Polymers - Eindhoven 21/06/07

RW - Durham 12/07/07

On ρ-percolation

and the number of directed paths.

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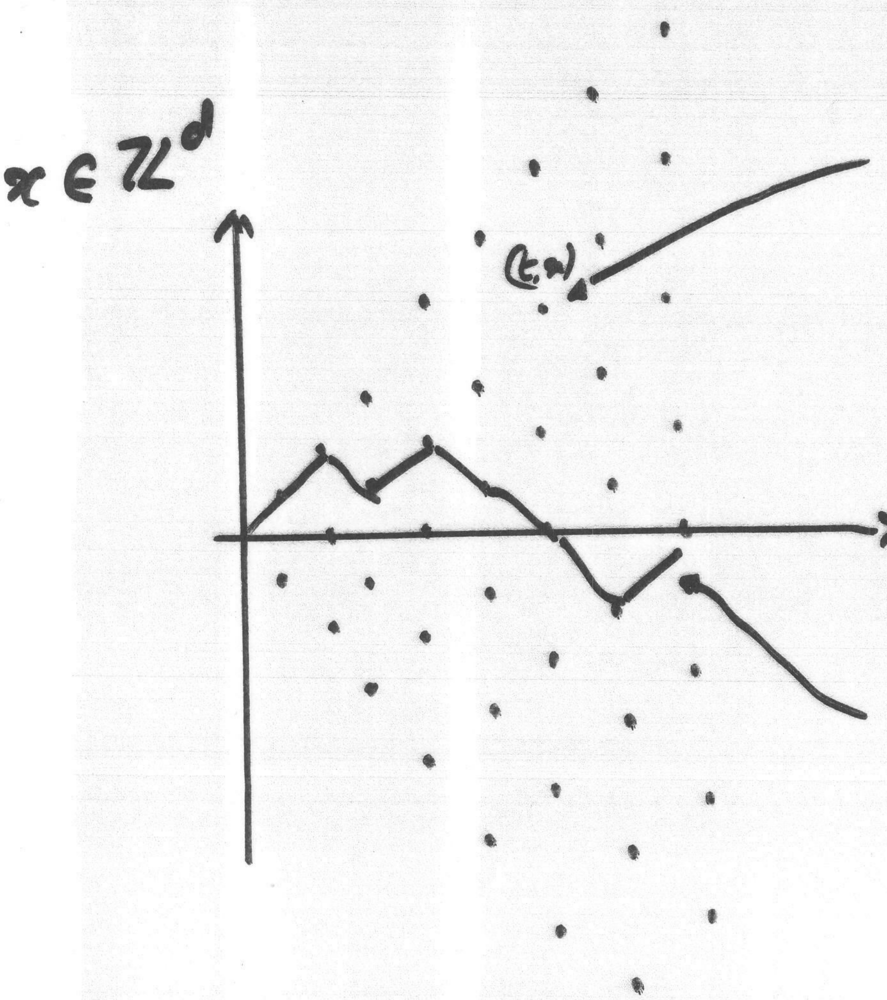


$\eta(t,x)$ i.i.d. Bernoulli(p).

path $S = (S_0 = 0, S_1, \ldots S_n) \in \mathcal{P}_n$.

$$H_n(S) = \sum_{t=1}^{n} \eta(t, S_t) = \#\text{ open vertices}$$

ρ - Percolation , $0 \le \rho \le 1$:

$$\rho\text{-Per} = \left\{ \exists S \in \mathcal{P}_\infty : \underline{\lim}_{n \to \infty} \frac{H_n(S)}{n} \geq \rho \right\}.$$

Menshikov - Zuev '93.

- $\exists \overrightarrow{p_c}(\rho, d) : \mathbb{P}(\rho\text{-Per}) = \begin{cases} 0 \\ 1 \end{cases}$ if $\begin{cases} p > \\ p < \end{cases} \overrightarrow{p_c}(\rho, d)$
- $\overrightarrow{p_c}(1, d) = \overrightarrow{p_c}(d)$

Kesten - Su '00 : asymptotics $_{d \to +\infty}$ of $\overrightarrow{p_c}(\rho, d)$.

are $\neq$ from the Tree.

$\longrightarrow$ Study :

$$Q_n(k) = \#\{ S \in \mathcal{P}_n : H_n(S) = k \}$$

$$R_n(\rho) = \#\{ S \in \mathcal{P}_n : H_n(S) \geq n\rho \}.$$

$\rho \geq p$.

Reprint Kesten - Sidoravicius '07

G.F. :

$$Z_n(\beta) = \sum_{S \in \mathcal{P}_n} e^{\beta H_n(S)}$$

partition function of Directed Polymers in Random Environment.

$\mathbb{E}$ w.r.t. η.

$$\Psi(\beta) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E} \ln Z_n$$ "Free energy"

$$= \text{a.s.-}\lim_{n\to\infty} \frac{1}{n} \ln Z_n$$

Time-constraints $\rho^+ = \lim_n \frac{1}{n} \max_S H_n(S)$

$\rho^- \quad \ldots \quad \min \ldots$

$(0 \leq \rho^- < p < \rho^+ \leq 1)$.

Legendre transform:

$$\Psi^*(\rho) = \sup_{\beta\in\mathbb{R}} \{\beta\rho - \Psi(\beta)\}$$

<u>th1</u>: $\forall \rho \in [0,1] \setminus \{\rho^+, \rho^-\}$,

$$\alpha(\rho) = \lim_{n\to\infty} \frac{1}{n} \ln R_n(\rho)$$ exists a.s.

and : $\alpha = -\Psi^*$.

α concave, $\in [0, \ln 2d] \cup \{-\infty\}$,

$\alpha(p) = \ln(2d)$

□ Sketch of proof:

The probability $\nu_n\{\rho\} := (2d)^{-n}\, Q_n(n\rho)$

has Laplace transform at $\beta \in \mathbb{R}$:

$$\frac{1}{n} \ln (2d)^{-n} Z_n(\beta) \xrightarrow[n \to \infty]{a.s.} -\ln(2d) + \varphi(\beta)$$

By **Gärtner-Ellis th.**, large deviations for ν_n: For $\rho \geqslant p$,

$$R_n(\rho) \leqslant \exp -n \left[\varphi^*(\rho) + o(1) \right]$$

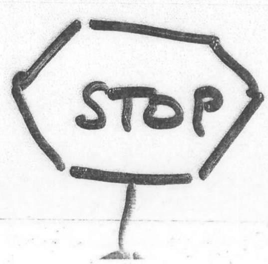
$$\geqslant \exp -n \left[\inf\{ \varphi^*(\rho') : \rho' \in \mathcal{F}, \rho' > \rho \} + o(1) \right]$$

$$\mathcal{F} = \left\{ \rho' : \exists \beta \ \forall r \neq \rho', \ \beta\rho' - \varphi^*(\rho') > \beta r - \varphi^*(r) \right\}$$

exposed points for φ^*.

Attempt: φ is differentiable ???

→ Conjecture !!



Assume $\rho = \dfrac{k\rho_1 + l\rho_2}{k+l} \in \mathcal{F}$:

Quasi Subadditivity: $\forall \varepsilon \; \exists \delta$:

$$\#\left\{ S \in \mathcal{P}_{n(k+l)} : \frac{H_{n(k+l)}(S)}{n(k+l)} \geq \rho - \varepsilon \right\} \geq$$

$$\#\left\{ S \in \mathcal{P}_{nk} : \frac{H_{nk}(S)}{nk} \in (\rho_1 - \delta, \rho_1 + \delta) \right\} \times$$

$$\min_{\|x\|_1 \leq nk} \#\left\{ S \in \mathcal{P}_{nl} : \frac{H_{nl}^{(x,nk)}(S)}{nl} \in (\rho_2 - \delta, \rho_2 + \delta) \right\}$$

potential with space/time shift environment

$\rightarrow$ Use lower board for **exposed** ρ_1, ρ_2 :

$$\underline{\lim}_{n \to \infty} \frac{1}{n} \ln R_n(\rho) \geq - \left[\frac{k}{k+l} \varphi^*(\rho_1 + \delta) + \frac{l}{k+l} \varphi^*(\rho_2 + \delta) \right]$$

and use that:

$$\frac{k}{k+l} \varphi^*(\rho_1) + \frac{l}{k+l} \varphi^*(\rho_2) \approx \varphi^*(\rho)$$

if ρ is in a linear piece !

$$\lambda(\beta) := \ln\left\{2d\left(1 + p\left[e^{\beta}-1\right]\right)\right\} \quad \text{"annealed"}$$

$$\lambda^*(\rho) = \rho \ln \frac{\rho}{2pd} + (1-\rho)\ln\frac{1-\rho}{2d(1-p)} .$$

<u>th 2</u>: 1) $\alpha(\rho) \leq -\lambda^*(\rho)$

2) $\mathcal{V}(p) = \{\rho : \alpha(\rho) = -\lambda^*(\rho)\}$ is an interval $\ni p$.

3) $d = 1$: $\mathcal{V}(p) = \{p\}$.

4) $d \geqslant 3$: $\mathcal{V}(p)$ is neighborhood of p.

5) $d \geqslant 3$:

$1-\pi_d :=$ Proba(SRW ever return at 0) > 0

$p > \pi_d \Longrightarrow [p, 1) \subset \mathcal{V}(p)$

$p < 1-\pi_d \Longrightarrow (0, p] \subset \mathcal{V}(p)$

6) $d \geqslant 2$:

$p < 1/2d \Longrightarrow \sup \mathcal{V}(p) < 1$

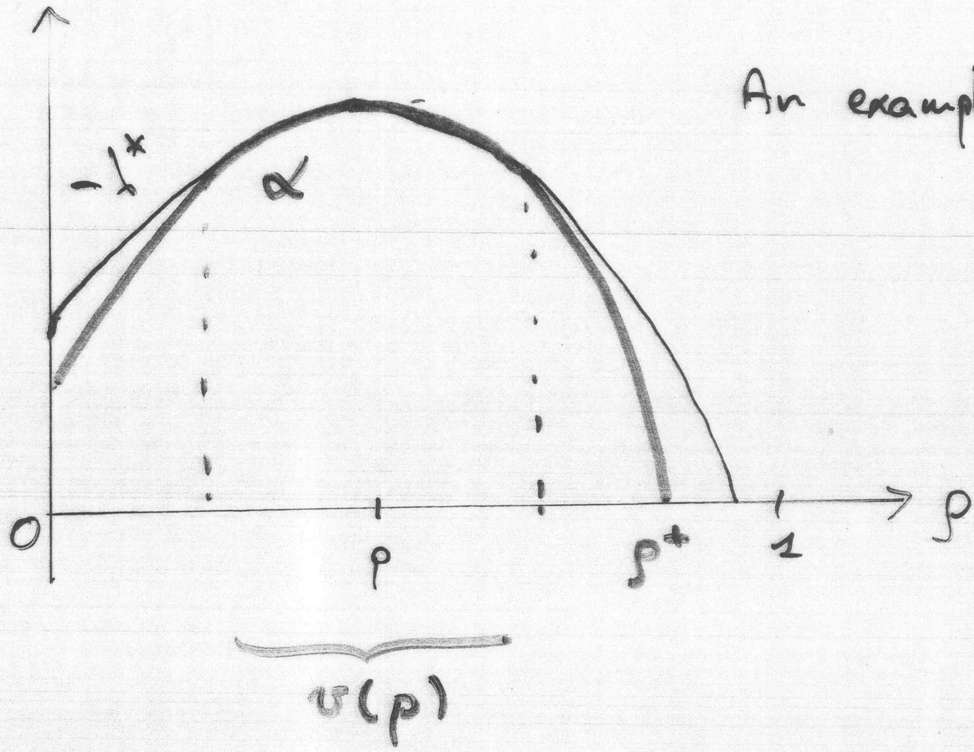
$p > 1 - 1/2d \Longrightarrow \inf \mathcal{V}(p) > 0$.

Proof: 1) Annealed bound for φ:

$$\mathbb{E}\ln Z_n \overset{\text{Jensen}}{\le} \ln \mathbb{E} Z_n = n\lambda(\beta)$$

$\Rightarrow$ $$\alpha(\rho) = -\varphi^*(\rho) \le -\lambda^*(\rho)$$

An example...



2) $\mathcal{W}(\rho) := \{\beta : \varphi(\beta) = \lambda(\beta)\}$ interval $\ni 0$.
by FKG. [C - Yoshida '06]

Duality: $$v(\rho) = \{\lambda'(\beta)\ ;\ \beta \in \mathcal{W}(\rho)\}.$$

$\to$ All properties follow from properties of φ.

Imbrie - Spencer '88, Bolthausen '89,
Albeverio - Zhou '96, R.Song '98, C - Vargas '06.

. It is not easy to have properties of α (not of λ^*...).

. Relate to the **polymer measure** :

$$\mu_n(s) = \frac{1}{Z_n} e^{\beta H_n(s)} \qquad s \in \mathcal{P}_n .$$

Markovian measure

<u>th 3</u> : α is differentiable on (β^-, β^+),
and φ is strictly convex.

$\Box$ Enough to prove strict convexity of φ : we show

$\forall K$ compact $\subset \mathbb{R}_+$ $\exists C$: $\mathbb{E}(\ln Z_n)''(\beta) \geq Cn$ $\forall \beta \in K$.

$\to$ Identity $\quad (\ln Z_n)'' = \mathrm{Var}_{\mu_n}(H_n) \quad , \forall \eta .$

$$\mathrm{Var}_{\mu_{2n}} H_{2n} \geq E_{\mu_{2n}} \mathrm{Var}_{\mu_{2n}}(H_{2n} \mid \Sigma^e)$$

Σ^e : $S_2, S_4, \ldots S_{2n}$

Markov $\Rightarrow$

$$= \sum_{t=1}^{n} E_{\mu_{2n}} \mathrm{Var}_{\mu_{2n}}\left(\eta(2t-1, S_{2t-1}) \mid \Sigma^e\right).$$

Let

$$M(t,\eta,x,y) = \left\{ \#\left\{ \eta(t,z) \,;\, z \text{ nearest neighbor to both } x,y \right\} \geq 2 \right\}$$

Key :

$\mathrm{Var}_{\mu_{2n}}\left(\eta(2t-1, S_{2t-1}) \mid \Sigma^e\right) \geq C\, \mathbb{1}\{M(2t-1, \eta, S_{2t-2}, S_{2t})

Play the same game with odd times :

$$\mathbb{E}\,\mathrm{Var}_{\mu_{2n}}(H_{2n}) \geqslant C \sum_{t=2}^{2n} \mathbb{E} \sum_{x,y} \mu_{2n}\left(S_{t-2}=x, S_{t}=y\right) \times \mathbb{1}_{M(t-1,\eta,x,y)}$$

$$\geqslant C' \sum_{t=2}^{2n} \mathbb{E} \sum_{x,y} \tilde{\mu}^{(t)}_{2n}(\ldots\ldots) \times \mathbb{1}_{M(\ldots\ldots)}$$

remove potential at time $t-1$

$$= C' \sum_{t} \sum_{x,y} \mathbb{E}\, \tilde{\mu}^{(t)}_{2n}(\ldots\ldots) \times \mathbb{P}\big(M(\ldots\ldots)\big)$$

$$\geqslant C' \sum_{t} \sum_{xy} \mathbb{E}\, \tilde{\mu}^{(t)}_{2n}(\ldots\ldots) \times 2p(1-p)\, \mathbb{1}_{\|x-y\|_\infty \leq}$$

$$\geqslant C'' \sum_{t} \sum_{xy} \mathbb{E}\, \mu_{2n}(\ldots\ldots) \times \mathbb{1}_{\|x-y\|_\infty \leq 1}$$

Finally,

$$\mathbb{E}\,\mathrm{Var}_{\mu_{2n}}(H_{2n}) \geqslant C'' \,\mathbb{E}\, E_{\mu_{2n}}\left[\sum_{t=2}^{2n} \mathbb{1}_{S_{t-1}-S_{t-2} \neq S_t - S_{t-1}}\right]$$

$$\geqslant C'' \,\delta\, n$$

(for $\beta \in$ compact.).

CQFD.

Sharp estimates

on $Q_n(k) = \#\{S : H_n(S) = k\}$, $d \geq 3$.

. Based on sharp asymptotics of Z_n: martingale

$$W_n(\beta) := Z_n(\beta)\, e^{-n\lambda(\beta)} \xrightarrow[n\to\infty]{a.s.} W_\infty(\beta).$$

[Imbrie – Spencer '88, Bolthausen '89 $\left\{ \begin{matrix} d \geq 3 \\ \text{"}\beta \text{ close to } p\text{"} \end{matrix} \right\}$ $\Rightarrow W_\infty(\beta) > 0$ a.s.

th. 4:

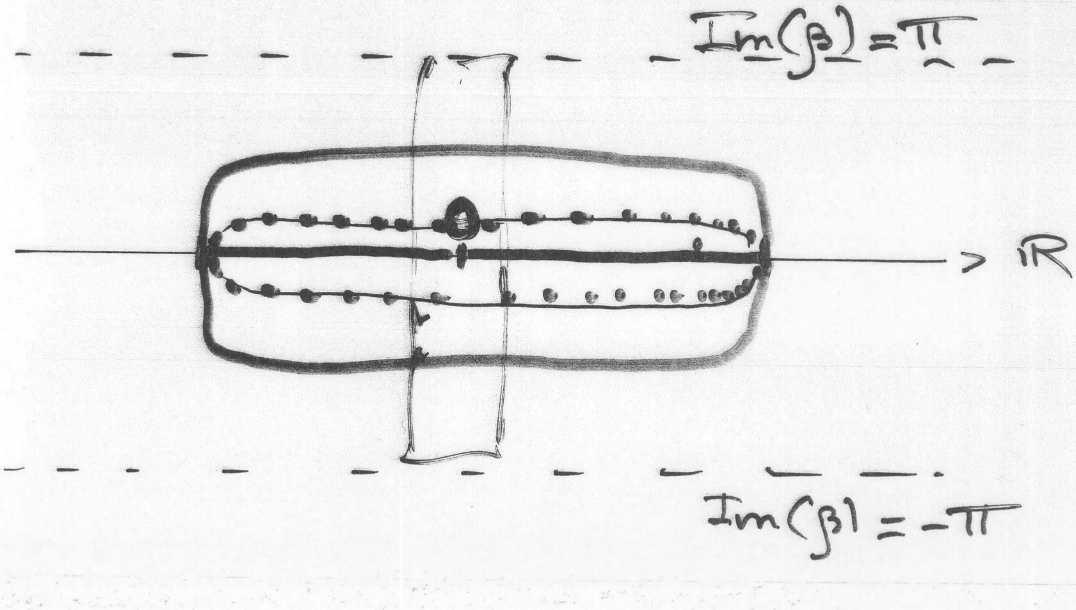
$\exists\, U$ neighborhood of 0 s.t. $\forall k_n$ with $\frac{k_n}{n} \to \rho \in U$,

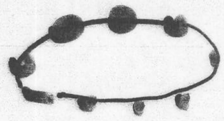
$$Q_n(k_n) = \sqrt{\frac{-\alpha''(\rho)}{2\pi n}} \cdot W_\infty(\beta(\rho)) \cdot e^{n\alpha\left(\frac{k_n}{n}\right)} \cdot (1+o(1))$$

with $o(1) \xrightarrow[n\to\infty]{} 0$ a.s., $\quad \beta(\rho) = \ln \frac{(1-p)\rho}{p(1-\rho)}$.

$\beta \in \mathbb{C}$

L^2-region



- In the $\mathbb{L}^2$**-region**, the martingale ($\beta \in$ real) converges in $\mathbb{L}^2$.
- In the **green region**, it converges in the sense of analytic functions [C-Yoshida '06].
- The dotted-green  is random, and $\log W_n(\beta)$ converges "analytically".
- Introduce **black** region  for further control.

$\cdots \mathrm{Im}(\beta) = +\pi/2$

$\cdots \mathrm{Im}(\beta) = -\pi/2$

$\longleftrightarrow$ small but fixed

$\longrightarrow$ $U :=$ **black** $\cap$ $\mathbb{L}^2$**-region**.

- Fourier Inversion Formula: $(\beta \in \mathbb{R})$

$$Q_n(k_n) = Z_n(\beta) e^{-\beta k_n} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} du \, \frac{Z_n(\beta + iu)}{Z_n(\beta)} e^{-ik_n u}$$

- Take $\beta = \underbrace{\beta(n, k_n)}_{\text{with } \beta(n,k)}$ maximizing:

$$\beta \longmapsto \beta k - \ln Z_n(\beta).$$

then, $\beta(n, k_n) \xrightarrow[n]{} \beta(\rho)$

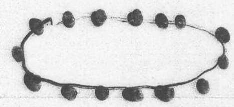
and

$$Z_n(\beta(n,k_n)) e^{-\beta(n,k_n) k_n} = W_\infty(\beta(\rho)) \cdot e^{n\alpha\left(\frac{k_n}{n}\right)} (1 + o(1)).$$

Work out the integral: with $\beta \equiv \beta(n, k_n)$,

$$\int_{-\pi}^{\pi} \frac{Z_n(\beta + iu)}{Z_n(\beta)} e^{-ik_n u} \, du = \int_{|u| \leq \varepsilon_n} \cdots + \int_{\varepsilon_n < |u| < \pi} \cdots$$

with $\varepsilon_n = \sqrt{\frac{\ln n}{n}}$.

- First term: $|u| \leq \varepsilon_n$, in the  region:

$$\log Z_n(\beta + iu) = \log Z_n(\beta) + iuk_n - \frac{u^2}{2} \hat{D}_n + \mathrm{Rest}_n .$$

since $\beta = \beta(n, k_n)$ maximizes

Var. of the tilted measure $\sim n \lambda''(\beta(\rho))$

$$|\mathrm{Rest}_n| \underset{\substack{\text{integral}\\\text{formula}}}{\overset{\text{Cauchy}}{\leq}} c\, n\, |u|^3 .$$

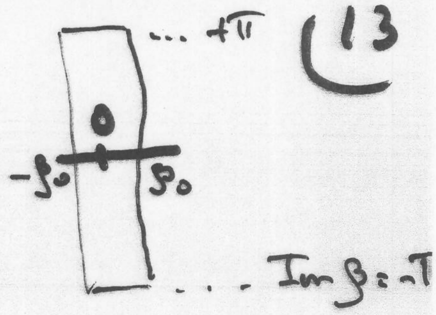
So, $$\int_{|u| \leq \varepsilon_n} \cdots = \sqrt{\frac{2\pi}{n \lambda''(\beta(\rho))}} \, (1 + o(1)) .$$

- It is enough to show **"locality property"**

$$\int_{\varepsilon_n < |u| \leq \pi} \cdots = o\left(\frac{1}{\sqrt{n}}\right)$$

meaning that the law of $H_n(\cdot)$ under the tilted measure does not concentrate on a sub-lattice of $\mathbb{Z}$!

Black region: we can fix $\beta_0 > 0$ s.t.



$\exists\, K > 0$ with

$$\left| \frac{Z_n(\beta + iu)}{Z_n(\beta)} \right| \leq e^{-Knu^2} + e^{-Kn} \qquad \begin{cases} -\beta_0 \leq \beta \leq \beta_0 \\ |u| \leq \pi \end{cases}, \quad \mathbb{P}\text{-a.s.}$$

$$\|$$

$$\left| E_{\mu_{2n}} e^{iuH_{2n}} \right| \leq E_{\mu_{2n}} \left| E_{\mu_{2n}} \left(e^{iuH_{2n}} \,\Big|\, \Sigma^e \right) \right|$$

even times

$$\overset{\text{Markov}}{=} E_{\mu_{2n}} \prod_{t=1}^{n} \left| E_{\mu_{2n}} \left(e^{iu\eta(2t-1, S_{2t-1})} \,\Big|\, \Sigma^e \right) \right|$$

$$\leq E_{\mu_{2n}} \prod_{t=1}^{n} e^{-Cu^2 \mathbb{1}_{M(2t-1, \eta, S_{2t-2}, S_{2t})}}$$

as above

$$= E_{\mu_{2n}} e^{-Cu^2 \sum_{t=1}^{n} \mathbb{1}_{M(2t-1, \eta, S_{2t-2}, S_{2t})}}$$

split according to $\sum_t > n\varepsilon$ or not.

$$\leq e^{-C\varepsilon u^2 n} + \mu_{2n}\left(\sum_{t=1}^{n} \mathbb{1}_{M(2t-1, \eta, S_{2t-2}, S_{2t})} \leq n\varepsilon \right)$$

exp. small under μ_{2n} when $\beta = 0$.
will remain so for small enough β.