Intersection of random walks in supercritical dimensions

Peter Mörters

University of Bath

joint work with Xia Chen (Knoxville)

Framework

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$$(X^{(1)}(n) \,:\, n \in \mathbb{N}), \dots, (X^{(p)}(n) \,:\, n \in \mathbb{N})$$

be $p \geq 2$ independent identically distributed random walks started in the origin and taking values in $\mathbb{Z}^d$.

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The number of intersections of these walks can be measured in two natural ways: The intersection local time of the random walks,

$$I := \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} \mathbf{1}\{X^{(1)}(i_1) = \cdots = X^{(p)}(i_p)\},$$

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counts the times when the paths intersect, whereas the intersection of the ranges

$$J := \sum_{x \in \mathbb{Z}^d} \mathbf{1}\{X^{(1)}(i_1) = \cdots = X^{(p)}(i_p) = x \text{ for some } (i_1, \ldots, i_p)\}$$

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where $\mathbb{G}(x) \approx (|x|+1)^{2-d}$ is the Green's function.

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Fact (Erdős and Taylor)

$$\mathbb{P}\{I < \infty\} = \mathbb{P}\{J < \infty\} = \begin{cases} 1 & \text{if } p(d-2) > d, \\ 0 & \text{otherwise.} \end{cases}$$

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From now on we assume that $p(d-2) > d$, i.e. we are in supercritical dimensions.

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Interestingly, the upper tails of J are substantially lighter. They show that, for all $\varepsilon > 0$ and all sufficiently large a,

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The challenging question lies in understanding the difference of these behaviours, providing sharp estimates for the tails, and understanding the underlying 'optimal strategies' for the random walks.

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$$\lim_{t\uparrow\infty} \frac{1}{t^{(d-2)/d}} \log \mathbb{P}\big\{\big|W_1^\varepsilon(\theta t) \cap W_2^\varepsilon(\theta t)\big| \geq t\big\} = -I_d^\varepsilon(\theta),$$

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This strongly suggests that, in the supercritical case $d \geq 5$,

$$\lim_{t\uparrow\infty} \frac{1}{t^{(d-2)/d}} \log \mathbb{P}\big\{\big|W_1^\varepsilon(\infty) \cap W_2^\varepsilon(\infty)\big| \geq t\big\} = -I_d^\varepsilon(\theta^*),$$

but their techniques do not allow the treatment of infinite times and this problem, like its discrete counterpart, remains open.

Main result

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Note that we are following the (slightly unusual) convention of not summing over the time $n = 0$, which has an influence on the value $\mathbb{G}(0)$.

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For every nonnegative $h \in L^q(\mathbb{Z}^d)$ a bounded, symmetric, positive operator

$$\mathfrak{A}_h \colon L^2(\mathbb{Z}^d) \to L^2(\mathbb{Z}^d)$$

is defined by

$$\mathfrak{A}_h g(x) = \sqrt{e^{h(x)} - 1} \sum_{y \in \mathbb{Z}^d} \mathbb{G}(x-y) g(y) \sqrt{e^{h(y)} - 1}\,.$$

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Our main result is formulated in terms of the spectral radius

$$\|\mathfrak{A}_h\| := \sup\left\{\langle g, \mathfrak{A}_h g\rangle \;:\; \|g\|_2 = 1\right\}$$

of the operator $\mathfrak{A}_h$.

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Theorem 1 (Chen, M 2007)

The upper tail behaviour of the intersection local time I is given as

$$\lim_{a\uparrow\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\big\{ I > a \big\} = -p \inf \big\{ \|h\|_q \,:\, h \geq 0 \text{ with } \|\mathfrak{A}_h\| \geq 1 \big\}.$$

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Remark: The optimal strategy for the random walks is to each spend about $a^{1/p}$ time units in a bounded domain which does not grow with a. Then we get $I \approx a$ from intersections in this domain alone. This strategy makes I large without making J large, thus explaining the different tail behaviour.

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By a Tauberian theorem for any nonnegative X,

$$\lim_{k\uparrow\infty} \frac{1}{k} \log E\Big[\frac{X^k}{(k!)^p}\Big] = -\kappa \quad \Longleftrightarrow \quad \lim_{a\uparrow\infty} \frac{1}{a^{1/p}} \log P\{X > a\} = -pe^{\kappa/p}.$$

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where $\mathcal{E}_m$ is the set of partitions $\{\pi_1,\dots,\pi_m\}$ of $\{1,\dots,k\}$ into m nonempty sets and $\mathcal{A}(\pi)$ is the set of tuples $(x_1,\dots,x_k)$ which are constant on the partitions.

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Let $A \subset \mathbb{Z}^d$ be finite. Then we can analyse expressions of the form

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- symmetry (and nothing more!) of the function $G\colon \mathbb{Z}^d \to (0,\infty)$.

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We obtain, for finite $A \subset \mathbb{Z}^d$ that

$$\lim_{k\to\infty} \frac{1}{k} \log \frac{1}{k!} \sum_{x_1,\dots,x_k\in A} \Big[\sum_{m=1}^{k} \sum_{\pi\in\mathcal{E}_m} \mathbf{1}_{\{(x_1,\dots,x_k)\in\mathcal{A}(\pi)\}} \sum_{\sigma\in\mathfrak{S}_m} \prod_{\ell=1}^{m} G\big(x_{\pi_{\sigma(\ell)}} - x_{\pi_{\sigma(\ell-1)}}\big)\Big]^p$$

$$= -p \log \inf \big\{ \|h\|_q \,:\, h \geq 0 \text{ with } \|\mathfrak{A}_h^A\| \geq 1 \big\},$$

where the self-adjoint operator $\mathfrak{A}_h^A \colon L^2(A) \to L^2(A)$ is defined by

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This suffices for the lower bound. The extension of the upper bound from finite sets A to the entire lattice is nontrivial, because the problem is not exponentially tight: Note that all shifts of A produce the same exponential decay of the upper tails of the intersection local times. To overcome this problem, we need to project the full problem onto a finite domain by wrapping it around a torus. The problem retains the given form, but with a different kernel G. We then let the period of the torus go to infinity.

Main result revisited

The upper tail behaviour of the intersection local time I is given as

$$\lim_{a\uparrow\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\big\{ I > a \big\} = -p \inf \big\{ \|h\|_q \,:\, h \geq 0 \text{ with } \|\mathfrak{A}_h\| \geq 1 \big\} .$$

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Remark: It is unsatisfactory that we cannot readily interpret the optimal h in the variational problem in a probabilistic manner. To some extent this is an artefact which is due to the discrete time structure of the random walk.

A related problem

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For comparison we therefore now look at independent continuous time random walks

$$(X^{(1)}(t) \,:\, t \geq 0), \ldots, (X^{(p)}(t) \,:\, t \geq 0)$$

and let A be their generator given by

$$Af(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x f(X_t) - f(x)}{t} \,.$$

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Again we ask for the upper tail behaviour.

A related problem

Theorem 2 (Chen, M 2007)

The upper tail behaviour of the intersection local time $\tilde{I}$ is given as

$$\lim_{a\uparrow\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\big\{\tilde{I} > a\big\} = -p \inf\left\{\big\|\sqrt{-A}g\big\|_2^2 \,:\, \|g\|_{2p} = 1\right\}.$$

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Remark: The optimal strategy for the random walks is to have a local time field like

$$\ell^{(j)}(x) := \int_0^\infty \mathbf{1}\{X^{(j)}(t) = x\} \approx a^{1/p} g^2(x),$$

which implies

$$\tilde{I} = \sum_{x\in\mathbb{Z}^d} \prod_{j=1}^{p} \ell^{(j)}(x) \approx \sum_{x\in\mathbb{Z}^d} \prod_{j=1}^{p} a^{1/p} g^2(x) = a.$$

The probability of a random walk achieving such a local time is

$$\approx \exp\left\{ -a^{1/p} \left\| \sqrt{-A} g \right\|_2^2 \right\},$$

which resembles the rate functions in Donsker-Varadhan theory.

How do the limits compare?

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Our proof follows a similar strategy as in the discrete time case, but there is now an simpler formula for the kth moments

$$\mathbb{E}\tilde{I}^k = \sum_{x_1,\dots,x_k \in \mathbb{Z}^d} \Big[\sum_{\sigma \in \mathfrak{S}_k} \prod_{\ell=1}^{k} \mathbb{G}(x_{\sigma(\ell-1)} - x_{\sigma(\ell)}) \Big]^p .$$

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From this we obtain

$$\lim_{a\uparrow\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\{\tilde{I} > a\} = -p \inf \{ \|h\|_q \,:\, h \geq 0 \text{ with } \|\mathfrak{B}_h\| \geq 1 \},$$

where the operator $\mathfrak{B}_h \colon L^2(\mathbb{Z}^d) \to L^2(\mathbb{Z}^d)$ is defined by

$$\mathfrak{B}_h g(x) = \sqrt{h(x)} \sum_{y \in \mathbb{Z}^d} \mathbb{G}(x-y) g(y) \sqrt{h(y)}$$

and the Green's function is

$$\mathbb{G}(x) = \int_0^\infty \mathbb{P}\{X(t) = x\}\, dt.$$

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$$\inf \Big\{ \|h\|_q \;:\; h \geq 0 \text{ with } \sup_{\|g\|_2=1} \langle g, \mathfrak{B}_h g \rangle \geq 1 \Big\}$$

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$$\inf\Big\{\|h\|_q \;:\; h\geq 0 \text{ with } \sup_{\|g\|_2=1}\langle g,\mathfrak{B}_h g\rangle\geq 1\Big\}$$
$$=\inf\Big\{b \;:\; \sup_{\substack{\|g\|_2=1\\ \|h\|_q=1}}\big\langle\sqrt{h}g,\mathfrak{G}\sqrt{h}g\big\rangle\geq 1/b\Big\}$$

where $\mathfrak{G}$ is the Green's operator

$$\mathfrak{G}f(x):=\sum_{y\in\mathbb{Z}^d}\mathbb{G}(x-y)f(y)\,.$$

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The maximiser f exists and satisfies $\rho f=\mathfrak{G}f^{2p-1}$. We obtain the final form from $-A\circ\mathfrak{G}=id$ as

$$1/\rho=-\sup\Big\{\langle f,Af\rangle \;:\; \|f\|_{2p}=1\Big\}=\inf\Big\{\big\|\sqrt{-A}f\big\|_2^2 \;:\; \|f\|_{2p}=1\Big\}.$$

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- Our approach allows a direct treatment of the infinite time horizon avoiding the use of Donsker-Varadhan theory.
- We believe that this method has potential to solve some hard problems related to the intersection of the ranges as well. This work is in progress.