Geometry of equations of Painlevé type

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Method of Integrabl Systems in Geometry,
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Today's talk are based on works listed below.

Papers

- K. Okamoto, Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé, Espaces des conditions initiales, Japan. J. Math. **5**, (1979), 1–79.
- M.-H. Saito, H. Umemura, *Painlevé equations and deformations of rational surfaces with rational double points.* Physics and combinatorics 1999 (Nagoya), 320–365, World Sci. Publishing, River Edge, NJ, 2001.
- H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. **220** (2001), 165–229.
- M-. H. Saito and T. Takebe, *Classification of Okamoto–Painlevé Pairs*, Kobe J. Math.**19**, No.1-2. (2002), 21–55.
- M.-H. Saito, T. Takebe and H. Terajima, Deformation of Okamoto-Painlevé pairs and Painlevé equations, J. Algebraic Geom. **11** (2002), no. 2, 311–362.
- M-. H. Saito, H. Terajima, Nodal curves and Riccati solutions of Painlevé equations, J. Math. Kyoto Univ. **44**, (2004), no. 3, 529–568.
- M.-H. Saito and H. Umemura, *Painlevé equations and deformations of rational surfaces with rational double points*,
- M. Inaba, K. Iwasaki and M.-H. Saito, *Bäcklund transformations of the sixth Painlevé equation in terms of Riemann-Hilbert Correspondence*, Internat. Math. Res. Notices **2004:1** (2004), 1–30.
- M. Inaba, K. Iwasaki and M.-H. Saito, *Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. Part I*, to appear in Publ. of RIMS (2006). (math.AG/0309342).
- ——— *Part II*, to appear in Advanced Studies in Pure Mathematics 42, 2006.
- M. Inaba, K. Iwasaki and M.-H. Saito, Dynamics of the Sixth Painlevé Equation, to appear in Angers proceedings, math.AG/0501007
- M. Inaba, *Moduli of parabolic connections on a curve and Riemann-Hilbert correspondence*, (math.AG/0602004).

- K. Iwasaki, *An Area-Preserving Action of the Modular Group on Cubic Surfaces and the Painlev´ e VI Equation*, Commun. Math. Phys. 242, 185219 (2003).
- K. Iwasaki, T. Uehara, *Periodic Solutions to Painlevé VI and Dynamical System on Cubic Surface*, (math.AG/0512583)
- K. Iwasaki, T. Uehara, *An ergodic study of Painlevé VI*, (math.AG/0604582)

The Purpose of Our Researches

We would like to:

- understand (partial or ordinary) algebraic differential eqautions of Painlevé type by means of geometry of the phase spaces and their relative compactifications.
- find more (partial or ordinary) algebraic differential equations of Painlevé type of higher orders and to classify all of them.

Two Main Strategy

- Strategy 1:
  Compactify the phase space by adding divisors on the boundary. Then analyse the order of poles of ODEs. Painlevé property of ODE imposes rather strong conditions on the order of poles. ($(n - \log)$-conditions).
- Conjecture for $(1 - \log)$-condition:
  For each ODE $\tilde{v}$ of Painlevé type, we can find a good model of family of compactifications of phase spaces, such that $\tilde{v}$ satisfy the $(1-\log)$-conditions on each boundary divisors.
- Resolution of accessible singularities:
  Under the $(1-\log)$-conditions, the accessible singularities can be considered as the zero of some vector bundles on the divisor. Then if there are no accessible singularities at all, divisor satisfies the Okamoto-Painlevé conditions. This fits into our notion of Okamoto-Painlevé pair.

- Related works: K.Okamoto, H.Sakai, S–Umeumura, S–Takebe–Terajima,

- Strategy 2:Moduli theoretic methods–Riemann–Hilbert correspondeces
  Construct the family of the moduli spaces $\mathcal{M} \longrightarrow T \times \Lambda$ (resp. $\overline{\mathcal{M}} \longrightarrow T \times \Lambda$) of stable parabolic connections and stable ϕ-parabolic connections. Moreover, we can construct the family of the moduli spaces $\mathcal{R}\mathbf{ep} \longrightarrow T \times \mathcal{A}$ of representations. Then we have the following Riemann-Hilbert correspondences,

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathbf{RH}} & \mathcal{R}\mathbf{ep} \\ {\scriptstyle \pi}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi} \\ T \times \Lambda & \xrightarrow{(1\times\mu)} & T \times \mathcal{A} \end{array} \tag{1}$$

- Fact: Painlevé equations $=$ Isomonodromic Flows:
  Painlevé equations can be derived from the isomonodromic flows on $\mathcal{M}$.
- Main Theorem:
  Riemann–Hilbert correspondences give proper surjective bimeromorphic analytic morphisms between fibers. This facts shows that the isomonodromic flows satisfies the Painlevé properties.
- Related works:
  Fuchs, Miwa-Jimbo-Ueno (1980– ??), K. Iwasaki(1990–), M.Inaba–K.Iwasaki–S (2003–), M. Inaba (2006), K. Iwasaki–T. Uehara(2005–).

Plan of Talk

- 1 Painlevé Property
- 2 Classification of ODEs with Painlevé Property of order ≤ 2. (due to Poincaré, Fuchs, Painlevé, Gambier).
- 3 Geometry of Spaces of initial Conditions, Okamoto–Painlevé pairs and $(1-\log)$-conditions
- 4 Isomonodromic deformation of Linear ODEs or stable parabolic connections.
- 5 Riemann-Hilbert correspondences.
- 6 Compactification of the moduli space of stable parabolic connections by stable ϕ-connections.
- 7 Geometry of Riemann-Hilbert correspodences. (Bäcklund transformations and Riccati solutions)

1. Painlevé Property

Algebraic ODE:

$$F(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \cdots, \frac{d^mx}{dt}) = 0 \tag{2}$$

where

$$F(t, x_0, x_1, x_2, \cdots, x_m) \in \mathbf{C}(t)[x_0, x_1, \cdots, x_m]$$

Cauchy Problem: Take

$$(t_0, \mathbf{c}_0) = (t_0, c_0, c_1, \cdots, c_m) \in \{(t_0, \mathbf{c}_0) \in \mathbf{C}^{m+2} | F = 0\}.$$

Find a solution $x(t) = \varphi(t; (t_0, \mathbf{c}_0))$ such that

$$\frac{d^i\varphi}{dt^i}(t_0) = c_i, \quad (i = 0, \ldots, m). \tag{3}$$

If the equation (2) is linear, we see that the singularity of the solution $x(t) = \varphi(t, (t_0, \mathbf{c}_0))$ can be detected from the equation itself and does not depend on the initial values.

Example 1.1. **Non-movable singularities**

Consider the linear ODEs and their solutions:

$$(t-a)\frac{dx}{dt} = 1. \Longrightarrow \quad x(t) = \log(t-a) + c_1$$

$$\frac{dx}{dt} = \frac{-x}{(t-a)^2}, \Longrightarrow \quad x(t) = c_2 e^{\frac{1}{t-a}}$$

Solutions have the singularities at $t = a$ which do not depend on the initial values ($=$ integral constants c_1, c_2). Such singularities are called non-movable singularities.

Example 1.2. **Movable singularities**

$' = \frac{d}{dx}$.

(1) $m \geq 2, \quad mx^{m-1}x' = 1 \Longrightarrow x = \sqrt[m]{t-c}$.

movable algebraic branched point.

(2) $x'' + (x')^2 = 0 \Longrightarrow x = \log(t - c_1) + c_2$.

movable logarithmic branched point.

(3) $(xx'' - (x')^2)^2 + 4x(x')^3 = 0 \Longrightarrow x = c_1 \exp(-1/(t - c_2))$.

movable essential singular point.

(4) $x' - x^2 = 0 \Longrightarrow x = \frac{-1}{t-c_1},$.

movable pole.

1.1. **Painlevé property**.

Definition 1.1. An algebraic ODE (2) has Painlevé property if the generic solution of (2) has only poles as its movable singularities.

Example 1.3. : The ODE for Weierstrass $\wp$ function has Painlevé property.

Assume that $g_2, g_3 \in \mathbf{C}, g_2^3 - 27g_3^2 \neq 0$.

$$(x')^2 = 4x^3 - g_2 x - g_3$$

The solutions are given by

$$x(t) = \wp(t - b)$$

where $\wp(t)$ is the Weierstrass $\wp$-function. The constant b can be determined by the initial condition, so the solution $x(t) = \wp(t - b)$ has movable poles of order 2 at $t \equiv b \mod \Lambda$, periods of the above elliptic curve, and no other singularity.

Example 2: Riccati equation

$$x' = a(t)x^2 + b(t)x + c(t). \tag{4}$$

By the change of unknown $x \longrightarrow u$,

$$\boxed{x = -\frac{1}{a(t)}\frac{d}{dt}\log(u) = -\frac{1}{a(t)}\frac{u'}{u}}, \tag{5}$$

the Riccati equation (4) is transformed into the linear equation

$$u'' - [\frac{a'(t)}{a(t)} + b(t)]u' + a(t)c(t)u = 0. \tag{6}$$

Hence the solutions $u(t)$ of (6) has only nonmovable singularities and only movable singularities of $x(t)$ is the zero of $u(t)$. Since the zero of $u(t)$ has a finite order, then the movable singularities of $x(t)$ are only poles.

Classification of 1st order ODE with Painlevé property

Theorem 1.1. (L. Fuchs, H. Poincaré, J. Malmquist, M. Matsuda). For $m = 1$, an algebraic ODE (2) has Painlevé property if and only if (2) can be transformed into one of the following equations:

(1) **Riccati equation**

$$x' = a(t)x^2 + b(t)x + c(t). \tag{7}$$

(2) **The equation of the Weierstrass $\wp$ function** .

$$(x')^2 = 4x^3 - g_2 x - g_3 \tag{8}$$

$(g_2, g_3 \in \mathbf{C}, g_2^3 - 27g_3^2 \neq 0)$.

(3) Or, one can integrate (2) algebraically.

I will give a very simple geometric proof for Theorem 1.1.

The case of order 2 –(original) Painlevé equations

Definition 1.2. Painlevé equation is a second order algebraic ODE of rational type, that is,

$$x^{''} = R(x, x', t), \quad R(x, y, t) \in \mathbf{C}(x, y, t) \tag{9}$$

satisfying Painlevé property.

Painlevé and his student B.O. Gambier showed that Painlevé equation reduces, by an approptiate transformation of the variables, to an equation which can be integrated by quadrature, or to a linear equation, or to P_J, $J = I, II, III, IV, V, VI$. (See Table 1). Here α, β, γ and δ are complex constants.

Painlevé–Gambier Classification

$$
\begin{aligned}
P_I: \quad \frac{d^2x}{dt^2} &= 6x^2 + t, \\
P_{II}: \quad \frac{d^2x}{dt^2} &= 2x^3 + tx + \alpha, \\
P_{III}: \quad \frac{d^2x}{dt^2} &= \frac{1}{x}\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} + \frac{1}{t}(\alpha x^2 + \beta) + \gamma x^3 + \frac{\delta}{x}, \\
P_{IV}: \quad \frac{d^2x}{dt^2} &= \frac{1}{2x}\left(\frac{dx}{dt}\right)^2 + \frac{3}{2}x^3 + 4tx^2 + 2(t^2 - \alpha)x + \frac{\beta}{x}, \\
P_V: \quad \frac{d^2x}{dt^2} &= \left(\frac{1}{2x} + \frac{1}{x-1}\right)\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} + \frac{(x-1)^2}{t^2}\left(\alpha x + \frac{\beta}{x}\right) \\
&\qquad + \gamma\frac{x}{t} + \delta\frac{x(x+1)}{x-1}, \\
P_{VI}: \quad \frac{d^2x}{dt^2} &= \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right)\left(\frac{dx}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)\left(\frac{dx}{dt}\right), \\
&\quad + \frac{x(x-1)(x-t)}{t^2(t-1)^2}\left[\alpha - \beta\frac{t}{x^2} + \gamma\frac{t-1}{(x-1)^2} + \left(\frac{1}{2} - \delta\right)\frac{t(t-1)}{(x-t)^2}\right].
\end{aligned}
$$

Table 1

2. Geometry of Spaces of initial Conditions, Okamoto–Painlevé pairs and $(1-\log)$-conditions

First, let us recall that each P_J is equivalent to a Hamiltonian system H_J

$$(H_J):\begin{cases}\dfrac{dx}{dt} = \dfrac{\partial H_J}{\partial y},\\[2mm] \dfrac{dy}{dt} = -\dfrac{\partial H_J}{\partial x},\end{cases} \tag{10}$$

$$
\begin{aligned}
H_I(x,y,t) &= \frac{1}{2}y^2 - 2x^3 - tx, \\
H_{II}(x,y,t) &= \frac{1}{2}y^2 - \left(x^2 + \frac{t}{2}\right)y - \left(\alpha + \frac{1}{2}\right)x, \\
H_{III}(x,y,t) &= \frac{1}{t}\left[2x^2y^2 - \left\{2\eta_\infty t x^2 + (2\kappa_0+1)x - 2\eta_0 t\right\}y + \eta_\infty(\kappa_0+\kappa_\infty)tx\right], \\
H_{IV}(x,y,t) &= 2xy^2 - \left\{x^2 + 2tx + 2\kappa_0\right\}y + \kappa_\infty x, \\
H_V(x,y,t) &= \frac{1}{t}\left[x(x-1)^2y^2 - \left\{\kappa_0(x-1)^2 + \kappa_t x(x-1) - \eta t x\right\}y + \kappa_\infty(x-1)\right], \\
&\quad \left(\kappa := \frac{1}{4}\left\{(\kappa_0+\kappa_t)^2 - \kappa_\infty^2\right\}\right), \\
H_{VI}(x,y,t) &= \frac{1}{t(t-1)}\left[x(x-1)(x-t)y^2 - \{\kappa_0(x-1)(x-t)\right. \\
&\quad \left. +\kappa_1 x(x-t) + (\kappa_t-1)x(x-1)\}y + \kappa(x-t)\right] \\
&\quad \left(\kappa := \frac{1}{4}\left\{(\kappa_0+\kappa_1+\kappa_t-1)^2 - \kappa_\infty^2\right\}\right).
\end{aligned}
$$

Table 2

Consider the Painlevé vector field

$$(H_J): \quad \boxed{v = \frac{\partial}{\partial t} + \frac{\partial H_J}{\partial y}\frac{\partial}{\partial x} - \frac{\partial H_J}{\partial x}\frac{\partial}{\partial y}} \tag{11}$$

This Painlevé vector field (H_J) is an algebraic regular vector field defined on the space $\mathbf{C}^2 \times B_J \quad \ni (x, y, t)$.
where $B_J = \mathbf{C}$, $\mathbf{C} \setminus \{0\}$ or $\mathbf{C} \setminus \{0, 1\}$.

$$
\begin{array}{ccccc}
v & \mathbf{C}^2 \times B_J & \hookrightarrow & \mathbf{P}^2 \times B_J & \tilde{v} \\
 & \pi \downarrow & & \downarrow & \\
 & B_J & = & B_J &
\end{array}
\tag{12}
$$

$L = \mathbf{P}^2 \setminus \mathbf{C}^2 \simeq \mathbf{P}^1$.

A rational vector field

$$
(H_J): \quad \boxed{\tilde{v} = \frac{\partial}{\partial t} + \frac{\partial H_J}{\partial y}\frac{\partial}{\partial x} - \frac{\partial H_J}{\partial x}\frac{\partial}{\partial y}}
$$

has the pole along $L \times B_J$.

Resolutions of Accessible Singularites

Okamoto's space of initial conditions

$$\begin{tikzcd} \tilde{v} & \mathbf{P}^2 \times B_J \arrow[d] & \mathcal{S} \arrow[l, "\tau"'] \arrow[dl, "\overline{\pi}"] & \tilde{v} \\ & B_J & & \end{tikzcd} \tag{13}$$

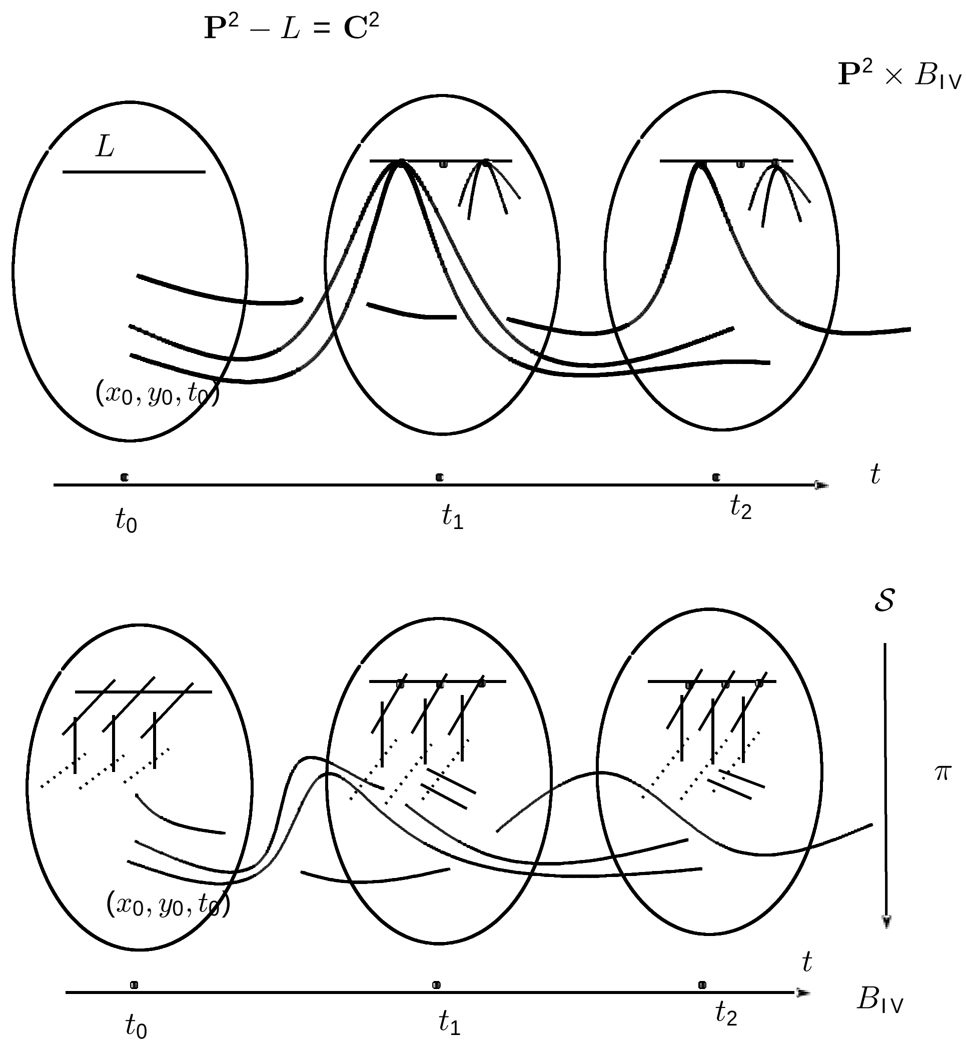


Figure 1. Example: Painlevé IV case.

Work of K. Okamoto, H. Sakai, S-Takebe, S-Takebe-Terajima

(Observations) After the resolutions of accessible singularities, we see that:

- $S = \mathcal{S}_t, t \in B_J$ is a rational surface which is 9-points blowings ups of $\mathbf{P}^2$.
- $S = \mathcal{S}_t$ has a global rational two forms ω such that the pole divisor Y of ω ($=$ anti-canonical divisor $-K_S$) satisfies the following Okamoto–Painlevé conditions. $-K_S = Y = \sum_{i=1}^r m_i Y_i$. $D = Y_{red} = \sum Y_i$

$$\boxed{\deg -(K_S)_{|Y_i} = -K_S \cdot Y_i = Y \cdot Y_i = 0 \quad 1 \leq^{\forall} i \leq r}. \qquad (14)$$

- Moreover the Painlevé vector field $\tilde{v}$ satisfies the $(1-\log)$-condition

$$\tilde{v} \in H^0(\mathcal{S}, \Theta(-\log \mathcal{D})(\mathcal{D})) \qquad (15)$$

where $\mathcal{D} = \mathcal{Y}_{red}$.

Main Questions

- Can one recover the Painlevé equations from the geometry of spaces of initial conditions ?
- What is the meaning of these two conditions?
- How are they essential for Painleve property?

2.1. **Definition of Okamoto–Painlevé pairs**.

Definition 2.1. Let (S, Y) be a pair of a complex projective smooth rational surface S and an anti-canonical divisor $Y \in |-K_S|$ of S. Let $Y = \sum_{i=1}^{r} m_i Y_i$ be the irreducible decomposition of Y. We call a pair

$$(S, Y)$$

a rational Okamoto–Painlevé Pair if for all $i, 1 \leq i \leq r$,

$$\deg(-K_S)_{|Y_i} = Y \cdot Y_i = \deg Y_{|Y_i} = 0. \tag{16}$$

(Okamoto–Painlevé condition).

Configuration of $-K_S = Y$ for a rational Okamoto– Painlevé pair (S, Y)

For a rational Okamoto–Painlevé pair (S, Y), let us set

$$-K_S = Y = \sum_{i=1}^{r} m_i Y_i.$$

One can show that

$$\boxed{\text{Config. of } Y \text{ is one as Kodaira–Néron's singular elliptic curves}}$$

$$\Longleftrightarrow$$

Okamoto–Painlevé conditions

$$\boxed{\deg -(K_S)_{|Y_i} = Y \cdot Y_i = 0 \quad \text{for all } i, \quad 1 \leq i \leq m}.$$

Moreover $r \leq 9$.

Classification of rational Okamoto–Painlevé pairs

Theorem 2.1. (Sakai, Saito–Takebe–Terajima)
Let (S, Y) be a rational Okamoto–Painlevé pair such that Y_{red} is a divisor with only normal crossings. Then the type of Y is same as one in the list of Table 3.

Y or $R(Y)$	$\tilde{E}_8$	$\tilde{D}_8$	$\tilde{E}_7$	$\tilde{D}_7$	$\tilde{D}_6$	$\tilde{E}_6$	$\tilde{D}_5$	$\tilde{D}_4$	$\tilde{A}_{r-1}$ $1 \leq r \leq 9$	$\tilde{A}_0{}^*$ $r = 1$
Kodaira's notation	II^*	I_4^*	III^*	I_3^*	I_2^*	IV^*	I_1^*	I_0^*	I_r	I_0
Painlevé equation	P_I	$P_{III}^{\tilde{D}_8}$	P_{II}	$P_{III}^{\tilde{D}_7}$	P_{III}	P_{IV}	P_V	P_{VI}	none	none
r	9	9	8	8	7	7	6	5	r	1

Table 3

Note that in Figure 2, the real line shows that a smooth rational curve $C \simeq \mathbf{P}^1$ with $C^2 = -2$ and the number near the each rational curve denotes the multiplicity in $Y = -K_S$.

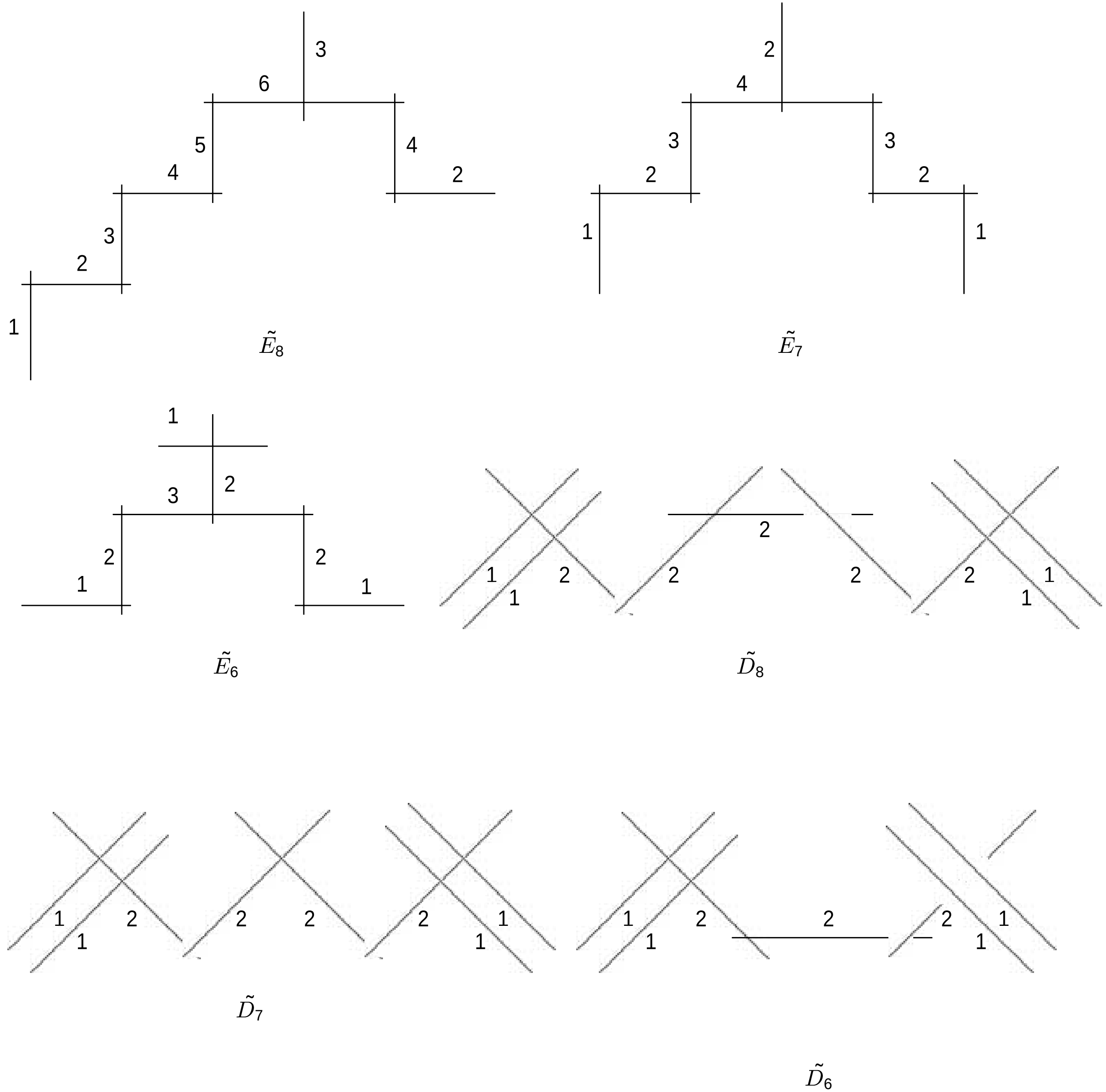
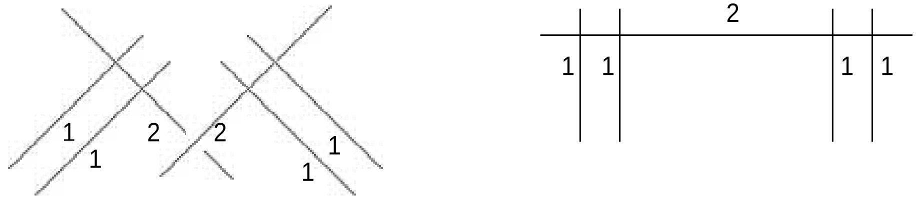


Figure 2



$\tilde{D}_5$ $\tilde{D}_4$

Figure 3

Geometric Picture of Painlevé Dynamics

Family of Okamoto–Painlevé pairs

$$\begin{array}{ccc} \mathcal{S}' & \hookrightarrow & \mathcal{S} \hookleftarrow \mathcal{D} \\ \downarrow \pi & \swarrow \overline{\pi} & \\ B_J \times \Lambda_J & & \end{array}$$

Here $\overline{\pi}$ is a smooth projective family of surfaces and $B_J \subset \operatorname{Spec} \mathbf{C}[t]$, $\Lambda_J \simeq \mathbf{C}^s$ and $\mathcal{D}$ is a flat family of normal crossing divisors.

- We can see that

$$\boxed{\tilde{v} \in H^0(\mathcal{S}, \Theta_{\mathcal{S}}(-log\mathcal{D})(\mathcal{D}))}$$

and

$$\overline{\pi}_*(\tilde{v}) = \frac{\partial}{\partial t}$$

- There exists rational relative two forms Ω on $\mathcal{S}$ such that supp of divisor $(\Omega) = \mathcal{D}$ and

$$\boxed{\iota_{\tilde{v}}(\Omega) = 0 \Longrightarrow \quad \tilde{v}\text{: non-autnomous Hamitonian system}}$$

- For each $(t_0, \lambda_0) \in B_J \times \Lambda_J$, the image of the Kodaira–Spencer map

$$\rho : T_{(t_0,\lambda_0)}(B_J) \longrightarrow H^1(\mathcal{S}_{(t_0,\lambda_0)}, \Theta_{\mathcal{S}_{(t_0,\lambda_0)}}(-\log \mathcal{D}_{(t_0,\lambda_0)}))$$

lies in the local cohomology group

$$\rho(\frac{\partial}{\partial t}) \in H^1_{\mathcal{D}_{(t_0,\lambda_0)}}(\mathcal{S}_{(t_0,\lambda_0)}, \Theta_{\mathcal{S}_{(t_0,\lambda_0)}}(-\log \mathcal{D}_{(t_0,\lambda_0)})) \simeq \mathbf{C}$$

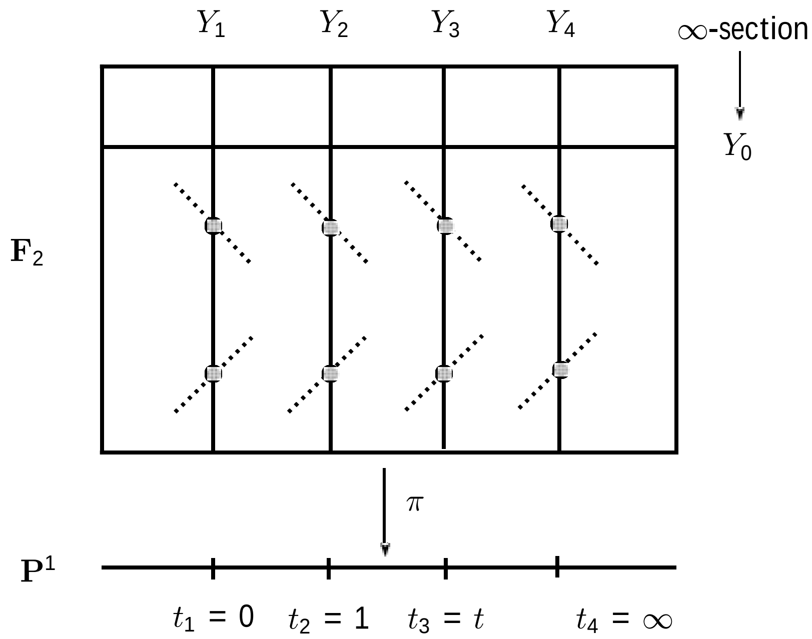


Figure 4. Okamoto-Painlevé pair of type $D_4^{(1)}$

3. $(n - \log)$-conditions

Consider an system of ODE on $\mathbf{C} \times \mathbf{C}^m$

$$\frac{dx_i}{dt} = a_i(t, x_i, \cdots, x_m), \quad 1 \leq i \leq m \tag{17}$$

$$\begin{array}{ccccc} B \times \mathbf{C}^m & \hookrightarrow & B \times S & \hookleftarrow & B \times D' = \mathcal{D} \\ \downarrow & & \downarrow & & \downarrow \\ B & = & B & = & B \end{array} \tag{18}$$

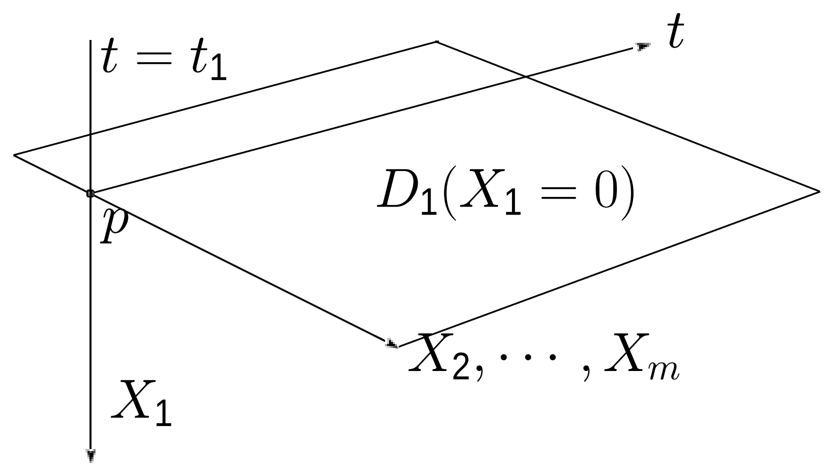


Figure 5. Coordinates on Boundary Divisors

$$\tilde{v} = \frac{\partial}{\partial t} + \frac{A_1}{X_1^{n_1}}\frac{\partial}{\partial X_1} + \sum_{i=2}^{m} \frac{A_i}{X_1^{n_i}}\frac{\partial}{\partial X_i} \tag{19}$$

$$\Theta_S(-\log D) = \{\theta \in \Theta_S,\ \theta \cdot I_D \subset I_D\} \tag{20}$$

$$\Theta_{B\times S}(-\log \mathcal{D}) = \{\theta \in \Theta_{B\times S},\ \theta \cdot I_{\mathcal{D}} \subset I_{\mathcal{D}}\} \tag{21}$$

Proposition 3.1. If

$$n_1 = \max_{1\leq i\leq m}(n_i) = n \geq 1 \tag{22}$$

there exists a solution curve of $\tilde{v}$ such that $p = (t, 0, \cdots, 0)$ is an movable branched point. So if $\tilde{v}$ satisfies the Painlevé property, we have

$$n_1 < \max_{1\leq i\leq m}(n_i) = n \tag{23}$$

or

$$\max_{1\leq i\leq m}(n_i) = n = 0, \tag{24}$$

that is, $\tilde{v}$ is regular along D_1.

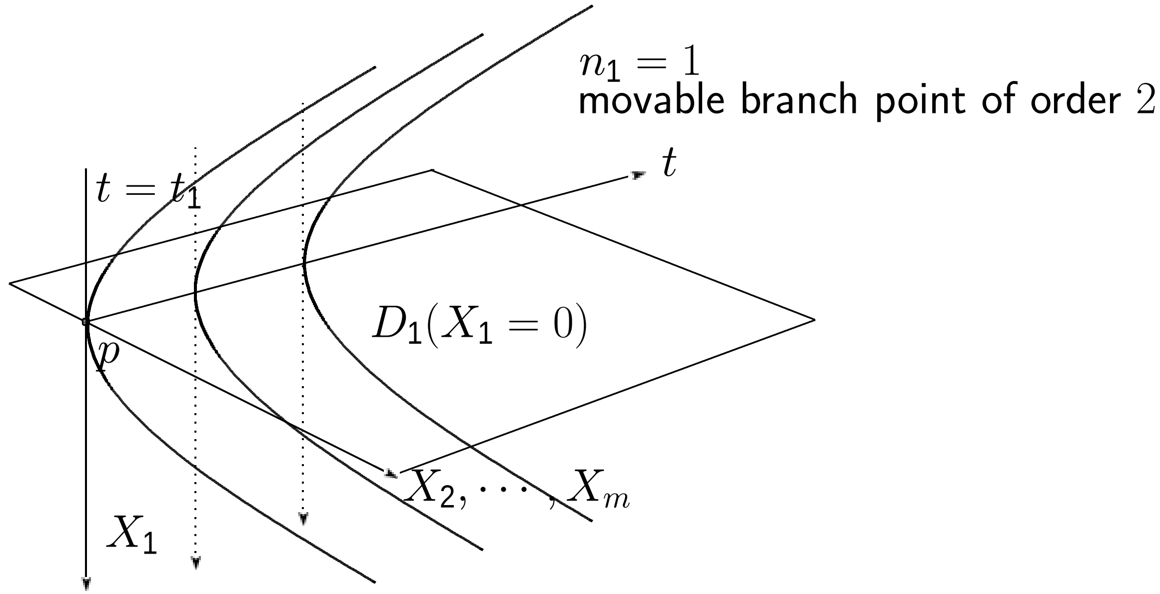


Figure 6

If $\tilde{v}$ does have poles of order n along $\mathcal{D} = B \times D_1$, but it does not have the algebraic branched points along $\mathcal{D} = B \times D_1$, then locally at the boudnary divisor, one can write $\tilde{v}$ as

$$\tilde{v} = \frac{\partial}{\partial t} + \frac{B_1}{X_1^{n-1}} \frac{\partial}{\partial X_1} + \sum_{i=2}^{m} \frac{B_i}{X_1^n} \frac{\partial}{\partial X_i} \tag{25}$$

Globally, this implies that:

$$\tilde{v} \in H^0(B \times S, \Theta_{B \times S}(-\log \mathcal{D})(n\mathcal{D})) \tag{26}$$

Definition 3.1. $\tilde{v}$ satifies $(n - \log)$-conditions if it satisfies the condition (26).

If $m = 1$ and $\tilde{v}$ satisfies the Painlevé property, $\tilde{v}$ must be regular everywhere.

Conjecture 3.1. If $\tilde{v}$ satisfies the Painlevé property, then after taking a suitable good model of the compactifications of the phase spaces, $\tilde{v}$ satisfies the $(1-\log)$-conditions along any divisor D.

$$\tilde{v} = \frac{\partial}{\partial t} + B_1 \frac{\partial}{\partial X_1} + \frac{B_2}{X_1}\frac{\partial}{\partial X_2} \tag{27}$$

Under the assumption that $(1-\log)$-conditions holds for along any irreducible components Y_i of Okamoto–Painlevé pair (S, Y), the conditions

$$-K_S \cdot Y_i = 0 \quad \Longleftrightarrow \quad \text{no accessible singular point on } Y_i$$

$$-K_S \cdot Y_i = \deg \Theta_{Y_i} \otimes N_{Y_i/S}$$

Proof of Theorem 1.1 (First order ODE with P.P.)

Let

$$\mathcal{C} = \cup_{t \in T} C_t = \cup_{t \in T} \{(x, y) \in \mathbf{C}^2 \quad | \quad F(t, x, y) = 0\}$$

be the family $\pi : \mathcal{C} \longrightarrow T = \mathrm{Spec}\,\mathbf{C}[t]$ of affine curves parametrized by $t \in T = \mathbf{C}$. Assume that C_t is smooth and irreducible for general $t \in T$. We can take the smooth relative compactification

$$\begin{array}{ccccccc} \mathcal{C} & \hookrightarrow & \overline{\mathcal{C}} & \hookleftarrow & \overline{\mathcal{C}}' & \hookleftarrow & \overline{\mathcal{C}}'' \\ \pi \downarrow & & \overline{\pi} \downarrow & & \downarrow & & f \downarrow \\ T & = & T & \hookleftarrow & T \setminus D & \hookleftarrow & T \setminus (D \cup D') = T'' \end{array} \tag{28}$$

D: the set of critical values of $\overline{\pi}$. The genus $g(\overline{C}_t)$ of curve $\overline{C}_t$ is constant. Algebraic ODE (2) $F(t, x, x') = 0$ defines a rational vector field on $\overline{\mathcal{C}}'$

$$v = \frac{\partial}{\partial t} + y \frac{\partial}{\partial x} \tag{29}$$

Delete the set $D' \subset T \setminus D$ of non-movable singularities of v, one can obtain the rational vector field v on $\overline{\mathcal{C}}''$.

$$\begin{array}{c} \overline{\mathcal{C}}'' \\ f \downarrow \\ T'' \end{array} \tag{30}$$

One can show that if the rational vector field v (29) satisfies the Painlevé property,

- v is a regular vector field on $\overline{\mathcal{C}}''$ (has no poles). (If v has a pole along a divisor, then v has a movable branced points along the divisor).
- and the moduli of $\overline{C}_t$ is constant. Consider the relative tangent sheaf

$$0 \longrightarrow \Theta_{\overline{\mathcal{C}}''/T''} \longrightarrow \Theta_{\overline{\mathcal{C}}''} \longrightarrow f^*\Theta_{T''} \longrightarrow 0$$

Note that $\Theta_{T''}$ is globally generated by $\frac{\partial}{\partial t}$.

Taking the direct images, we have

$$f_*(\Theta_{\overline{\mathcal{C}}"}) \longrightarrow \Theta_{T"} \xrightarrow{\rho} R^1 f_*(\Theta_{\overline{\mathcal{C}}''/T"})$$

where ρ is the Kodaira–Spencer map and the image $\rho(\frac{\partial}{\partial t})$ is in $R^1 f_*(\Theta_{\overline{\mathcal{C}}''/T"})$. The regular vector field v is a global section of $\Theta_{\overline{\mathcal{C}}"}$ such that $f_*(v) = \rho(\frac{\partial}{\partial t})$. Hence such v exists if and only if Kodaira-Spencer map ρ is zero. Now the moduli of $\overline{C}_t$ is constant.

	$\overline{C}_t$	$g(\overline{C}_t)$	ODE
Case (1)	$\mathbf{P}^1$	0	Riccati
Case (2)	E (elliptic curve)	1	ODE for $\wp$
Case (3)	a curve of genus ≥ 2	≥ 2	alg. integrable

Works in New, in Progress and in Future

(1) DS-hierarchy with similarity reduction $\Longrightarrow$ Painlevé equations (Noumi–Yamada, S. Kakei, T. Suzuki, K. Fuji, $\cdots$.)

(2) Coupled Painlevé system and Higher ordered Painlevé equations with affine Weyl group symmetries. (Sasano)

(3) Dynamical Systems associated to Painlevé VI via Riemann-Hilbert correspondences. (K. Iwasaki and T. Uehara (2005—))

4. Strategy 2: Moduli of stable parabolic connections and Riemann–Hilbert correspondences

• Translations of the terminology

Analysis	Geometry
C: a compact R. surface of genus g	C: a nonsing. proj. curve of genus g
$\mathbf{t} = (t_1, \cdots, t_n)$; n-distinct pts on C	$\mathbf{t} = (t_1, \cdots, t_n)$; n-distinct pts on C
$\frac{d\mathbf{x}}{dz} = \sum_{i=1}^{n} \frac{A_i(z)}{z-t_i}\mathbf{x}$ Linear D.E. on C with at most regular sing. at $\mathbf{t}$.	$\nabla : E \longrightarrow E \otimes \Omega^1_C(D(\mathbf{t}))$ A connection on vect. bdl E of rank r on C with at most 1^{st} order poles at $\mathbf{t}$.
$\lambda_j^{(i)}$:Eigenvalues of $A_i(t_i)$	$\lambda_j^{(i)}$: Eigenvalues of $\mathrm{res}_{t_i}(\nabla) \in \mathrm{End}(E_{\|t_i})$
Time varaiables $(s_1, \ldots, s_{3g-3}, t_1, \ldots, t_n)$	$T = \mathcal{M}_{g,n} = \{(C, \mathbf{t})\}$ Moduli of n-pointed curves of genus g
Space of initial conditions $S_{(C,\mathbf{t},\boldsymbol{\lambda})}$	Moduli space of stable parabolic connections $\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t})_{\boldsymbol{\lambda}}$
Phase space $\mathcal{S} \longrightarrow T \times \Lambda^r_n$	Family of moduli spaces $\mathcal{M} \longrightarrow T \times \Lambda^r_n$
Riemann-Hilbert correspondence	$\mathbf{RH}_{\boldsymbol{\lambda}} : \mathcal{M}^{\boldsymbol{\alpha}}_{\boldsymbol{\lambda}} \longrightarrow R_{\mathbf{a}}$
Isomonodromic deformations of L.D.E.	Pullback of local constant section
Schlessinger equation	Zero curvature equations on $\mathcal{M}$

• Translations of Properties

Analysis	Geometry
Painlevé property	Properness + Surjectivity of $\mathbf{RH}_{\boldsymbol{\lambda}} : \mathcal{M}^{\boldsymbol{\alpha}}_{\boldsymbol{\lambda}} \longrightarrow R_{\mathbf{a}}$
Symmetry (Bäklund transformation)	Elementary transformations of s.p. conn.
Simple reflections in Bäcklund transf.	Special Birational map (Flop) $\tilde{s} : \mathcal{M} \cdots \longrightarrow \mathcal{M}$ appeared in the resol. of simult. sing. of $R_{\mathbf{a}}$
Hamitonian Structures	Symplectic str. on $\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t})_{\boldsymbol{\lambda}}$ on $R_{\mathbf{a}}^{smooth}$ and $\mathbf{RH}_{\boldsymbol{\lambda}}$ is a symmplectic map
Spacial solutions like Riccati solution	Singylarities of $R_{\mathbf{a}}$
Poincaré return map or non-linear monodromy of equations of Painlevé type	Natural actions of $\pi_1(\mathcal{M}^{\circ}_{g,n}, *)$ on isomonodromic flows, $\mathbf{R}_{(C_0, \mathbf{t}_0), \mathbf{a}}$ and on $\mathcal{M}^{\boldsymbol{\alpha}}((C_0, \mathbf{t}_0))_{\boldsymbol{\lambda}}$
τ-functions	Sections of the determinant line bundle on $\mathcal{M}$ which are flat on isomonod. flows

Stable Parabolic connections

Setting

Fix the following data

$$(C, \mathbf{t}, (L, \nabla_L), (\lambda_j^{(i)})) \tag{31}$$

which consists of

- C : a complex smooth projective curve of genus g,
- $\mathbf{t} = (t_1, \cdots, t_n)$: a set of n-ditinct points on C.
  (Put $D(\mathbf{t}) = t_1 + \cdots + t_n$).
- (L, ∇_L): a line bundle on C with a logarithmic connection

$$\nabla_L : L \longrightarrow L \otimes \Omega^1_C(D(\mathbf{t})).$$

- $\boldsymbol{\lambda} = (\lambda_j^{(i)})_{1 \leq i \leq n, 0 \leq j \leq r-1} \in \mathbf{C}^{nr}$ such that $\sum_{j=0}^{r-1} \lambda_j^{(i)} = \mathrm{res}_{t_i}(\nabla_L)$.

Moduli space of stable parabolic connections

We can consider the moduli space of <u>**stable parabolic**</u> connection on C with logarithmic singularities at $D(\mathbf{t})$:

$$\mathcal{M}^{\boldsymbol{\alpha}}(C,\mathbf{t},L)_{\boldsymbol{\lambda}} = \{(E,\nabla_E,\{l_j^{(i)}\}_{1\leq i\leq n, 0\leq j\leq r-1},\Psi)\}/\simeq \quad (32)$$

- E : a vector bundle of rank r on C
- $\nabla : E \longrightarrow E\otimes\Omega_C(D(\mathbf{t}))$:a logarithmic connection
- $\Psi : \wedge^r E \xrightarrow{\simeq} L$: a horizontal isomorphism (Fixing the determinant)
- $E_{|t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r = 0$: a filtration of the fiber at t_i such that $\dim\left(l_j^{(i)}/l_{j+1}^{(i)}\right) = 1$ such that

$$\left(\mathrm{res}_{t_i}(\nabla) - \lambda_j^{(i)} Id\right)(l_j^{(i)}) \subset l_{j+1}^{(i)}$$

α-stability

Take a sequence of rational numbers $\boldsymbol{\alpha} = (\alpha_j^{(i)})_{1 \le j \le r}^{1 \le i \le n}$ such that

$$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \cdots < \alpha_r^{(i)} < 1 \tag{33}$$

for $i = 1, \ldots, n$ and $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for $(i, j) \neq (i', j')$. We choose $\boldsymbol{\alpha} = (\alpha_j^{(i)})$ sufficiently generic. Let $(E, \nabla, \{l_*^{(i)}\}_{1 \le i \le n})$ be a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection, and $F \subset E$ a nonzero subbundle satisfying $\nabla(F) \subset F \otimes \Omega_C^1(D(\mathbf{t}))$. We define integers $\mathrm{len}(F)_j^{(i)}$ by

$$\mathrm{len}(F)_j^{(i)} = \dim(F|_{t_i} \cap l_{j-1}^{(i)})/(F|_{t_i} \cap l_j^{(i)}). \tag{34}$$

Note that $\mathrm{len}(E)_j^{(i)} = \dim(l_{j-1}^{(i)}/l_j^{(i)}) = 1$ for $1 \le j \le r$.

Definition 4.1. A parabolic connection $(E, \nabla, \{l_*^{(i)}\}_{1 \le i \le n})$ is $\boldsymbol{\alpha}$-stable if for any proper nonzero subbundle $F \subsetneq E$ satisfying

$\nabla(F) \subset F \otimes \Omega^1_C(D(\mathbf{t}))$, the inequality

$$\frac{\deg F + \sum_{i=1}^{m} \sum_{j=1}^{r} \alpha_j^{(i)} \operatorname{len}(F)_j^{(i)}}{\operatorname{rank} F} < \frac{\deg E + \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_j^{(i)} \operatorname{len}(E)_j^{(i)}}{\operatorname{rank} E} \tag{35}$$

holds.

Moduli space of SL_r-rep. of the fundamental group

Take the categorical quotient of affine variety

$$\mathbf{Rep}(C, \mathbf{t}, r) = \{\rho : \pi_1(C \setminus D_{\mathbf{t}}) \longrightarrow SL_r(\mathbf{C})\}//Ad(SL_r(\mathbf{C})) \tag{36}$$

($\rho_1, \rho_2 \in \mathrm{Hom}(\pi_1(C \setminus D(\mathbf{t})), SL_r(\mathbf{C}))$ are Jordan equivalent iff $\mathrm{sem}(\rho_1) \simeq \mathrm{sem}(\rho_2)$).

Fix:

$$\mathbf{a} = \left(a_j^{(i)}\right)_{1 \leq i \leq n, 1 \leq j \leq r-1} \in \mathcal{A}_{r,n} = \mathbf{C}^{n(r-1)}$$

Then we define another moduli space of SL_r-representations with fixed characteristic polynomial of monodromies around t_i:

$$\mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}} = \left\{[\rho] \in \mathbf{Rep}(C, \mathbf{t}, r), \det(sI_r - \rho(\gamma_i)) = \chi_{\mathbf{a}^{(i)}}(s)\right\}$$

where

$$\boxed{\chi_{\mathbf{a}^{(i)}}(s) = s^r + a_{r-1}^{(i)} s^{r-1} + \cdots + a_1^{(i)} s + (-1)^r.}$$

Riemann-Hilbert correspondence

Assume that $r \geq 2$, $n \geq 1$ and $nr - 2r - 2 > 0$ when $g = 0$, $n \geq 2$. (Moreover the weight $\boldsymbol{\alpha}$ is generic). Then the Riemann-Hilbert correspondence

$$\mathbf{RH}_{(C,\mathbf{t},\boldsymbol{\lambda})} : \mathcal{M}^{\boldsymbol{\alpha}}(C,\mathbf{t},L)_{\boldsymbol{\lambda}} \longrightarrow \mathbf{Rep}(C,\mathbf{t},r)_{\mathbf{a}} \tag{37}$$

can be defined by

$$\boxed{(E, \nabla_E, \{l_j^{(i)}\}, \Psi) \mapsto \ker(\nabla^{an}_{|C \setminus D_{\mathbf{t}}})}$$

where

$$\chi_{\mathbf{a}^{(i)}}(s) = \prod_{j=0}^{r-1}(s - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)}))$$

Note that

$$\dim \mathcal{M}^{\boldsymbol{\alpha}}(C,\mathbf{t},L)_{\boldsymbol{\lambda}} = (r-1)(2(r+1)(g-1) + rn)$$

Fundamental Results

Theorem 4.1. (Inaba-Iwasaki-Saito ($r = 2, g = 0, n \geq 4$), Inaba (general case)) Under the notation as above, we have the following.

(1) The moduli space $\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}}$ is a **nonsingular algebraic manifold with a natural symplectic structure**.

(2) The modulis space $\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}}$ has a natural compactification $\overline{\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}}}$ which is the moduli space of the ϕ-stable parabolic connections.

Theorem 4.2. (Inaba-Iwasaki-Saito ($r = 2, g = 0, n \geq 4$), Inaba (general case)): Under the conditions above, the Riemann-Hilbert correspondense

$$\mathbf{RH}_{C,t,\boldsymbol{\lambda}} : \mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}} \longrightarrow \mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}} \tag{38}$$

is a **proper surjective bimeromorphic** map. Hence the Riemann-Hilbert correspondence gives an (analytic) resolution of singularities. Moreover $\mathbf{RH}_{C,t,\boldsymbol{\lambda}}$ preserves the symplectic structures on $\mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}}$ $\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}}$.

Remark 4.1.

- $\mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}}$ is an affine scheme which may have singularities for special $\mathbf{a}$.
- In the case of $g = 0$, we can show that $d\omega = 0$. Moreover, we expect that $d\omega = 0$ in general.

Varying time $(C, \mathbf{t})$ and parameter $\boldsymbol{\lambda}$, a

Consider the open set of the moduli space of n-pointed curves of genus g

$$M^o_{g,n} = \{(C, \mathbf{t}) = (C, t_1, \cdots, t_n), t_i \neq t_j, i \neq j\}$$

and the universal curve $\pi : \mathcal{C} \longrightarrow M^o_{g,n}$. Fixing a relative line bundle L for π with logarithmic connection ∇_L we can obtain the family of moduli spaces over $M^o_{g,n} \times \Lambda(L)$

$$\begin{array}{c} \mathcal{M}^{\boldsymbol{\alpha}}_{g,n}(L) \\ \downarrow \pi_n \\ M^o_{g,n} \times \Lambda(L) \end{array} \tag{39}$$

such that $\boxed{\pi_n^{-1}((C, \mathbf{t}, L, \boldsymbol{\lambda})) = \mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}}}$

We can also construct the fiber space

$$\begin{array}{c} \mathcal{R}\mathbf{ep}_g^{r,n} \\ \\ \downarrow \phi_g^{r,n} \\ \\ M_{g,n}^o \times \mathcal{A}_{r,n} \end{array} \tag{40}$$

such that

$$(\phi_g^{r,n})^{-1}((C, \mathbf{t}, \mathbf{a})) = \mathbf{Rep}(C, \mathbf{t}, SL_r)_{\mathbf{a}}.$$

Riemann-Hilbert corr. in family

We can obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}^{\boldsymbol{\alpha}}(L) & \xrightarrow{\mathbf{RH}_n} & \mathcal{R}\mathbf{ep}_g^{r,n} \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \phi_g^{r,n}} \\ M_{g,n}^o \times \Lambda(L) & \xrightarrow{(1\times\mu_{r,n})} & M_{g,n}^o \times \mathcal{A}_{r,n} \end{array} \tag{41}$$

where $\mu_{r,n}$ can be given by the relations

$$\chi_{\mathbf{a}}(s) = \prod_{j=0}^{r-1}(s - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)}))$$

that is, $a_k^{(i)}$ are $(\pm 1)\times$ k^{th} fundamental symmetric functions of $\exp(-2\pi\sqrt{-1}\lambda_j^{(i)})$.

Geometric Isomonodromic Deform. of L.D.E.

The case of generic exponents $\boldsymbol{\lambda}$

Fix a generic $\boldsymbol{\lambda} \in \Lambda(L)$ and set $\mathbf{a} = \mu_{r,n}(\boldsymbol{\lambda})$ so that

$$\mathbf{RH}_{C,t,\boldsymbol{\lambda}} : \mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}} \xrightarrow{\simeq} \mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}}$$

is an analytic isomorphism for any $(C, \mathbf{t}) \in M^o_{g,n}$.

- $\boxed{\text{Algebraic structure of } \mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}} \text{ \underline{does not change} under variation of } (C, \mathbf{t}) \text{, that is, } \mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}} \simeq \mathbf{Rep}(C_0, \mathbf{t}_0, r)_{\mathbf{a}}.}$

- $\boxed{\text{Algebraic structure of } \mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}} \text{ \underline{change} under variation of } (C, \mathbf{t}).}$

Taking the universal covering map $\tilde{M}^o_{g,n} \longrightarrow M^o_{g,n}$, and pulling back we obtain the diagram:

$$
\begin{tikzcd}
\widehat{\mathcal{M}}^{\boldsymbol{\alpha}}_{g,n}(L)_{\boldsymbol{\lambda}} \arrow[r, "\mathbf{RH}_{n,\boldsymbol{\lambda}}", "\simeq"'] \arrow[d, "(\tilde{\pi}_n)_{\boldsymbol{\lambda}}"'] & \left(\widehat{\mathcal{R}ep}^{r,n}_g\right)_{\mathbf{a}} \simeq \mathbf{Rep}(C_0, \mathbf{t}_0, r)_{\mathbf{a}} \times \tilde{M}^o_{g,n} \arrow[d, "\tilde{\phi}^{r,n}_{g,\mathbf{a}}"] \\
\tilde{M}^o_{g,n} \times \{\boldsymbol{\lambda}\} \arrow[r, "(1\times\mu_{r,n})"] & \tilde{M}^o_{g,n} \times \ \mathbf{a}.
\end{tikzcd}
$$

Since $\tilde{\phi}^{r,n}_{g,\mathbf{a}}$ is isomorphic to product family, it has a unique constant section $s_{\mathbf{x}}$ passing through a point $\mathbf{x} \in \mathbf{Rep}(C_0, \mathbf{t}_0, r)_{\mathbf{a}} \times \{\mathbf{t}_0\}$.

Pulling back the section $\{s_{\mathbf{x}}\}_{\mathbf{x} \in \mathbf{Rep}(C_0,\mathbf{t}_0,r)_{\mathbf{a}} \times \{\mathbf{t}_0\}}$ via $\mathbf{RH}_{\boldsymbol{\lambda}}$, we obtain the set of analytic sections of $(\tilde{\pi}_n)_{\boldsymbol{\lambda}} : \widehat{\mathcal{M}}^{\boldsymbol{\alpha}}_{g,n}(L)_{\boldsymbol{\lambda}} \to \tilde{M}^o_{g,n} \times \{\boldsymbol{\lambda}\}$

$$\{\tilde{s_{\mathbf{x}}}\}_{\mathbf{x} \in \mathbf{Rep}(C_0,\mathbf{t}_0,r)_{\mathbf{a}} \times \{\mathbf{t}_0\}}.$$

The family of sections $\{\tilde{s}_{\mathbf{x}}\}_{\mathbf{x}}$ gives the splitting homomorphism

$$\tilde{v}_{\boldsymbol{\lambda}} : (\tilde{\pi}_n)^*_{\boldsymbol{\lambda}}(T_{\mathbb{M}^o_{g,n}\times\{\boldsymbol{\lambda}\}}) \longrightarrow T_{\mathcal{M}^{\hat{\boldsymbol{\alpha}}}_{g,n}(L)_{\boldsymbol{\lambda}}}$$

for the natural homomorphism $T_{\mathcal{M}^{\hat{\boldsymbol{\alpha}}}_{g,n}(L)_{\boldsymbol{\lambda}}} \longrightarrow (\tilde{\pi}_n)^*_{\boldsymbol{\lambda}}(T_{\mathbb{M}^o_{g,n}} \times \{\boldsymbol{\lambda}\})$. Then the subbundle

$$\mathcal{IF}_{g,n,\boldsymbol{\lambda}} = \tilde{v}_{\boldsymbol{\lambda}}((\tilde{\pi}_n)^*_{\boldsymbol{\lambda}}(T_{\mathbb{M}^o_{g,n}\times\{\boldsymbol{\lambda}\}})) \subset T_{\mathcal{M}^{\hat{\boldsymbol{\alpha}}}_{g,n}(L)_{\boldsymbol{\lambda}}}. \tag{42}$$

Take any local generators of the tangent sheaf of $T_{\mathbb{M}^o_{g,n}}$

$$\langle \frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_N} \rangle.$$

where $N = 3g-3+n = \dim \mathbb{M}^o_{g,n}$. Then setting $v_i(\boldsymbol{\lambda}) := v_{\boldsymbol{\lambda}}(\frac{\partial}{\partial q_i})$, we obtain the integrable differential system on $\mathcal{M}^{\hat{\boldsymbol{\alpha}}}_{g,n}(L)_{\boldsymbol{\lambda}}$

$$\mathcal{IF}_{g,n,\boldsymbol{\lambda}} \simeq \langle v_1(\boldsymbol{\lambda}), \dots, v_N(\boldsymbol{\lambda}) \rangle.$$

(locally).

Case of special exponents $\boldsymbol{\lambda}$

- When the exponents $\boldsymbol{\lambda}$ is special, the R.H. corr.

$$\mathbf{RH}_{n,\boldsymbol{\lambda}} : \mathcal{M}^{\hat{\boldsymbol{\alpha}}}_{g,n}(L)_{\boldsymbol{\lambda}} \longrightarrow \left(\hat{\mathcal{R}}ep^{r,n}_{g}\right)_{\mathbf{a}}$$

  contracts some subvatieties to the singular locus on $\left(\hat{\mathcal{R}}ep^{r,n}_{g}\right)_{\mathbf{a}}$

- However, by Hartogs' theorem, we can extend the isomonodromic foliation $\mathcal{IF}_{g,n,\boldsymbol{\lambda}}$ to the total space $\mathcal{M}^{\hat{\boldsymbol{\alpha}}}_{g,n}(L)_{\boldsymbol{\lambda}}$.

Painlevé Property of Isomonodromic Flows

Theorem 4.3. (Inaba-Iwasaki-S, Part I (2003) and II(2006), Inaba(2006)).
The isomonodromic flows $\mathcal{IF}_{\boldsymbol{\lambda}}$ satisfies the Painlevé property for all exponents $\boldsymbol{\lambda}$.

Hamiltonian strucure of Isomonodromic Flows

Theorem 4.4. (Inaba-Iwasaki-S, Part I (2003) and II(2006), Inaba(2006)).
The isomonodromic flows $\mathcal{IF}_{\boldsymbol{\lambda}}$ can be written in a Hamiltonian system locally

- In the case of generic $\boldsymbol{\lambda}$, the differential system on $\mathcal{M}^{\boldsymbol{\alpha}}_{g,n}(L)_{\boldsymbol{\lambda}}$

$$\mathcal{IF}_{g,n,r} := \langle v_1(\boldsymbol{\lambda}), \dots , v_N(\boldsymbol{\lambda})\rangle.$$

  has cleary solution manifolds or integrable manifolds $=$ the images of $M^{o}_{g,n}$ by $\{\tilde{s}_{\mathbf{x}}\}_{\mathbf{x}}$. By construction,

These integrable submanifolds are isomonodromic flow of connections.

- Even in the case of special $\boldsymbol{\lambda}$, **the properness** of $\mathbf{RH}_{\boldsymbol{\lambda},n}$ implies the theorem.
- $\mathcal{IF}_{(0,4,2)}$ is equivalent to a **Painlevé VI equation**.
- $\mathcal{IF}_{(0,n,2)}$ with $n \geq 5$ are **Garnier systems**.

Parabolic connections of rank 2 on $\mathbf{P}^1$.

Let $n \geq 3$ and set

$$T_n = \{(t_1, \ldots, t_n) \in (\mathbf{P}^1)^n \quad | \quad t_i \neq t_j, (i \neq j)\}, \tag{43}$$

$$\Lambda_n = \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n\}. \tag{44}$$

Fixing a data $(\mathbf{t}, \boldsymbol{\lambda}) = (t_1, \ldots, t_n, \lambda_1, \ldots, \lambda_n) \in T_n \times \Lambda_n$, we define a reduced divisor on $\mathbf{P}^1$ as

$$D(\mathbf{t}) = t_1 + \cdots + t_n. \tag{45}$$

Moreover we fix a line bundle L on $\mathbf{P}^1$ with a logarithmic connection $\nabla_L : L \longrightarrow L \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$.

Definition 4.2. A (rank 2) $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection on $\mathbf{P}^1$ with the determinant (L, ∇_L) is a quadruplet $(E, \nabla, \varphi, \{l_i\}_{1 \leq i \leq n})$ which consists of

(1) a rank 2 vector bundle E on $\mathbf{P}^1$,
(2) a logarithmic connection $\nabla : E \longrightarrow E \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$
(3) a bundle isomorphism $\varphi : \wedge^2 E \xrightarrow{\simeq} L$
(4) one dimensional subspace l_i of the fiber E_{t_i} of E at t_i, $l_i \subset E_{t_i}$, $i = 1, \ldots, n$, such that
  (a) for any local sections s_1, s_2 of E,

$$\varphi \otimes id(\nabla s_1 \wedge s_2 + s_1 \wedge \nabla s_2) = \nabla_L(\varphi(s_1 \wedge s_2)),$$

  (b) $l_i \subset \mathrm{Ker}(\mathsf{res}_{t_i}(\nabla) - \lambda_i)$, that is, λ_i is an eigenvalue of the residue $\mathsf{res}_{t_i}(\nabla)$ of ∇ at t_i and l_i is a one-dimensional eigensubspace of $\mathsf{res}_{t_i}(\nabla)$.

The set of local exponents $\boldsymbol{\lambda} \in \Lambda_n$

Note that a data $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n \simeq \mathbf{C}^n$ specifies the set of eigenvalues of the residue matrix of a connection ∇ at $\mathbf{t} = (t_1, \ldots, t_n)$, which will be called a set of local exponents of ∇.

Definition 4.3. A set of local exponents $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n$ is called special if

(1) $\boldsymbol{\lambda}$ is resonant, that is, for some $1 \leq i \leq n$,

$$2\lambda_i \in \mathbf{Z}, \tag{46}$$

(2) or $\boldsymbol{\lambda}$ is reducible, that is, for some $(\epsilon_1, \ldots, \epsilon_n) \in \{\pm 1\}^n$

$$\sum_{i=1}^{n} \epsilon_i \lambda_i \in \mathbf{Z}. \tag{47}$$

If $\boldsymbol{\lambda} \in \Lambda_n$ is not special, $\boldsymbol{\lambda}$ is said to be generic.

Parabolic degrees and $\boldsymbol{\alpha}$-stability

Let us fix a series of positive rational numbers $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{2n})$, which is called a weight, such that

$$0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_i < \cdots < \alpha_{2n} < \alpha_{2n+1} = 1. \tag{48}$$

For a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection on $\mathbf{P}^1$ with the determinant (L, ∇_L), we can define the parabolic degree of $E = (E, \nabla, \varphi, l)$with respect to the weight $\boldsymbol{\alpha}$ by

$$\begin{aligned} \operatorname{pardeg}_{\boldsymbol{\alpha}} E &= \deg E + \sum_{i=1}^{n} \big(\alpha_{2i-1} \dim E_{t_i}/l_i + \alpha_{2i} \dim l_i\big) \quad (49) \\ &= \deg L + \sum_{i=1}^{n} (\alpha_{2i-1} + \alpha_{2i}). \end{aligned}$$

Let $F \subset E$ be a rank 1 subbundle of E such that $\nabla F \subset F \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$. We define the parabolic degree of $(F, \nabla_{|F})$ by

$$\operatorname{pardeg}_{\boldsymbol{\alpha}} F = \deg F + \sum_{i=1}^{n} \left(\alpha_{2i-1} \dim F_{t_i}/l_i \cap F_{t_i} + \alpha_{2i} \dim l_i \cap F_{t_i} \right) \tag{50}$$

Definition 4.4. Fix a weight $\boldsymbol{\alpha}$. A $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection (E, ∇, φ, l) on $\mathbf{P}^1$ with the determinant (L, ∇_L) is said to be $\boldsymbol{\alpha}$-stable (resp. $\boldsymbol{\alpha}$-semistable) if for every rank-1 subbundle F with $\nabla(F) \subset F \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$

$$\operatorname{pardeg}_{\boldsymbol{\alpha}} F < \frac{\operatorname{pardeg}_{\boldsymbol{\alpha}} E}{2}, \quad (\text{resp.} \operatorname{pardeg}_{\boldsymbol{\alpha}} F \leq \frac{\operatorname{pardeg}_{\boldsymbol{\alpha}} E}{2}). \tag{51}$$

(For simplicity, "$\boldsymbol{\alpha}$-stable" will be abbreviated to "stable").

We define the coarse moduli space by

$$M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)=\left\{(E,\nabla,\varphi,l);\ \begin{array}{l}\text{an } \boldsymbol{\alpha}\text{-stable } (\mathbf{t},\boldsymbol{\lambda})\text{-parabolic} \\ \text{connection with} \\ \text{the determinant } (L,\nabla_L)\end{array}\right\}/\text{isom.} \tag{52}$$

Stable parabolic ϕ-connections

If $n \geq 4$, the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ never becomes projective nor complete. In order to obtain a compactification of the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$, we will introduce the notion of a stable parabolic ϕ-connection, or equivalently, a stable parabolic Λ-triple. Again, let us fix $(\mathbf{t}, \boldsymbol{\lambda}) \in T_n \times \Lambda_n$ and a line bundle L on $\mathbf{P}^1$ with a connection $\nabla_L : L \to L \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$.

Definition 4.5. The data $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}_{i=1}^n)$ is said to be a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic ϕ-connection of rank 2 with the determinant (L, ∇_L) if E_1, E_2 are rank 2 vector bundles on $\mathbf{P}^1$ with $\deg E_1 = \deg L$, $\phi : E_1 \to E_2$, $\nabla : E_1 \to E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ are morphisms of sheaves, $\varphi : \bigwedge^2 E_2 \xrightarrow{\sim} L$ is an isomorphism and $l_i \subset (E_1)_{t_i}$ are one dimensional subspaces for $i = 1, \dots, n$ such that

(1) $\phi(fa) = f\phi(a)$ and $\nabla(fa) = \phi(a) \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_{\mathbf{P}^1}$, $a \in E_1$,

(2) $(\varphi \otimes \mathrm{id})(\nabla(s_1) \wedge \phi(s_2) + \phi(s_1) \wedge \nabla(s_2)) = \nabla_L(\varphi(\phi(s_1) \wedge \phi(s_2)))$ for $s_1, s_2 \in E_1$ and

(3) $(\mathrm{res}_{t_i}(\nabla) - \lambda_i \phi_{t_i})|_{l_i} = 0$ for $i = 1, \dots, n$.

Remark 4.2. Assume that two vector bundles E_1, E_2 and morphisms $\phi : E_1 \to E_2$, $\nabla : E_1 \to E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ satisfying $\phi(fa) = f\phi(a)$, $\nabla(fa) = \phi(a) \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_{\mathbf{P}^1}$, $a \in E_1$ are given. If ϕ is an isomorphism, then $(\phi \otimes \mathrm{id})^{-1} \circ \nabla : E_1 \to E_1 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ becomes a connection on E_1.

Fix rational numbers $\alpha'_1, \alpha'_2, \ldots, \alpha'_{2n}, \alpha'_{2n+1}$ satisfying

$$0 \le \alpha'_1 < \alpha'_2 < \cdots < \alpha'_{2n} < \alpha'_{2n+1} = 1$$

and positive integers β_1, β_2. Setting $\boldsymbol{\alpha}' = (\alpha'_1, \ldots, \alpha'_{2n}), \boldsymbol{\beta} = (\beta_1, \beta_2)$, we obtain a weight $(\boldsymbol{\alpha}', \boldsymbol{\beta})$ for parabolic ϕ-connections.

Definition 4.6. Fix a sufficiently large integer γ. Let

$$(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}_{i=1}^{n})$$

be a parabolic ϕ-connection. For any subbundles $F_1 \subset E_1$, $F_2 \subset E_2$ satisfying $\phi(F_1) \subset F_2$, $\nabla(F_1) \subset F_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$, we define

$$\mu((F_1, F_2))_{\boldsymbol{\alpha}'\boldsymbol{\beta}} = \frac{1}{\beta_1 \operatorname{rank}(F_1) + \beta_2 \operatorname{rank}(F_2)} (\beta_1 (\deg F_1(-D(\mathbf{t})))$$

$$+ \beta_2 (\deg F_2 - \gamma \operatorname{rank}(F_2)) + \sum_{i=1}^{n} \beta_1 (\alpha'_{2i-1} d_{2i-1}(F_1) + \alpha'_{2i} d_{2i}(F_1))$$

where $d_{2i-1}(F) = \dim((F_1)_{t_i}/l_i \cap (F_1)_{t_i})$, $d_{2i}(F_1) = \dim((F_1)_{t_i} \cap l_i)$.

A parabolic ϕ-connection $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}_{i=1}^n)$ is said to be $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-stable (resp. $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-semistable) if for any subbundles $F_1 \subset E_1$, $F_2 \subset E_2$ satisfying $\phi(F_1) \subset F_2$, $\nabla(F_1) \subset F_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ and $(F_1, F_2) \neq (E_1, E_2), (0,0)$, the inequality

$$\mu((F_1, F_2))_{\boldsymbol{\alpha}'\boldsymbol{\beta}} < \mu((E_1, E_2))_{\boldsymbol{\alpha}'\boldsymbol{\beta}}, \quad \text{(resp.} \quad \mu((F_1, F_2))_{\boldsymbol{\alpha}'\boldsymbol{\beta}} \leq \mu((E_1, E_2))_{\boldsymbol{\alpha}'\boldsymbol{\beta}}.) \tag{53}$$

We define the coarse moduli space of $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic ϕ-connections with the determinant (L, ∇_L) by

$$\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L) := \left\{(E_1, E_2, \phi, \nabla, \varphi, \{l_i\})\right\} / \mathrm{isom}. \tag{54}$$

For a given weight $(\boldsymbol{\alpha}', \boldsymbol{\beta})$ and $1 \leq i \leq 2n$, define a rational number α_i by

$$\alpha_i = \frac{\beta_1}{\beta_1 + \beta_2} \alpha'_i. \tag{55}$$

Then $\boldsymbol{\alpha} = (\alpha_i)$ satisfies the condition

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{2n} < \frac{\beta_1}{(\beta_1 + \beta_2)} < 1, \tag{56}$$

hence $\boldsymbol{\alpha}$ defines a weight for parabolic connections. It is easy to see that if we take γ sufficiently large $(E, \nabla, \varphi, \{l_i\})$ is $\boldsymbol{\alpha}$-stable if and only if the associated parabolic ϕ-connection $(E, E, \mathrm{id}_E, \nabla, \varphi, \{l_i\})$ is stable with respect to $(\boldsymbol{\alpha}', \boldsymbol{\beta})$. Therefore we see that the natural map

$$(E, \nabla, \varphi, \{l_i\}) \mapsto (E, E, \mathrm{id}_E, \nabla, \varphi, \{l_i\}) \tag{57}$$

induces an injection

$$M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \hookrightarrow \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L). \tag{58}$$

Conversely, assuming that $\boldsymbol{\beta} = (\beta_1, \beta_2)$ are given, for a weight $\boldsymbol{\alpha} = (\alpha_i)$ satisfying the condition (56), we can define $\alpha'_i = \alpha_i \frac{\beta_1 + \beta_2}{\beta_1}$

for $1 \le i \le 2n$. Since $0 \le \alpha'_1 < \alpha'_2 < \cdots < \alpha'_{2n} = \alpha_{2n}\frac{\beta_1+\beta_2}{\beta_1} < 1$, $(\boldsymbol{\alpha}', \boldsymbol{\beta})$ give a weight for parabolic ϕ-connections.

Moreover, considering the relative setting over $T_n \times \Lambda_n$, we can define two families of the moduli spaces

$$\overline{\pi}_n : \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \longrightarrow T_n \times \Lambda_n, \quad \pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n \times \Lambda_n \quad (59)$$

such that the following diagram commutes;

$$\begin{array}{ccc} M_n^{\boldsymbol{\alpha}}(L) & \stackrel{\iota}{\hookrightarrow} & \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \overline{\pi}_n} \\ T_n \times \Lambda_n & = & T_n \times \Lambda_n. \end{array} \quad (60)$$

Here the fibers of π_n and $\overline{\pi}_n$ over $(\mathbf{t}, \boldsymbol{\lambda}) \in T_n \times \Lambda_n$ are

$$\pi_n^{-1}(\mathbf{t}, \boldsymbol{\lambda}) = M^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L), \quad \overline{\pi}_n^{-1}(\mathbf{t}, \boldsymbol{\lambda}) = \overline{M^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L). \quad (61)$$

Riemann-Hilbert correspondence

$$\begin{array}{ccc} M_n^{\boldsymbol{\alpha}}(L) & \xrightarrow{\mathbf{RH}_n} & \mathcal{R}_n \\ {\scriptstyle \pi_n}\big\downarrow & & \big\downarrow{\scriptstyle \phi_n} \\ T'_n \times \Lambda_n & \xrightarrow{(1\times\mu_n)} & T'_n \times \ \mathcal{A}_n. \end{array} \tag{62}$$

Here, we have $1 \times \mu_n$ $(1 \times \mu_n)(\mathbf{t}, \boldsymbol{\lambda}) = (\mathbf{t}, \mathbf{a})$

$$\boxed{a_i = 2\cos 2\pi\lambda_i} \quad \text{for } 1 \leq i \leq n. \tag{63}$$

The case of $n=4$ (The Painlevé VI case).

Theorem 4.5. Take $L = \mathcal{O}_{\mathbf{P}^1}(-1)$ with a natural connection.

(1) For a suitable choice of a weight $\boldsymbol{\alpha}'$, the morphism

$$\overline{\pi}_4 : \overline{M_4^{\boldsymbol{\alpha}'}}(-1) \longrightarrow T_4 \times \Lambda_4$$

is $\underline{\text{projective and smooth}}$. Moreover for any $(\mathbf{t}, \boldsymbol{\lambda}) \in T_4 \times \Lambda_4$ the fiber $\overline{\pi}_4^{-1}(\mathbf{t}, \boldsymbol{\lambda}) := \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$ is irreducible, hence a smooth projective surface.

(2) Let $\mathcal{D} = \overline{M_4^{\boldsymbol{\alpha}'}}(-1) \setminus M_4^{\boldsymbol{\alpha}}(-1)$ be the complement of $M_4^{\boldsymbol{\alpha}}(-1)$ in $\overline{M_4^{\boldsymbol{\alpha}'}}(-1)$. (Note that $\boldsymbol{\alpha} = \boldsymbol{\alpha}'/2$). Then $\mathcal{D}$ is a flat reduced divisor over $T_4 \times \Lambda_4$.

(3) For each $(\mathbf{t}, \boldsymbol{\lambda})$, set

$$\overline{S}_{\mathbf{t},\boldsymbol{\lambda}} := \overline{\pi}_4^{-1}(\mathbf{t}, \boldsymbol{\lambda}) := \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1).$$

Then $\overline{S}_{\mathbf{t},\boldsymbol{\lambda}}$ is a smooth projective surface which can be obtained by blowing-ups at 8 points of the Hirzeburch surface $\mathbf{F}_2 = \mathrm{Proj}(\mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1})$ of degree 2. The surface has a unique effective anti-canonical divisor $-K_{S_{\mathbf{t},\boldsymbol{\lambda}}} = \mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}$ whose support is $\mathcal{D}_{\mathbf{t},\boldsymbol{\lambda}}$. Then the pair

$$(\overline{S}_{\mathbf{t},\boldsymbol{\lambda}}, \mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}) \tag{64}$$

is an Okamoto-Painlevé pair of type $D_4^{(1)}$. That is, the anti-canonical divisor $\mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}$ consists of 5-nodal rational curves whose configuration is same as Kodaira–Néron degenerate elliptic curves of type $D_4^{(1)}$ ($=$Kodaira type I_0^*). Moreover we have $(M_4^{\boldsymbol{\alpha}}(-1))_{\mathbf{t},\boldsymbol{\lambda}} = (\overline{M_4^{\boldsymbol{\alpha}'}}(-1))_{\mathbf{t},\boldsymbol{\lambda}} \setminus \mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}$.

Okamoto Painlevé pair of type P_{VI}

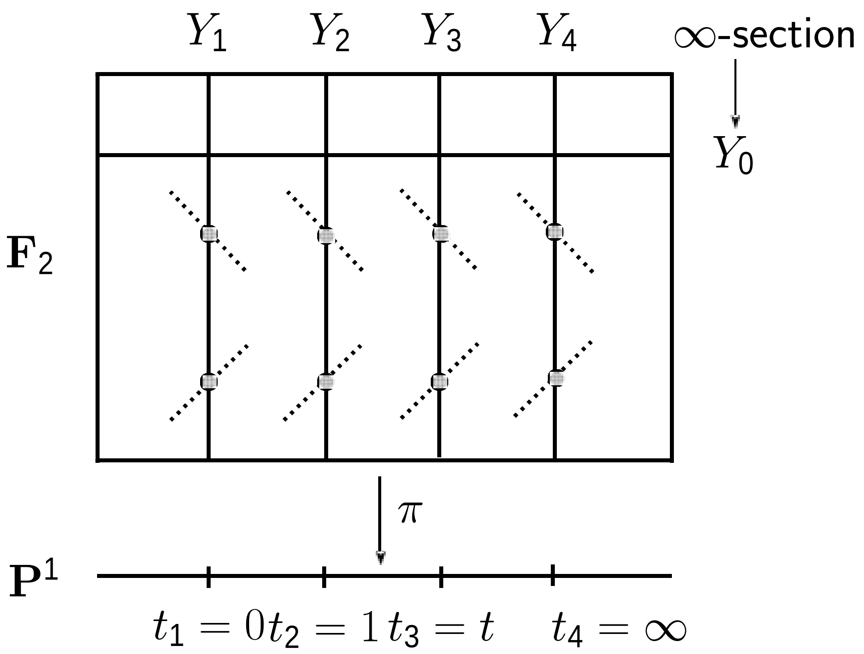


Figure 7. Okamoto-Painlevé pair of type $D_4^{(1)}$

Proposition 4.1. The invariant ring $(R_3)^{Ad(SL_2(\mathbf{C}))}$ is generated by seven elements $x_1, x_2, x_3, a_1, a_2, a_3, a_4$ and there exist a relation

$$f(\mathbf{x}, \mathbf{a}) = x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(\mathbf{a})x_1 - \theta_2(\mathbf{a})x_2 - \theta_3(\mathbf{a})x_3 + \theta_4(\mathbf{a}), \tag{65}$$

where we set

$$\begin{aligned} \theta_i(\mathbf{a}) &= a_ia_4 + a_ja_k, \quad (i,j,k) = \text{a cyclic permutation of } (1,2,3), \\ \theta_4(\mathbf{a}) &= a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4. \end{aligned}$$

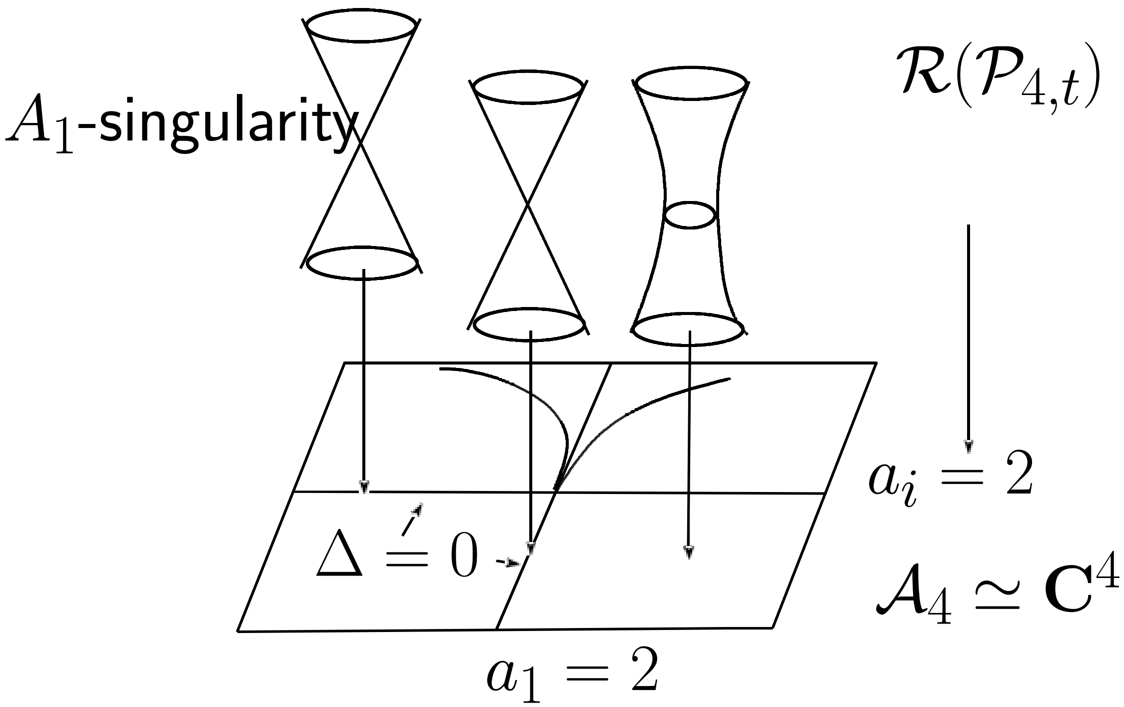
Therefore we have an isomorphism

$$(R_3)^{Ad(SL_2(\mathbf{C}))} \simeq \mathbf{C}[x_1, x_2, x_3, a_1, a_2, a_3, a_4]/(f(\mathbf{z}, \mathbf{a})).$$

Hence

$$\mathbf{Rep}(\mathbf{P}^1, (t_1, t_2, t_3, t_4), 2) = \mathrm{Spec}\,(R_3)^{Ad(SL_2(\mathbf{C}))}$$

is isomorphic to an affine cubic.



The family of affine cubic surfaces

$$
\begin{array}{ccc}
\tilde{M}_4^{\boldsymbol{\alpha}}(-1) & \xrightarrow{\mathbf{RH}_4} & \tilde{\mathcal{R}}_4 \simeq \tilde{T}_4 \times \mathbf{Rep}(\mathbf{P}^1, (t_1, t_2, t_3, t_4), 2) \\
\tilde{\pi}_4 \Big\downarrow & & \Big\downarrow \tilde{\phi}_4 \\
\tilde{T}_4 \times \Lambda_4 & \xrightarrow{(1 \times \mu_n)} & \tilde{T}_4 \times \mathcal{A}_4.
\end{array}
\tag{66}
$$

$$
\boxed{a_i = 2\cos 2\pi\lambda_i} \quad \text{for } 1 \le i \le 4. \tag{67}
$$

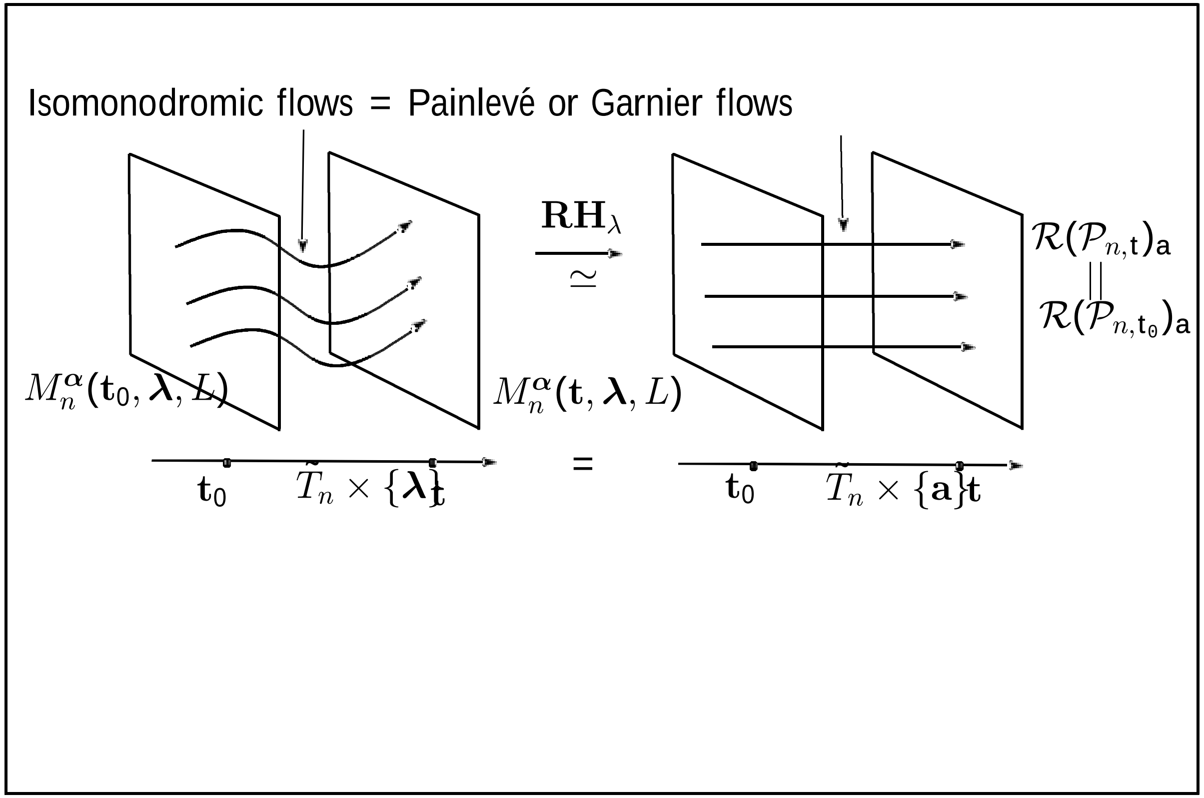


Figure 8. Riemann-Hilbert correspondence and isomonodromic flows for generic $\boldsymbol{\lambda}$

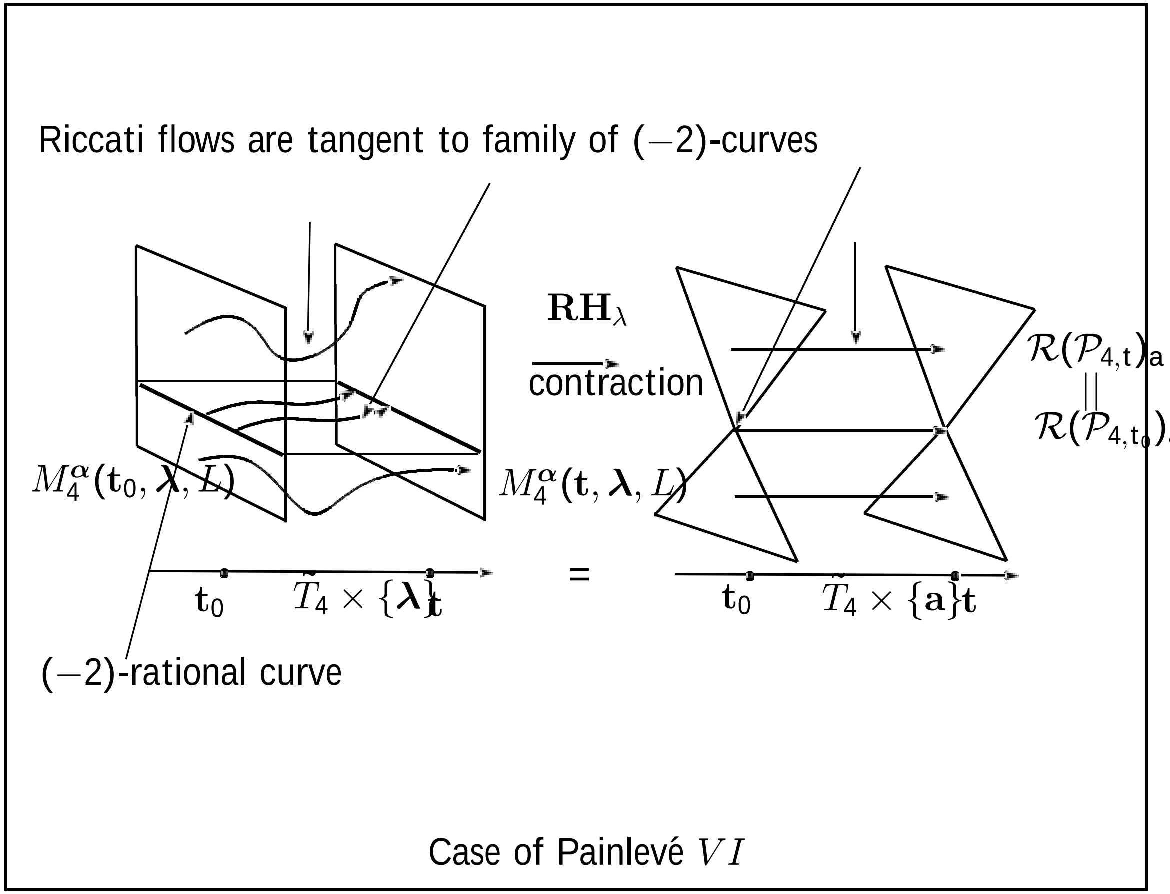


Figure 9. Riemann-Hilbert correspondence and isomonodromic flows for special $\boldsymbol{\lambda}$

Hamiltonian systems of Painlevé P_{VI}

P_{VI} is equivalent to a Hamiltonian system H_{VI}.

$$(H_{VI}) : \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H_{VI}}{\partial y}, \\[2ex] \dfrac{dy}{dt} = -\dfrac{\partial H_{VI}}{\partial x}, \end{cases}$$

Hamiltionian in suitable coordinates can be given

$$H_{VI} = H_{VI}(x, y, t, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbf{C}(t)[x, y, \lambda_i]$$

$$\begin{aligned} H_{VI}(x, y, t) \ &= \ \frac{1}{t(t-1)} \left[x(x-1)(x-t)y^2 - \{2\lambda_1(x-1)(x-t) \right. \\ &\left. \quad +2\lambda_2 x(x-t) + (2\lambda_3 - 1)x(x-1)\} \, y + \lambda(x-t) \right] \end{aligned}$$

$$\left(\lambda := \left\{(\lambda_1 + \lambda_2 + \lambda_3 - 1/2)^2 - \lambda_4^2\right\}\right).$$

Bäcklund transformations for Painlevé VI.

- $P_{VI}(\boldsymbol{\lambda})$ have non-trivial birational automorphisms which are called **Bäcklund transformations**. The group of all **Bäcklund transformations** form the affine Weyl group W of type $D_4^{(1)}$.

Proposition 4.2. The group of Bäcklund transformations which can be obtained from elementary transformations of stable parabolic connections is a proper subgroup of $W(D_4^{(1)})$ whose index is finite. The invloution s_0 of $W(D_4^{(1)})$ is not in the group.

The case of connection with irregular singular points

Pailevé equation	Order of pole at $t = 0$	$t = 1$	$t = \infty$
Painlevé V	≤ 1	≤ 1	2
IV	≤ 1	0	3
III	2	0	2
II	0	0	4
I	0	0	4 (ramified)

Bäcklund tranformations for P_{VI}

- **–Symmety of Affine Weyl group** $W(D_4^{(1)})$
  - $W(D_4^{(1)}) = \langle s_0, s_1, \cdots, s_4 \rangle$ acts on $\Lambda_{VI} = \mathbf{C}^4$ by

$$s_i(\lambda_j) = (-1)^{\delta_{ij}} \lambda_j, i = 1, \cdot, 4.$$

$$s_0(\lambda_j) = \lambda_j - \frac{1}{2} \sum_{j=1}^{4} \lambda_j + \frac{1}{2}.$$

- **Fact:**
  - (Bäcklund transformation). The actions of $W(D_4^{(1)})$ on Λcan be lifted to birational actions on $\overline{\mathcal{S}'}$ which prserve $\tilde{v}$.

$$\begin{array}{ccc} \overline{\mathcal{S}'} & \cdots\overset{\tilde{s}_i}{\to} & \overline{\mathcal{S}'} \\ \downarrow & & \downarrow \\ T \times \Lambda_4 & \xrightarrow{s_i} & T \times \Lambda_4 \end{array}$$

$$\boxed{\tilde{s}_{i*}(\tilde{v}) = \tilde{v}}$$

$\boxed{\text{Problem}}$

- What is a geometric origin of Bäcklund transformations ?

$\boxed{\text{Answer}}$

- $s_i, i = 1, \cdots, 4$ are easy. Elementary transformations.
- Except s_0, we can almost explain the geometric origin.

Riccati solution for Painlevé equations and Raional curves

- **Riccati equation :**

$$x' = a(t)x^2 + b(t)x + c(t).$$

- **Example** P_{IV}**)**

$$\begin{cases} \dfrac{dx_0}{dt} = 4x_0y_0 - x_0^2 - 2tx_0 - 2\kappa_0 \\ \\ \dfrac{dy_0}{dt} = -2y_0^2 + 2(x_0+t)y_0 - \kappa_\infty \end{cases} . \tag{68}$$

- When $\kappa_0 = 0$, $x_0 \equiv 0$ satisfies first equation automatically. The second equation becomes Riccati equation:

$$\frac{dy_0}{dt} = -2y_0^2 + 2ty_0 - \kappa_\infty$$

- When $\kappa_\infty = 0$, $y_0 \equiv 0$ satisfies the second equation automatically, then first equation becomes

$$\frac{dx_0}{dt} = -x_0^2 - 2tx_0 - 2\kappa_0$$

- Even when $\kappa_0 = \kappa_\infty$, setting $x_0y_0 - \kappa_0 = 0$, we have a Riccati equation.

- **Phase space of Riccati equations**

$$\mathbf{P}^1 \times T$$

- **Saito–Terajima, J. of Kyoto Math. (2004)**

$$\boxed{\text{Riccati solutions}} \Longleftrightarrow \boxed{\quad C = \mathbf{P}^1 \subset \mathcal{S}'_{t,\boldsymbol{\lambda}}, C^2 = -2}$$

- **N. A. Lukashevich and A. I. Yablonski**, **A.S. Fokas and M.J. Ablowitz**, **Watanabe**.

  For $\boldsymbol{\lambda} \in \Lambda_4$, $\mathcal{S}'_{t,\boldsymbol{\lambda}}$ contains $\mathbf{P}^1$ if and only if $\boldsymbol{\lambda}$ lies on a reflection hyperplane with respect to the affine Weyl group actions on Λ_4.

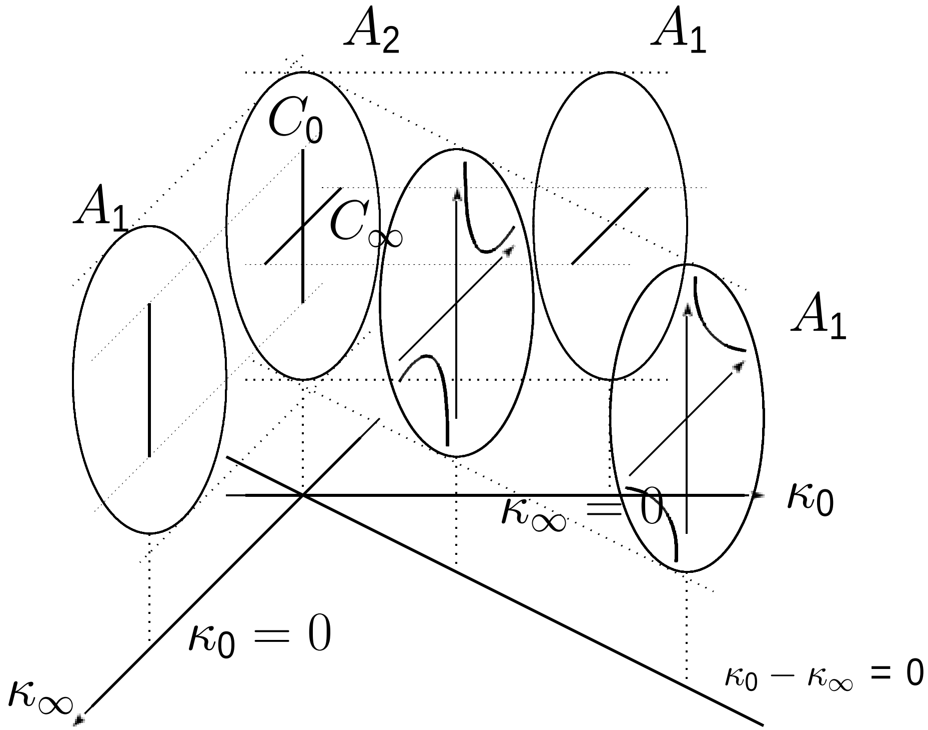


Figure 10. A Confluence of Nodal Curves in the case $\tilde{E}_6$ (P_{IV}).