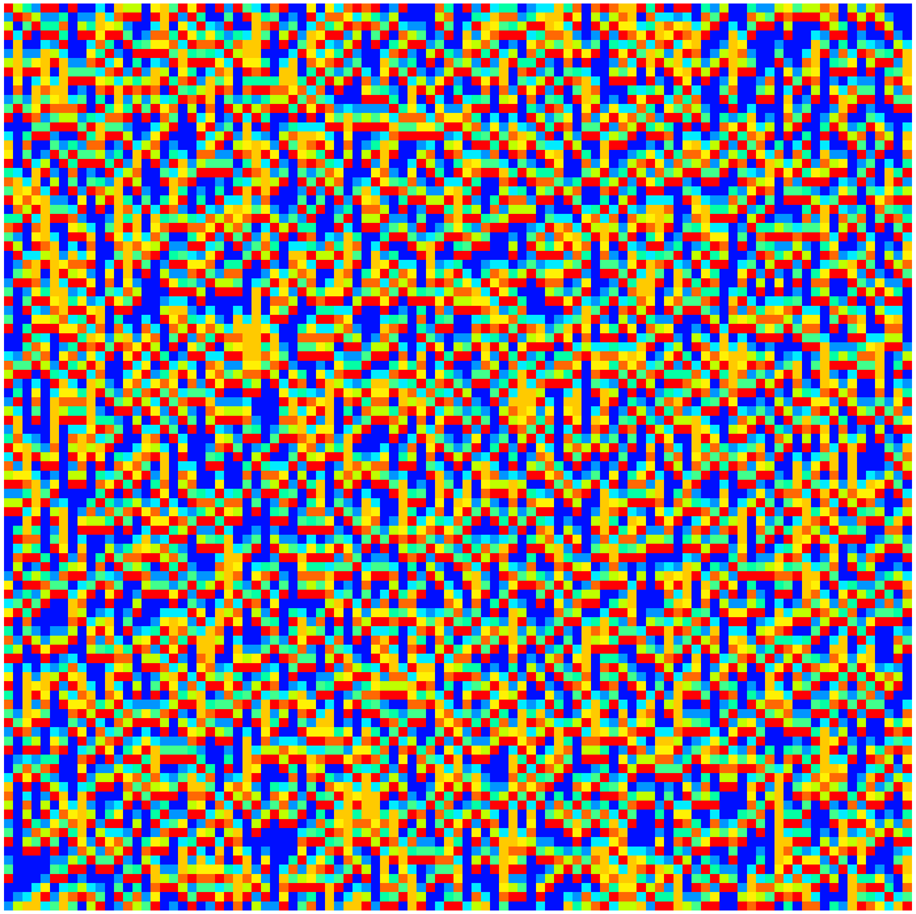
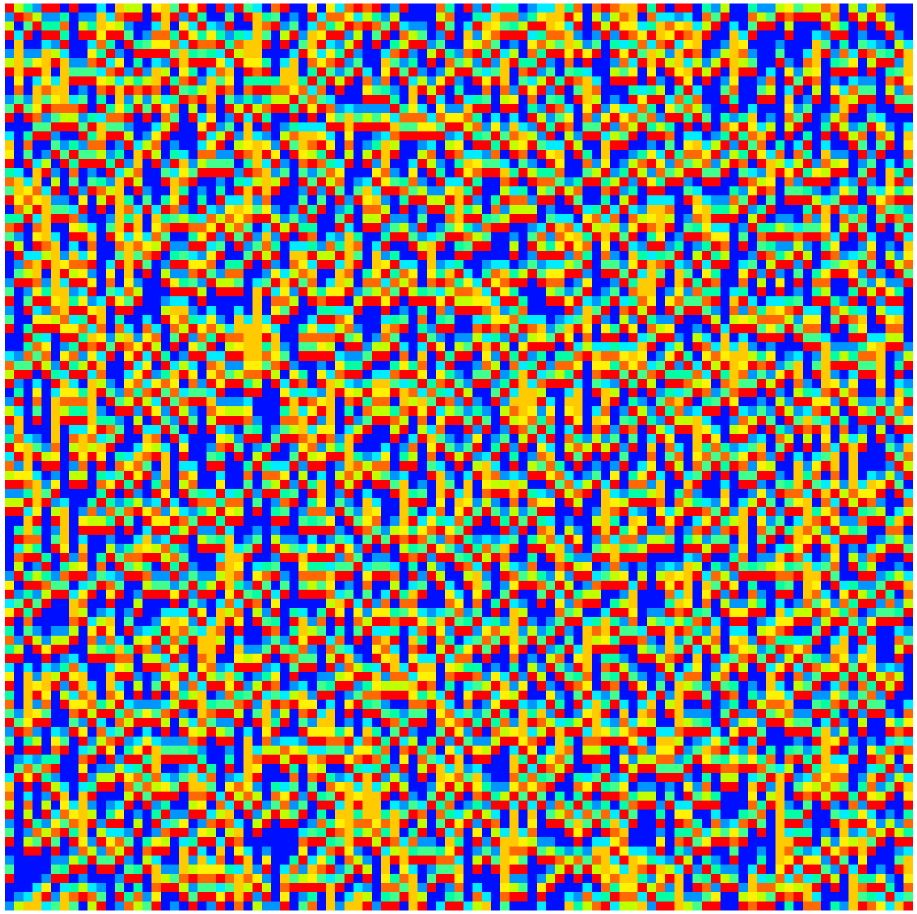
-1.0 -0.8 -0.6 -0.4 -0.2 0.00 +0.2 +0.4 +0.6 +0.8 +1.0

$$\Phi_{n+1}^{i} = (1-a)T(\Phi_n^i) + \frac{a}{2}(T(\phi_n^{i-1}) + T(\Phi_n^{i+1}))$$

$i \rightarrow$ $n \downarrow$

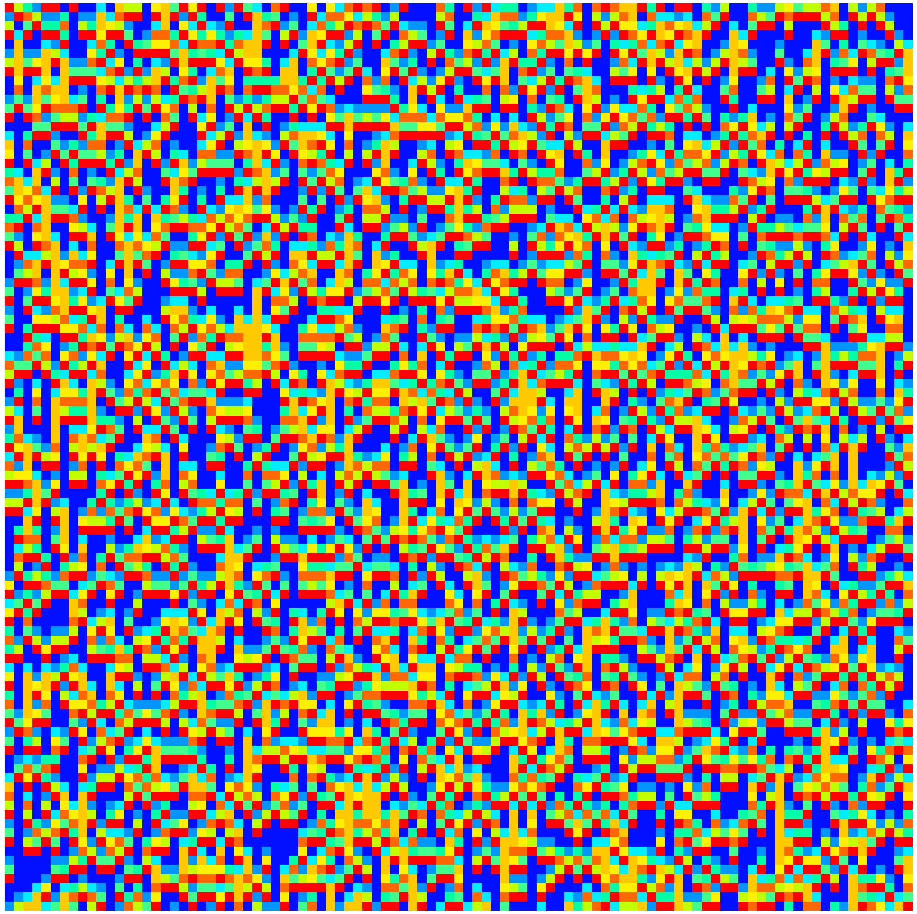


a=0

(larger lattice)  a=0 (larger lattice)  a=0

Ulam map conjugated to tent map, iterates satisfy a Central Limit Theorem for a=0:

$\frac{1}{\sqrt{M}}\sum_{n=1}^{M}\Phi_n^i \longrightarrow$ Gaussian $(M \longrightarrow \infty)$

(larger lattice)  a=0

Ulam map conjugated to tent map, iterates satisfy a Central Limit Theorem for a=0:

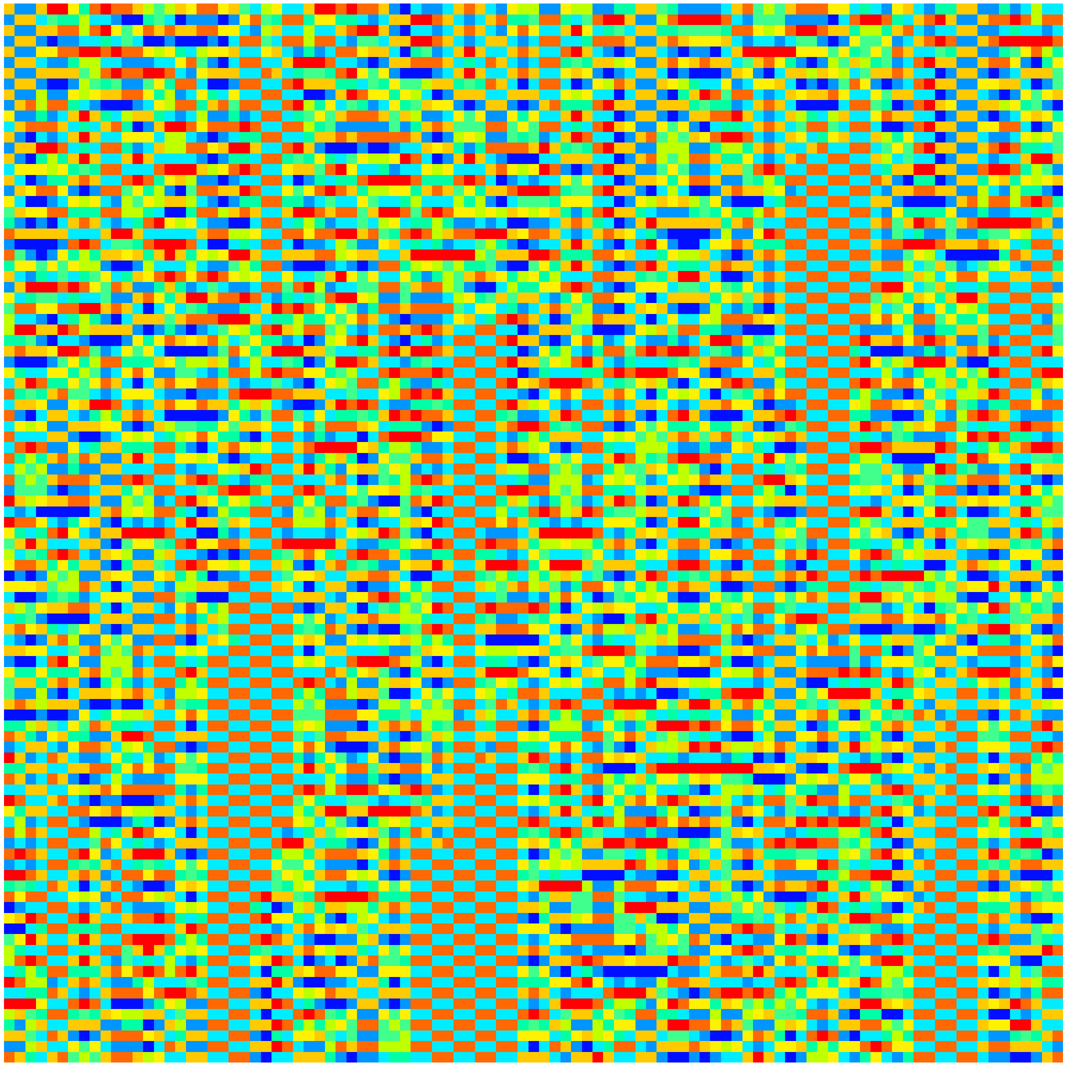
$\frac{1}{\sqrt{M}}\sum_{n=1}^{M}\Phi_n^i \longrightarrow$ Gaussian $(M \longrightarrow \infty)$

But there are complicated higher-order correlations, see C.B., Nonlinearity 4, 1131 (1991)

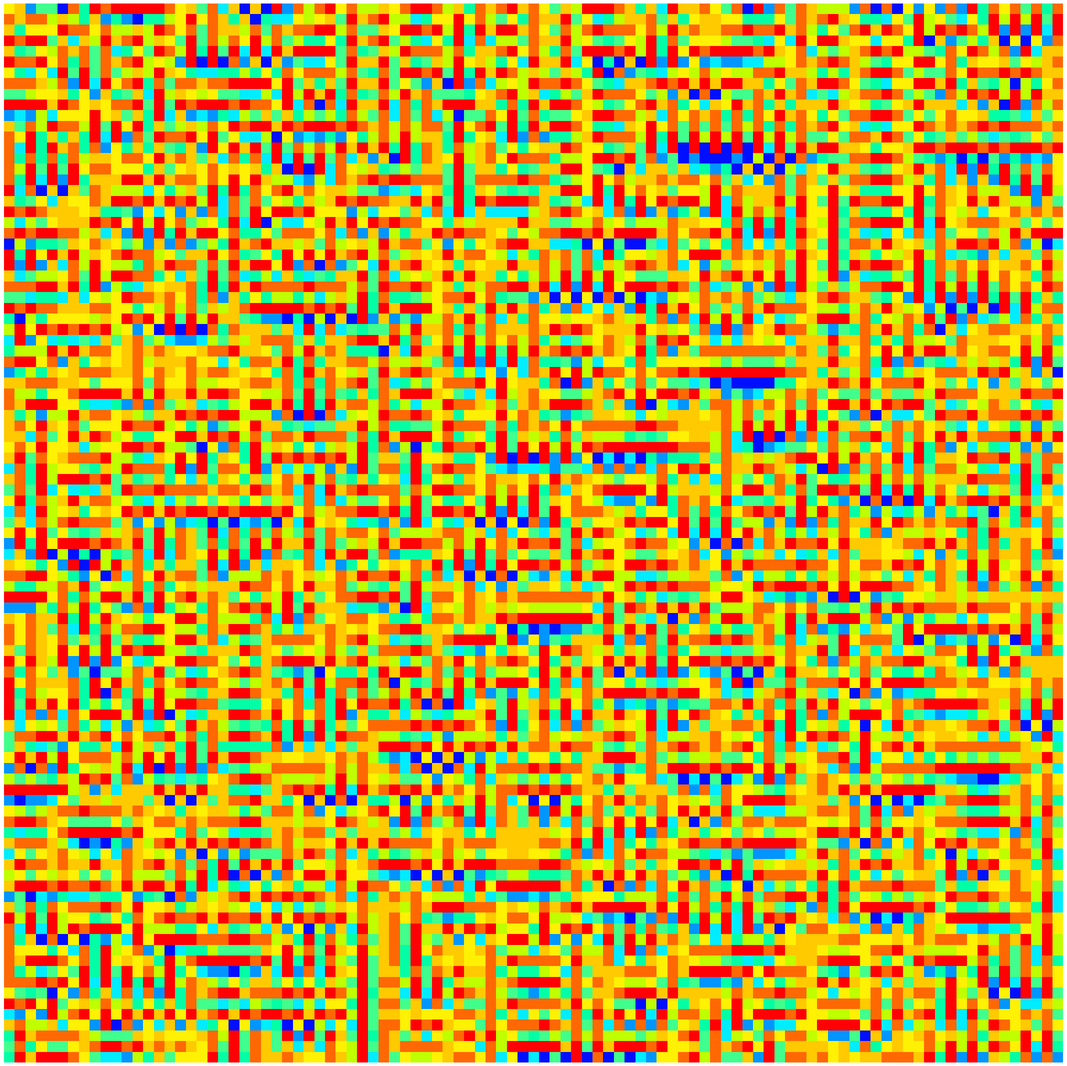
n-point functions $\langle \Phi_{n_1}^i \Phi_{n_2}^i \cdots \Phi_{n_r}^i \rangle$ do not factorize, even for $a = 0$.

These correlations are 'scaled away' for $M \longrightarrow \infty$.

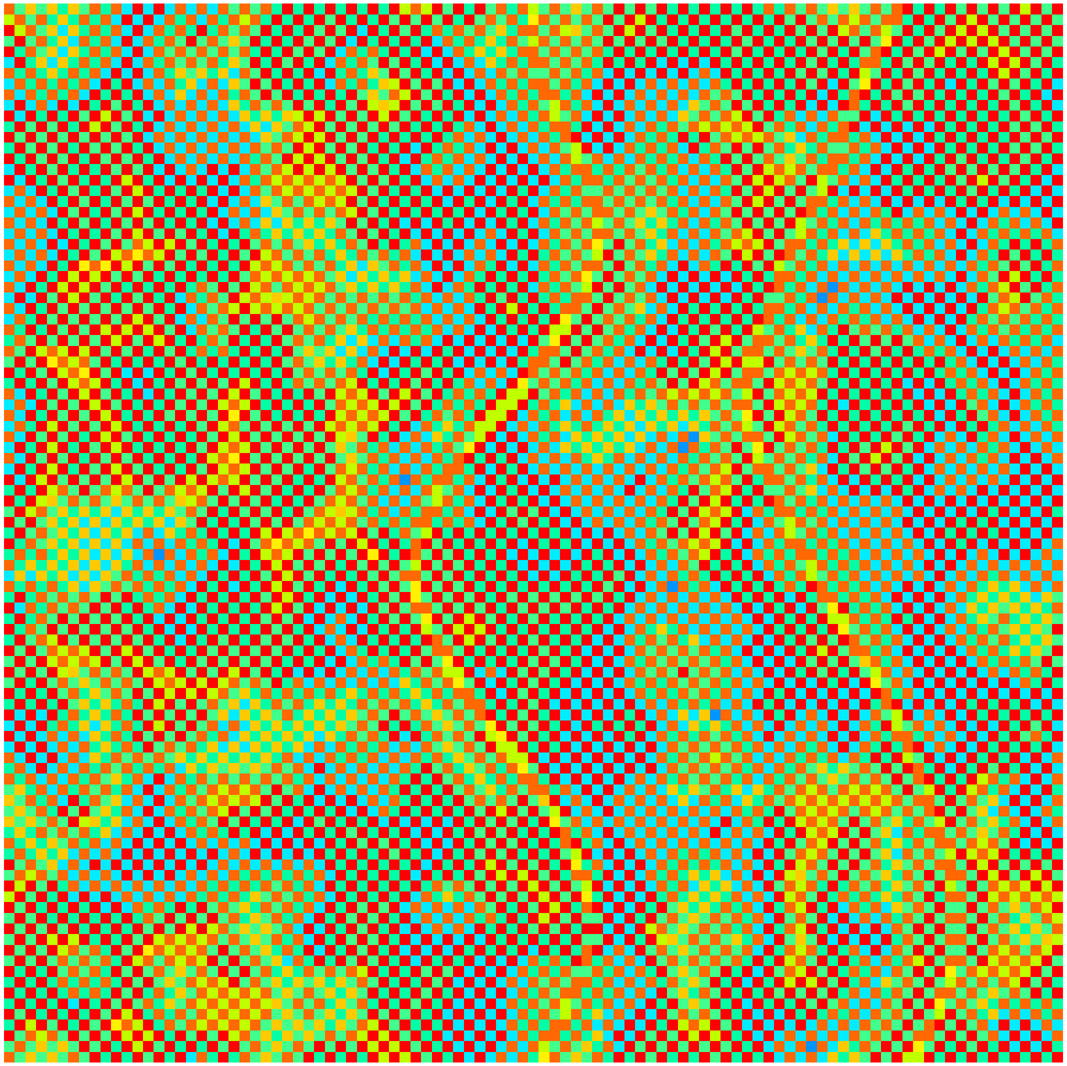
$$\Phi_{n+1}^{i} = (1-a)T(\Phi_n^i) + \frac{a}{2}(T(\phi_n^{i-1}) + T(\Phi_n^{i+1}))$$



$a = 0.375$

$\boldsymbol{a = 1}$ 

a=0.5, 2-dim lattice



(snapshot at fixed time n)

2 Diffusively coupled Tschebyscheff maps

Lots of results for CMLs consisting of hyperbolic (uniformly expanding) maps

(hyperbolicity in 1-d case: $|slope| > 1$) (e.g. work by Keller, Kuenzle, Jarvenpaa, Baladi, Rugh, MacKay, Bunimovich, Just, Pesin,...)

but much less is known for nonhyperbolic situations, though some promising steps have been made (Chaté, Torcini, Ruffo, ...)

We are particularly interested in cases where the local map exhibits strongest possible chaotic behaviour and small coupling, e.g. Tchebyscheff maps T_N of N-th order:

$$T_2(\Phi) = 2\Phi^2 - 1 \tag{1}$$

$$T_3(\Phi) = 4\Phi^3 - 3\Phi \tag{2}$$

$$... = ... \tag{3}$$

$$T_N(\Phi) = \cos(N \arccos \Phi) \tag{4}$$

conjugated to a Bernoulli shift of N symbols (generalized tent maps with $|slope| = N$ having $\sim N/2$ maxima).

2 Diffusively coupled Tschebyscheff maps

Lots of results for CMLs consisting of hyperbolic (uniformly expanding) maps

(hyperbolicity in 1-d case: $|slope| > 1$) (e.g. work by Keller, Kuenzle, Jarvenpaa, Baladi, Rugh, MacKay, Bunimovich, Just, Pesin,...)

but much less is known for nonhyperbolic situations, though some promising steps have been made (Chaté, Torcini, Ruffo, ...)

We are particularly interested in cases where the local map exhibits strongest possible chaotic behaviour and small coupling, e.g. Tchebyscheff maps T_N of N-th order:

$$T_2(\Phi) = 2\Phi^2 - 1 \quad (1)$$

$$T_3(\Phi) = 4\Phi^3 - 3\Phi \quad (2)$$

$$... = ... \quad (3)$$

$$T_N(\Phi) = \cos(N \arccos \Phi) \quad (4)$$

conjugated to a Bernoulli shift of N symbols (generalized tent maps with $|slope| = N$ having $\sim N/2$ maxima).

This conjugacy is 'destroyed' for finite coupling $a > 0$ in the CML.

3 Observed scaling behavior of invariant density

Single Tchebyscheff map: Invariant density given by $\rho_0(x) = \frac{1}{\pi\sqrt{1-\phi^2}}$.

CML with $a = 0$: The invariant density for all M lattice sites is (of course) given by

$$\rho_0(\Phi^1, \Phi^2, \dots, \Phi^M) = \prod_{i=1}^{M} \frac{1}{\pi\sqrt{1-(\Phi^i)^2}} \tag{5}$$

3 Observed scaling behavior of invariant density

Single Tchebyscheff map: Invariant density given by $\rho_0(x) = \frac{1}{\pi\sqrt{1-\phi^2}}$.

CML with $a = 0$: The invariant density for all M lattice sites is (of course) given by

$$\rho_0(\Phi^1, \Phi^2, \ldots, \Phi^M) = \prod_{i=1}^{M} \frac{1}{\pi\sqrt{1-(\Phi^i)^2}} \tag{5}$$

Note that this is like a generalized canonical ensemble (product of q-exponentials) in nonextensive statistical mechanics with $q = 3$, energy $\epsilon = \frac{1}{2}\Phi^2$, $\beta = 1$.

(recall $e_q(x) := (1 + (q-1)x)^{-\frac{1}{q-1}}$, hence $\rho_0(\phi) = \frac{1}{\pi} e_q^{-\beta\epsilon}$ (formal analogy only))

3 Observed scaling behavior of invariant density

Single Tchebyscheff map: Invariant density given by $\rho_0(x) = \frac{1}{\pi\sqrt{1-\phi^2}}$.

CML with $a = 0$: The invariant density for all M lattice sites is (of course) given by

$$\rho_0(\Phi^1, \Phi^2, \ldots, \Phi^M) = \prod_{i=1}^{M} \frac{1}{\pi\sqrt{1-(\Phi^i)^2}} \tag{5}$$

Note that this is like a generalized canonical ensemble (product of q-exponentials) in nonextensive statistical mechanics with $q = 3$, energy $\epsilon = \frac{1}{2}\Phi^2$, $\beta = 1$.

(recall $e_q(x) := (1 + (q-1)x)^{-\frac{1}{q-1}}$, hence $\rho_0(\phi) = \frac{1}{\pi} e_q^{-\beta\epsilon}$ (formal analogy only))

For finite $a > 0$ the density changes and is not a product of single-site densities any more.

Still one can define the 1-point density $\rho_a(\Phi)$ at each lattice site as a marginal density (integrating the joint density over all but one lattice site).

We are interested in averages of arbitrary single-site test functions $h(\Phi^i)$:

$$\langle h(\Phi) \rangle_a = \lim_{M \to \infty, J \to \infty} \frac{1}{MJ} \sum_{n=1}^{M} \sum_{i=1}^{J} h(\Phi_n^i). \tag{6}$$

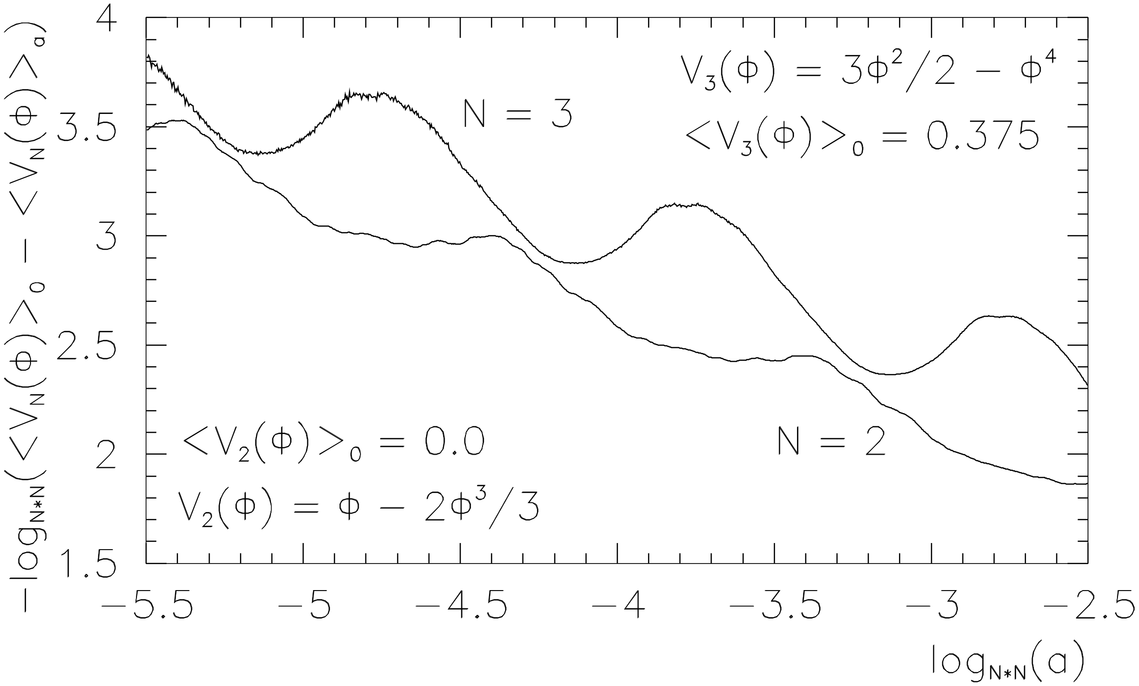
We are interested in averages of arbitrary single-site test functions $h(\Phi^i)$:

$$\langle h(\Phi) \rangle_a = \lim_{M\to\infty, J\to\infty} \frac{1}{MJ} \sum_{n=1}^{M} \sum_{i=1}^{J} h(\Phi_n^i). \tag{6}$$

For $a \longrightarrow 0$ one numerically observes the scaling behaviour

$$\langle h(\Phi) \rangle_a - \langle h(\Phi) \rangle_0 = \sqrt{a} \cdot F^{(N)}(\log a) \tag{7}$$

where $F^{(N)}$ is a periodic function of $\log a$ with period $\log N^2$.



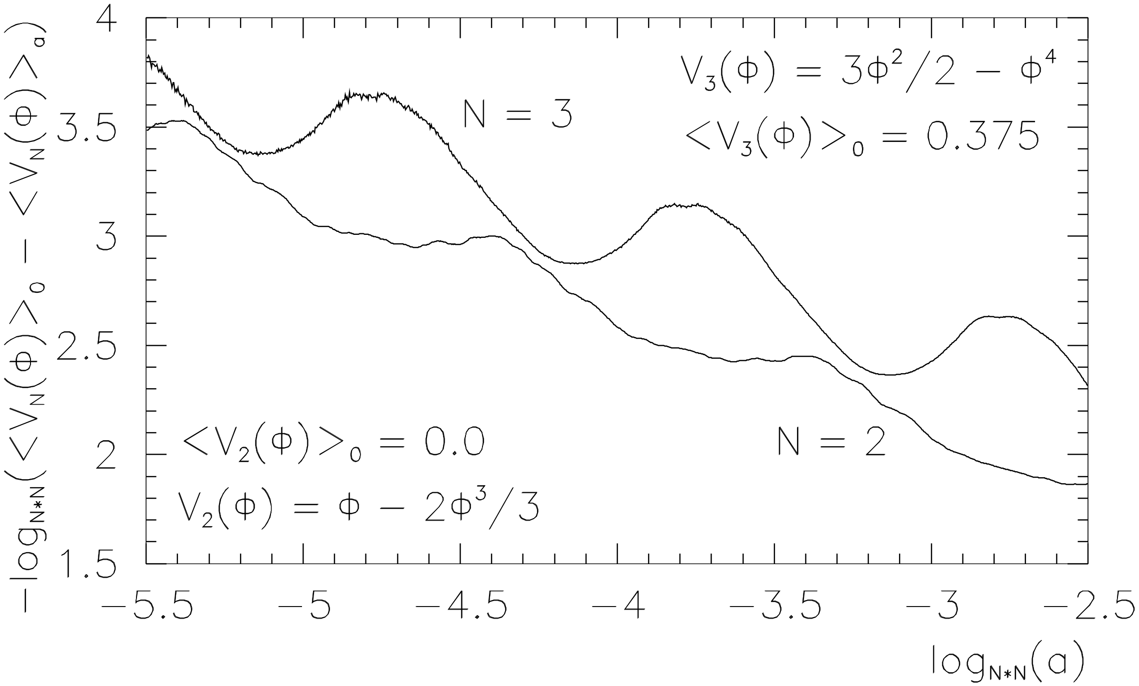
We are interested in averages of arbitrary single-site test functions $h(\Phi^i)$:

$$\langle h(\Phi)\rangle_a = \lim_{M\to\infty, J\to\infty} \frac{1}{MJ} \sum_{n=1}^{M} \sum_{i=1}^{J} h(\Phi_n^i). \tag{6}$$

For $a \longrightarrow 0$ one numerically observes the scaling behaviour

$$\langle h(\Phi)\rangle_a - \langle h(\Phi)\rangle_0 = \sqrt{a} \cdot F^{(N)}(\log a) \tag{7}$$

where $F^{(N)}$ is a periodic function of $\log a$ with period $\log N^2$.



chosen test functions in the above plot :
$h(\Phi) = \Phi - \frac{2}{3}\Phi^3$ $(N=2)$, $h(\Phi) = \frac{3}{2}\Phi^2 - \Phi^4$ $(N=3)$

Not only scaling behaviour in the parameter space but also in the phase space:

Near the left edge of the interval $[-1, 1]$ we may write $\Phi = ay - 1$ and observe the scaling behaviour

$$\rho_a(ay - 1) = a^{-1/2} g(y) \tag{8}$$

where the function g is independent of a for small a.

Not only scaling behaviour in the parameter space but also in the phase space:

Near the left edge of the interval $[-1, 1]$ we may write $\Phi = ay - 1$ and observe the scaling behaviour

$$\rho_a(ay - 1) = a^{-1/2} g(y) \tag{8}$$

where the function g is independent of a for small a.

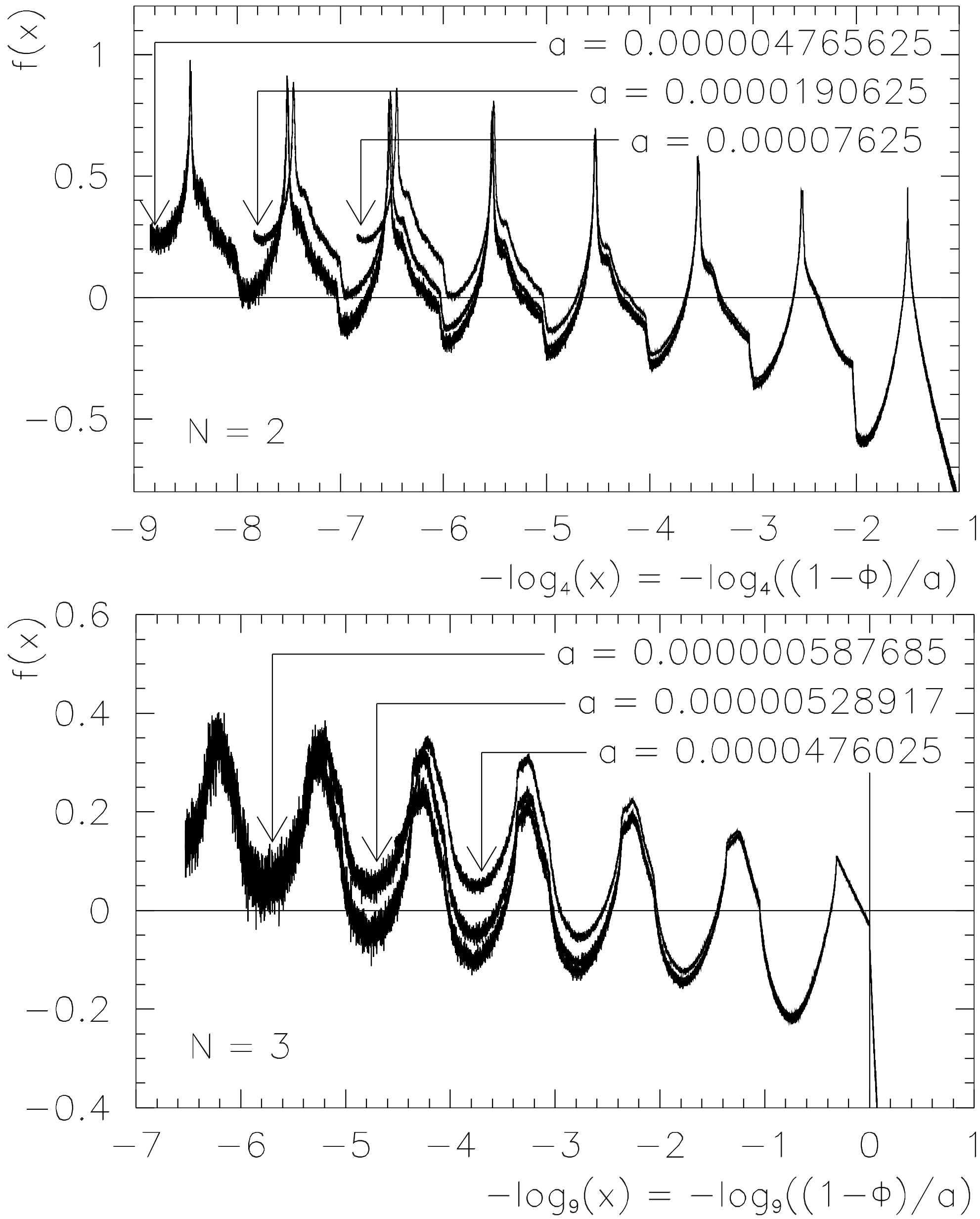
At the right edge, writing $\Phi = 1 - ax$, one observes

$$\rho_a(1 - ax) = \rho_0(1 - ax) + \frac{1}{2} a^{-1/2} x^{-1} f(x) \tag{9}$$

where f is independent of a for small a. Moreover, f exhibits log-periodic oscillations

$$f(N^2 x) = f(x) \tag{10}$$

over a large region of the phase space.


f(x)
a = 0.000004765625
a = 0.0000190625
a = 0.00007625
N = 2
1
0.5
0
−0.5
−9 −8 −7 −6 −5 −4 −3 −2 −1
−log4(x) = −log4((1−Φ)/a)
f(x)
a = 0.000000587685
a = 0.00000528917
a = 0.0000476025
N = 3
0.6
0.4
0.2
0
−0.2
−0.4
−7 −6 −5 −4 −3 −2 −1 0 1
−log9(x) = −log9((1−Φ)/a)

4 Perturbative results for the invariant 1-point density

Final result of a longer calculation ($N = 2$):

Left edge: In leading order in a

$$\rho_a(-1 + ay) = \frac{1}{\pi\sqrt{2a}} \int_{1-y}^{1} \frac{\rho_{00}(z)dz}{\sqrt{y - 1 + z}} \quad (11)$$

$$\rho_{00}(z) = \frac{2}{\pi^2} K(\sqrt{1 - z^2})\theta(1 - z^2) \quad (12)$$

where $K(x)$ is the complete elliptic integral of the first kind.

right edge:

$$\rho_a(1-ax) = \sum_{p=1}^{\infty} \rho_a^{(p)}(1-ax) \tag{13}$$

where

$$\rho_a^{(p)}(1-ax) = \frac{1}{4^p \pi \sqrt{2a}} \int \frac{\rho_0(\phi_+)d\phi_+ \rho_0(\phi_-)d\phi_-}{\sqrt{x/4^p + r_2^p(\phi_+) + r_2^p(\phi_-)}}. \tag{14}$$

Here the function $r_2^p(\phi)$ is defined as follows:

$$r_2^p(\phi) = \frac{1}{2}\sum_{q=0}^{p} \frac{T_{2^q}(\phi) - 1}{2^{2q}}. \tag{15}$$

Limits of the two integrations in Eq. (14) given by the condition that $|\phi_\pm| \leq 1$ and that the argument of the square root should always be positive.

right edge:

$$\boldsymbol{\rho_a(1-ax)=\sum_{p=1}^{\infty}\rho_a^{(p)}(1-ax)} \tag{13}$$

where

$$\boldsymbol{\rho_a^{(p)}(1-ax)=\frac{1}{4^p\pi\sqrt{2a}}\int\frac{\rho_0(\phi_+)d\phi_+\rho_0(\phi_-)d\phi_-}{\sqrt{x/4^p+r_2^p(\phi_+)+r_2^p(\phi_-)}}.} \tag{14}$$

Here the function $\boldsymbol{r_2^p(\phi)}$ is defined as follows:

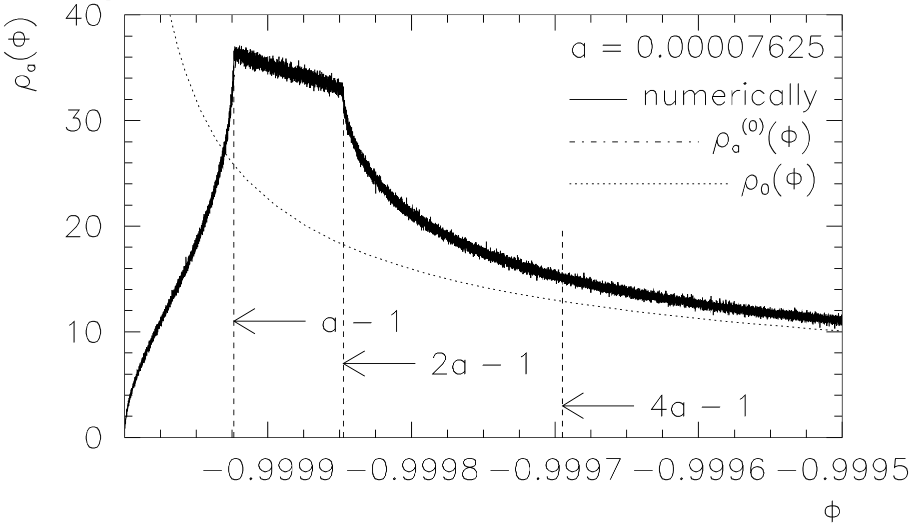
$$\boldsymbol{r_2^p(\phi)=\frac{1}{2}\sum_{q=0}^{p}\frac{T_{2^q}(\phi)-1}{2^{2q}}.} \tag{15}$$

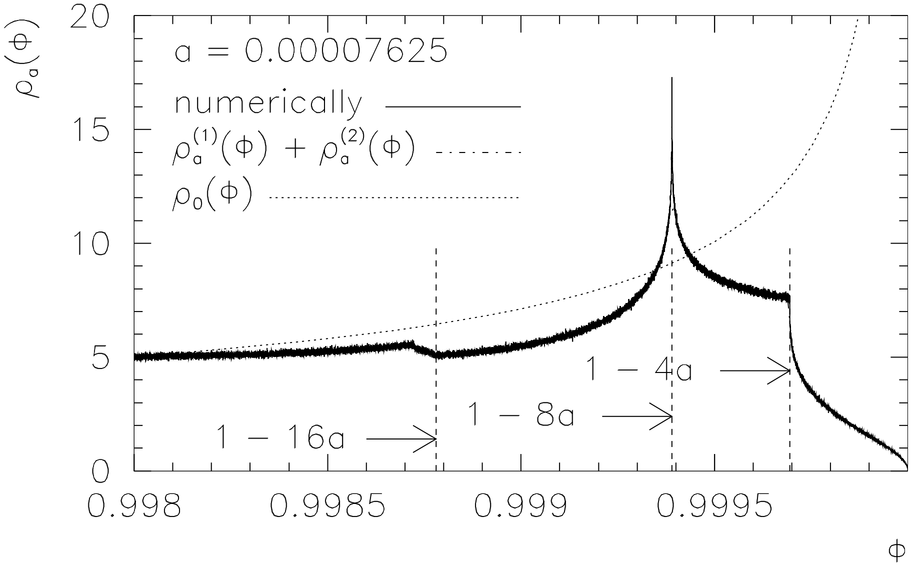
Limits of the two integrations in Eq. (14) given by the condition that $\boldsymbol{|\phi_\pm|\le 1}$ and that the argument of the square root should always be positive.

Techniques: Start from perturbed 1-dimensional map, apply Perron-Frobenius and convolution techniques, iterate result

S. Groote, C.B., nlin.CD/0603397 (2006)

Comparison with numerics:





Along similar lines, we obtain for $N = 3$

$$\rho_a^{(p)}(1 - ax) = \frac{2}{9^p 3\pi\sqrt{2a}} \int \frac{\rho_0(\phi_+)d\phi_+ \rho_0(\phi_-)d\phi_-}{\sqrt{x/9^p + r_3^p(\phi_+) + r_3^p(\phi_-)}}, \tag{16}$$

with

$$r_3^p(\phi) = \frac{1}{2}\sum_{q=0}^{p} \frac{T_{3^q}(\phi) - 1}{9^q}. \tag{17}$$

The density is symmetric, i.e.

$$\rho_a^{(p)}(ax - 1) = \rho_a^{(p)}(1 - ax). \tag{18}$$

For general N:

$$\rho_a^{(p)}(1 - ax) \sim \frac{1}{\sqrt{a}} \int \frac{\rho_0(\phi_+)d\phi_+ \rho_0(\phi_-)d\phi_-}{\sqrt{x/N^{2p} + r_N^p(\phi_+) + r_N^p(\phi_-)}}, \tag{19}$$

with

$$r_N^p(\phi) = \frac{1}{2}\sum_{q=0}^{p} \frac{T_{N^q}(\phi) - 1}{N^{2q}}. \tag{20}$$

Along similar lines, we obtain for $N = 3$

$$\rho_a^{(p)}(1 - ax) = \frac{2}{9^p 3\pi\sqrt{2a}} \int \frac{\rho_0(\phi_+)d\phi_+ \rho_0(\phi_-)d\phi_-}{\sqrt{x/9^p + r_3^p(\phi_+) + r_3^p(\phi_-)}}, \tag{16}$$

with

$$r_3^p(\phi) = \frac{1}{2}\sum_{q=0}^{p} \frac{T_{3^q}(\phi) - 1}{9^q}. \tag{17}$$

The density is symmetric, i.e.

$$\rho_a^{(p)}(ax - 1) = \rho_a^{(p)}(1 - ax). \tag{18}$$

For general N:

$$\rho_a^{(p)}(1 - ax) \sim \frac{1}{\sqrt{a}} \int \frac{\rho_0(\phi_+)d\phi_+ \rho_0(\phi_-)d\phi_-}{\sqrt{x/N^{2p} + r_N^p(\phi_+) + r_N^p(\phi_-)}}, \tag{19}$$

with

$$r_N^p(\phi) = \frac{1}{2}\sum_{q=0}^{p} \frac{T_{N^q}(\phi) - 1}{N^{2q}}. \tag{20}$$

Using these results, one can prove the existence of the log-periodic oscillations both in phase and parameter space.

5 Physical applications for nonhyperbolic CMLs in quantum field theories and cosmology

How can chaotic coupled map lattices be relevant in quantum field theories?

5 Physical applications for nonhyperbolic CMLs in quantum field theories and cosmology

How can chaotic coupled map lattices be relevant in quantum field theories?

Stochastic Quantization.

Consider classical field decribed by an action $S[\varphi]$. Classical field equation:

$$\frac{\delta S}{\delta \varphi} = 0 \tag{21}$$

meaning: Action has an extremum.

5 Physical applications for nonhyperbolic CMLs in quantum field theories and cosmology

How can chaotic coupled map lattices be relevant in quantum field theories?

Stochastic Quantization.

Consider classical field decribed by an action $S[\varphi]$. Classical field equation:

$$\frac{\delta S}{\delta \varphi} = 0 \tag{21}$$

meaning: Action has an extremum.

Parisi-Wu (1981): Obtain 2nd quantized equation of motion by considering a Langevin equation in fictitious time s:

$$\frac{\partial}{\partial s}\varphi(x,s) = -\frac{\delta S}{\delta \varphi}(x,s) + L(x,s) \tag{22}$$

$x = (x^1, x^2, x^3, x^4) = x^\mu$ point in Euclidean space-time

$x^4 = t$ physical time

$L(x,s)$ spatio-temporal Gaussian white noise

$$\langle L(x,s)\rangle = 0 \tag{23}$$
$$\langle L(x,s)L(x',s')\rangle = 2\delta(x-x')\delta(s-s') \tag{24}$$

Parisi and Wu: Quantum mechanical expectations = expectations of Langevin process for $s \longrightarrow \infty$.

Example: φ^4-theory

Action:

$$S[\varphi] = \int d^4x \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4}\varphi^4\right) \tag{25}$$

Classical field equation:

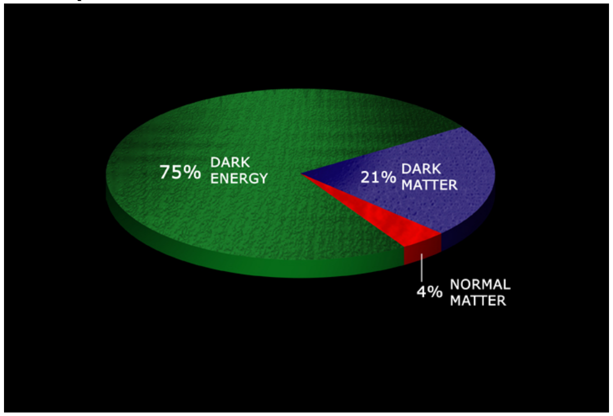
$$(-\partial^2 + m^2)\varphi(x) + \lambda\varphi^3(x) = 0 \tag{26}$$

2nd quantized version:

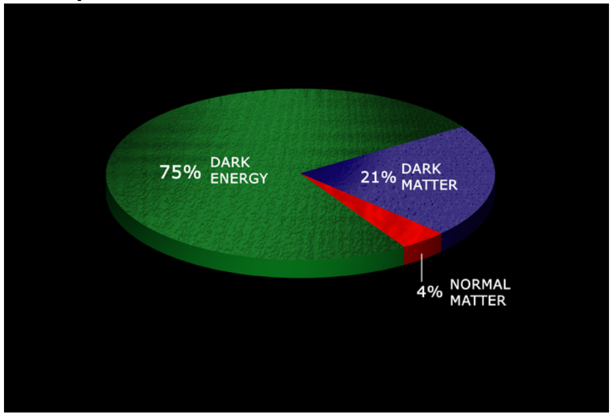
$$\frac{\partial}{\partial s}\varphi(x,s) = (\partial^2 - m^2)\varphi(x,s) - \lambda\varphi^3(x,s) + L(x,s) \tag{27}$$

Now construct a chaotic dark energy model based on a stochastically quantized scalar field (C. B., Phys. Rev. D 69, 123515 (2004))

Now construct a chaotic dark energy model based on a stochastically quantized scalar field (C. B., Phys. Rev. D 69, 123515 (2004))



Now construct a chaotic dark energy model based on a stochastically quantized scalar field (C. B., Phys. Rev. D 69, 123515 (2004))



Quantized scalar field φ in Robertson-Walker metric:

$$\frac{\partial}{\partial s}\varphi = \ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) + L(s,t), \tag{28}$$

where H is the Hubble parameter, V is the potential under consideration and $L(s,t)$ is Gaussian white noise t physical time, s fictitious time. Discretize

$$s = n\tau \tag{29}$$

$$t = i\delta \tag{30}$$

τ: fictitious time lattice constant, δ: physical time lattice constant. We obtain

$$\frac{\varphi_{n+1}^{i} - \varphi_{n}^{i}}{\tau} = \frac{1}{\delta^2}(\varphi_n^{i+1} - 2\varphi_n^{i} + \varphi_n^{i-1}) + 3\frac{H}{\delta}(\varphi_n^{i} - \varphi_n^{i-1}) + V'(\varphi_n^{i}) + noise \tag{31}$$

This can be written as the following recurrence relation for the field φ_n^i

$$\varphi_{n+1}^i = (1-\alpha)\left\{\varphi_n^i + \frac{\tau}{1-\alpha}V'(\varphi_n^i)\right\} + 3\frac{H\tau}{\delta}(\varphi_n^i - \varphi_n^{i-1}) + \frac{\alpha}{2}(\varphi_n^{i+1} + \varphi_n^{i-1}) + \tau \cdot noi \tag{32}$$

where a dimensionless coupling constant α is introduced as

$$\alpha := \frac{2\tau}{\delta^2}. \tag{33}$$

This can be written as the following recurrence relation for the field φ_n^i

$$\varphi_{n+1}^i = (1-\alpha)\left\{\varphi_n^i + \frac{\tau}{1-\alpha}V'(\varphi_n^i)\right\} + 3\frac{H\tau}{\delta}(\varphi_n^i - \varphi_n^{i-1}) + \frac{\alpha}{2}(\varphi_n^{i+1} + \varphi_n^{i-1}) + \tau \cdot noi \tag{32}$$

where a dimensionless coupling constant α is introduced as

$$\alpha := \frac{2\tau}{\delta^2}. \tag{33}$$

Introduce dimensionless field variable Φ_n^i by writing $\varphi_n^i = \Phi_n^i p_{max}$, where p_{max} is some (so far) arbitrary energy scale. $\Longrightarrow$

$$\Phi_{n+1}^i = (1-\alpha)T(\Phi_n^i) + \frac{3}{2}H\delta\alpha(\Phi_n^i - \Phi_n^{i-1}) + \frac{\alpha}{2}(\Phi_n^{i+1} + \Phi_n^{i-1}) + \tau \cdot noise, \tag{34}$$

where the local map T is given by

$$T(\Phi) = \Phi + \frac{\tau}{p_{max}(1-\alpha)}V'(p_{max}\Phi). \tag{35}$$

Note that a symmetric diffusively coupled map lattice (Kaneko 1984)

$$\Phi^i_{n+1} = (1-\alpha)T(\Phi^i_n) + \frac{\alpha}{2}(\Phi^{i+1}_n + \Phi^{i-1}_n) + \tau \cdot noise \tag{36}$$

is obtained if $H\delta << 1$, equivalent to

$$\delta << H^{-1} \tag{37}$$

Note that a symmetric diffusively coupled map lattice (Kaneko 1984)

$$\Phi_{n+1}^{i} = (1-\alpha)T(\Phi_n^i) + \frac{\alpha}{2}(\Phi_n^{i+1} + \Phi_n^{i-1}) + \tau \cdot noise \tag{36}$$

is obtained if $H\delta << 1$, equivalent to

$$\delta << H^{-1} \tag{37}$$

The main result of our consideration is that iteration of a coupled map lattice of the form (36) with a given map T has physical meaning: It means that one is considering the second-quantized dynamics of a self-interacting real scalar field φ with a force V' given by

$$V'(\varphi) = \frac{1-\alpha}{\tau}\left\{-\varphi + p_{max}T\left(\frac{\varphi}{p_{max}}\right)\right\}. \tag{38}$$

Integration yields

$$V(\varphi) = \frac{1-\alpha}{\tau}\left\{-\frac{1}{2}\varphi^2 + p_{max}\int d\varphi\, T\left(\frac{\varphi}{p_{max}}\right)\right\} + const. \tag{39}$$

In terms of the dimensionless field Φ this can be written as

$$V(\varphi) = \frac{1-\alpha}{\tau}p_{max}^2\left\{-\frac{1}{2}\Phi^2 + \int d\Phi T(\Phi)\right\} + const. \tag{40}$$

Lattice constant τ should be small, in order to approximate the continuum theory, which is ordinary quantum field theory. Typical choice $\tau \sim 1/m_{Pl}^2$.

Distinguished example of a φ^4-theory generating strongest possible chaotic behaviour:

$$\boldsymbol{\Phi_{n+1} = T_{-3}(\Phi_n) = -4\Phi_n^3 + 3\Phi_n} \tag{41}$$

on the interval $\boldsymbol{\Phi} \in [-1, 1]$. $\boldsymbol{T_{-3}}$ is the negative third-order Tchebyscheff map, a standard example of a map exhibiting strongly chaotic behaviour. It is conjugated to a Bernoulli shift. The corresponding potential is given by

$$\boldsymbol{V_{-3}(\varphi) = \frac{1-\alpha}{\tau}\left\{\varphi^2 - \frac{1}{p_{max}^2}\varphi^4\right\} + const,} \tag{42}$$

or, in terms of the dimensionless field $\boldsymbol{\Phi}$,

$$\boldsymbol{V_{-3}(\varphi) = \frac{1-\alpha}{\tau}p_{max}^2(\Phi^2 - \Phi^4) + const.} \tag{43}$$

We obtain by second quantization a field $\boldsymbol{\varphi}$ that rapidly fluctuates in fictitious time on some finite interval, provided that initially $\boldsymbol{\varphi_0} \in [-\boldsymbol{p_{max}}, \boldsymbol{p_{max}}]$.

Of physical relevance are the expectations of suitable observables with respect to the ergodic chaotic dynamics. For example, the expectation $\langle V_{-3}(\varphi)\rangle$ of the potential is a possible candidate for vacuum energy in our universe. One obtains

$$\langle V_{-3}(\varphi)\rangle = \frac{1-\alpha}{\tau} p_{max}^2 (\langle \Phi^2 \rangle - \langle \Phi^4 \rangle) + const. \tag{44}$$

For uncoupled Tchebyscheff maps ($\alpha = 0$), expectations of any observable A can be evaluated as the ergodic average

$$\langle A \rangle = \int_{-1}^{+1} A(\Phi) d\mu(\Phi), \tag{45}$$

with the natural invariant measure being given by

$$d\mu(\Phi) = \frac{d\Phi}{\pi\sqrt{1-\Phi^2}} \tag{46}$$

From eq. (46) one obtains $\langle \Phi^2 \rangle = \frac{1}{2}$ and $\langle \Phi^4 \rangle = \frac{3}{8}$, thus

$$\langle V_{-3}(\varphi)\rangle = \frac{1}{8}\frac{p_{max}^2}{\tau} + const. \tag{47}$$

Alternatively, we may consider the positive Tchebyscheff map $T_3(\Phi) = 4\Phi^3 - 3\Phi$. This basically exhibits the same dynamics as T_{-3}, up to a sign. Repeating the same calculation we obtain

$$V_3(\varphi) = \frac{1-\alpha}{\tau}\left\{-2\varphi^2 + \frac{1}{p_{max}^2}\varphi^4\right\} + const \tag{48}$$

and

$$V_3(\varphi) = \frac{1-\alpha}{\tau} p_{max}^2(-2\Phi^2 + \Phi^4). \tag{49}$$

For the expectation of the vacuum energy one gets

$$\langle V_3(\varphi)\rangle = \frac{1-\alpha}{\tau} p_{max}^2(-2\langle\Phi^2\rangle + \langle\Phi^4\rangle) + const, \tag{50}$$

which for $\alpha = 0$ reduces to

$$\langle V_3(\varphi)\rangle = -\frac{5}{8}\frac{p_{max}^2}{\tau} + const. \tag{51}$$

Symmetry considerations between T_{-3} and T_3 suggest to take the additive constant $const$ as

$$const = +\frac{1-\alpha}{\tau} p_{max}^2 \frac{1}{2}\langle\Phi^2\rangle. \tag{52}$$

One obtains the fully symmetric equation

$$\langle V_{\pm 3}(\varphi)\rangle = \pm\frac{1-\alpha}{\tau}p_{max}^2\left\{-\frac{3}{2}\langle\Phi^2\rangle + \langle\Phi^4\rangle\right\}, \tag{53}$$

which for $\alpha \longrightarrow 0$ reduces to

$$\langle V_{\pm 3}(\varphi)\rangle = \pm\frac{p_{max}^2}{\tau}\left(-\frac{3}{8}\right). \tag{54}$$

One obtains the fully symmetric equation

$$\langle V_{\pm 3}(\varphi)\rangle = \pm\frac{1-\alpha}{\tau}p_{max}^2\left\{-\frac{3}{2}\langle\Phi^2\rangle + \langle\Phi^4\rangle\right\}, \tag{53}$$

which for $\alpha \longrightarrow 0$ reduces to

$$\langle V_{\pm 3}(\varphi)\rangle = \pm\frac{p_{max}^2}{\tau}\left(-\frac{3}{8}\right). \tag{54}$$

The simplest model for dark energy in the universe, as generated by a chaotic φ^4-theory, would be to identify $\frac{3}{8}p_{max}^2/\tau = \rho_\Lambda$, the constant vacuum energy density corresponding to a classical cosmological constant Λ, which stays constant during the expansion of the universe.

For a more sophisticated model (including late-time symmetry breaking due to structure formation, and tracking behaviour in the early universe) see C.B., Phys. Rev. D 69, 123515 (2004)

Some interesting aspects of this model

- Vacuum fluctuations underlying dark energy are produced by a deterministic chaotic noise field (a CML) evolving in fictitious time
- Field (almost) conjugated to a Bernoulli shift today. Dynamics given by a CML of diffusively coupled 3rd-order Tchebyscheff map, coupling $a \sim (m/m_{Pl})^2 \sim 10^{-50}$. In the very early universe, coupling can be significantly larger.

Some interesting aspects of this model

- Vacuum fluctuations underlying dark energy are produced by a deterministic chaotic noise field (a CML) evolving in fictitious time
- Field (almost) conjugated to a Bernoulli shift today. Dynamics given by a CML of diffusively coupled 3rd-order Tchebyscheff map, coupling $a \sim (m/m_{Pl})^2 \sim 10^{-50}$. In the very early universe, coupling can be significantly larger.
- Could these chaotic noise fluctuations help to 'derive' ordinary statistical mechanics, by coupling them as a small Langevin-like noise term to ordinary matter?
- Could these or similar types of fluctuations produce measurable effects in laboratory experiments (C.B., M.C.Mackey, Phys. Lett. B 605, 295 (2005))?

6 Summary

- Behaviour of nonhyperbolic CMLs much more complicated than that of hyperbolic ones.
- Scaling with $\sqrt{a}$, logperiodic oscillations , ...
- Analytical perturbative treatment possible for diffusively coupled Tchebyscheff maps of N-th order

  S. Groote, C.B.,nlin.CD/0603397
- Diffusively CMLs do have interesting applications in stochastically quantized field theories, in particular for theories that require a cutoff.

  C.B., Spatio-temporal Chaos and Vacuum Fluctuations of Quantized Fields, World Scientific (2002)
- A chaotic dark energy model, leading to finite vacuum energy, is based on a CML of diffusively coupled 3rd-order Tchebyscheff maps

  C.B., Phys. Rev. D 69, 123515 (2004)