Adaptive solution of eigenvalue problems for PDEs

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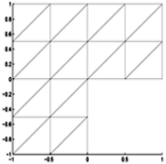
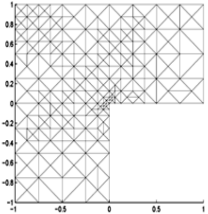
- Motivation.
- Model problem.
- A priori and a posteriori error estimation.
- Convergence of adaptive method for eigenvalue problems.
- New AFEM algorithm for elliptic eigenvalue problem.
- Numerical examples.

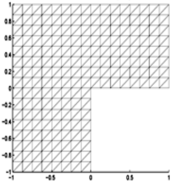
Too large error



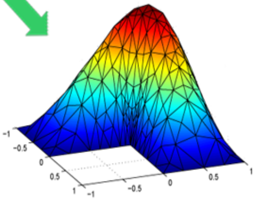
AFEM



$$-\Delta u = \lambda u \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$



High computational effort



Second order elliptic eigenvalue problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary (i.e. polygonal domain if $d = 2$), $\mathcal{A}$ piecewise $W^{1,\infty}(\Omega)$ uniformly positive definite symmetric matrix-valued function i.e.

$$a_1|\xi|^2 \leqslant \mathcal{A}(x)\xi \cdot \xi \leqslant a_2|\xi|^2, \forall \xi \in \mathbb{R}^d, \ \forall x \in \Omega,$$

and $\mathcal{B}$ a scalar function such that

$$b_1 \leqslant \mathcal{B}(x) \leqslant b_2 \quad \text{for some } b_1, b_2 > 0.$$

PDE formulation of elliptic eigenvalue problem

$$\begin{aligned} -\nabla \cdot (\mathcal{A}\nabla u) &= \lambda \mathcal{B} u \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Let $a, b : \mathcal{H}_0^1 \times \mathcal{H}_0^1 \to \mathbb{R}$ be the bilinear forms defined by

$$\begin{aligned} a(u,v) &:= \int_\Omega \mathcal{A}\nabla u \cdot \nabla v, \\ b(u,v) &:= \int_\Omega \mathcal{B} uv. \end{aligned}$$

a and b induce norms of the form

$$\begin{aligned} \|u\|_a &:= a(u,u)^{\frac{1}{2}}, \quad u \in \mathcal{H}_0^1(\Omega) \\ \|u\|_b &:= b(u,u)^{\frac{1}{2}}, \quad u \in L^2(\Omega). \end{aligned}$$

Because of certainly chosen $\mathcal{A}$, $\mathcal{B}$, $\|\cdot\|_a \simeq \|\cdot\|_{\mathcal{H}_0^1(\Omega)}$ and $\|\cdot\|_b \simeq \|\cdot\|_\Omega$.

Weak formulation for eigenvalue problem

Weak formulation for continuous eigenvalue problem

$$\begin{cases} a(u,v) &= \lambda b(u,v), \qquad \forall v \in \mathcal{H}_0^1(\Omega) \\ \|u\|_b &= 1. \end{cases}$$

Weak formulation for discrete eigenvalue problem

Given the finite-dimensional subspace $V_h \in V$, we get the discrete eigenvalue problem of the following form

$$\begin{cases} a(u_h,v_h) &= \lambda_h b(u_h,v_h), \qquad \forall v_h \in V_h \subset \mathcal{H}_0^1(\Omega) \\ \|u_h\|_b &= 1. \end{cases}$$

PDE formulation

$$-\Delta u = \lambda u \in \Omega \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega$$

$\mathcal{A} \equiv I$ and $\mathcal{B} \equiv 1$.

Continuous variational formulation

$$a(u, v) := \lambda(u, v) \quad \forall v \in \mathcal{H}_0^1(\Omega)$$

with

$$a(u, v) := \int_\Omega \nabla u \, \nabla v \, dx, \quad (u, v) := \int_\Omega u \, v \, dx.$$

Discrete variational formulation

$$a(u_h, v_h) = \lambda_h(u_h, v_h), \quad \forall v_h \in V_h \subset \mathcal{H}_0^1(\Omega).$$

Let $V_h = span\{\varphi_1^h, \ldots, \varphi_n^h\}$. Then discrete eigenvalue problem can be written as algebraic eigenvalue problem

$$A_h x_h = \lambda_h B_h x_h.$$

A priori error estimation

Proving convergence of FEM approximations and determing convergence rate when the mesh is refined.

A posteriori error estimation

Obtaining two-sided computable error bounds based on the data and the discrete solution.
Computable quantity that indicates distribution of the error.

A priori error estimates [RaviartThomas83]

$$\begin{aligned} \|u-u_h\|_a &\leqslant Ch^r, \\ \|u-u_h\| &\leqslant Ch^r\|u-u_h\|_a, \\ |\lambda-\lambda_h| &\leqslant C\|u-u_h\|_a^2. \end{aligned}$$

$u \in \mathcal{H}^{1+r}(\Omega),\ r \in (0,1]$; C – constant depending on λ and $\mathcal{T}_h$.

A priori error estimates [KnyazevOsborn06]

$$\begin{aligned} 0 \leqslant \frac{\lambda_j-\lambda_{jh}}{\lambda_j} &\leqslant \|(I-\tilde{Q}+\tilde{P}_{1,\dots,j-1})u_j\|^2 \\ &\leqslant \left(1+\frac{\|\left(I-\tilde{Q}\right)T\tilde{P}_{1,\dots,j-1}\|^2}{\min_{i=1,\dots,j-1}|\lambda_{jh}-\lambda_j|^2}\right)\sin^2\angle(u_j,\tilde{U}). \end{aligned}$$

T – sp. operator on $\mathcal{H}$; $\tilde{U}=\mathrm{span}\{u_{1h},\dots,u_{nh}\}$; $\tilde{Q}$ – orthogonal projector onto $\tilde{U}$;

$\tilde{P}_{1,\dots,j-1}$ orthogonal projector onto $\tilde{U}_{1,\dots,j-1}$; $\angle(u_j,\tilde{U})$ largest principal angle.

Simple bound

$$|x^H Ax - y^H Ay| \leqslant spr(A) sin^2\theta(x,y)$$

$\mathcal{X}, \mathcal{Y}$ – one-dimensional trial subspace, perturbed $\mathcal{X}$, spanned by unit vectors x, y; A - Hermitian matrix.

Generalizations for subspaces $\mathcal{X}$ and $\mathcal{Y}$

$$|\lambda\left(X^H AX\right) - \lambda\left(Y^H AY\right)| \prec_w spr(A) sin^2\theta(\mathcal{X},\mathcal{Y})$$

dim$(\mathcal{X}) = dim(\mathcal{Y})$; $\mathcal{X}, \mathcal{Y}$ A-invariant; X, Y – o.n. basis.

Error bounds for FEM

$$\begin{aligned} 0 \quad &\leqslant \quad \Lambda\left(\left(P_{\mathcal{X}}A\right)|_{\mathcal{X}}\right) - \Lambda_{dim\mathcal{X}}\left(\left(P_{\mathcal{Y}}A\right)|_{\mathcal{Y}}\right) \\ &\prec_w \quad \left(\Lambda\left(\left(P_{\mathcal{X}}A\right)|_{\mathcal{X}}\right) - \lambda_{\min(\mathcal{X}+\mathcal{Y})}\right) sin^2\Theta(\mathcal{X},\mathcal{Y}) \end{aligned}$$

$A : \mathcal{H} \to \mathcal{H}$; $\mathcal{X}, \mathcal{Y}, \mathcal{X} + \mathcal{Y}$ finite dimensional subspaces of $\mathcal{H}$; $P_{\mathcal{X}}$ orthogonal projector onto $\mathcal{X}$;

$(P_{\mathcal{X}}A)|_{\mathcal{X}}$ restriction of operator $P_{\mathcal{X}}A$ to its invariant subspace $\mathcal{X}$; $\Theta(\mathcal{X},\mathcal{Y})$ vector of principal angles.

Residual-based error estimator [DuránPadraRodríguez03]

$$\eta_h := \left(\sum_{T \in \mathcal{T}} h_T^2 \lambda_h^2 \|u_h\|^2_{L^2(T)} + \frac{1}{2} \sum_{E \in \mathcal{E}} h_E \| \left[\frac{\partial u_h}{\partial \nu_E} \right] \|^2_{L^2(E)} \right)^{\frac{1}{2}}.$$

Averaging estimator [MaoShenZhou06]

$$\mu_h := \left(\sum_{T \in \mathcal{T}} h_T^2 ||\lambda_h u_h - \beta u_h + div(A(\nabla u_h))||^2 + ||A(\nabla u_h) - \nabla u_h||^2 \right)^{\frac{1}{2}}.$$

h_T – element diameter; h_E – edge diameter;
$A(\nabla u_h) := \sum_{z \in \mathcal{N}_h} \frac{1}{|\omega_z|} \left(\int_{\omega_z} \nabla u_h dx \right) \varphi_z;$
$\left[\frac{\partial u_h}{\partial \nu_E} \right] := (\nabla u_h \mid_{T_2} (x) - \nabla u_h \mid_{T_1} (x)) \nu_E.$

Edge-based residual estimator [CarstensenGedicke08]

$$\eta_h := \left(\sum_{E \in \mathcal{E}} h_E \left\| \left[\frac{\partial u_h}{\partial \nu_E} \right] \right\|^2_{L^2(E)} \right)^{\frac{1}{2}} .$$

Averaging estimator [CarstensenGedicke08]

$$\mu_h := \left(\sum_{T \in \mathcal{T}} \| A(\nabla u_h) - \nabla u_h \|^2_{L^2(T)} \right)^{\frac{1}{2}} .$$

Global convergence [CarstensenGedicke08]

The sequence of discrete eigenvalues (λ_h) converges toward some eigenvalue λ of the continuous problem. Each subsequence (u_{h_j}) of discrete eigenvectors has a further subsequence which converges toward some u in V and u is an eigenvector of λ.

- S.Giani, I.G.Graham 2007,
- C.Carstensen, J.Gedicke 2008 (without inner node property),
- E.M.Garau, P.Morin, C.Zuppa 2008.

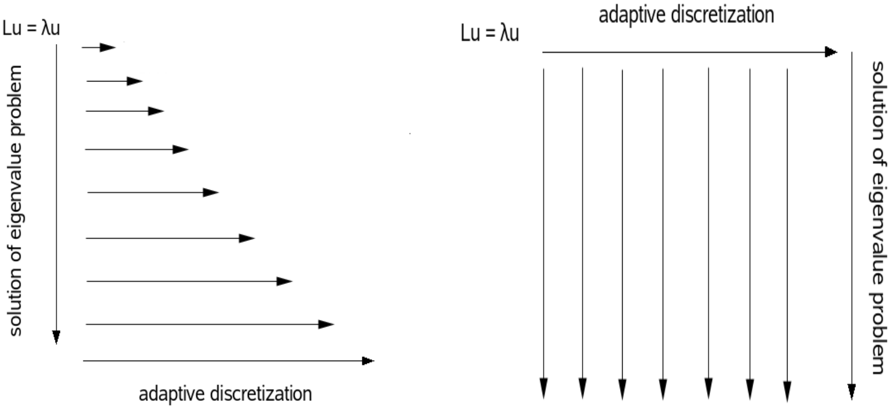
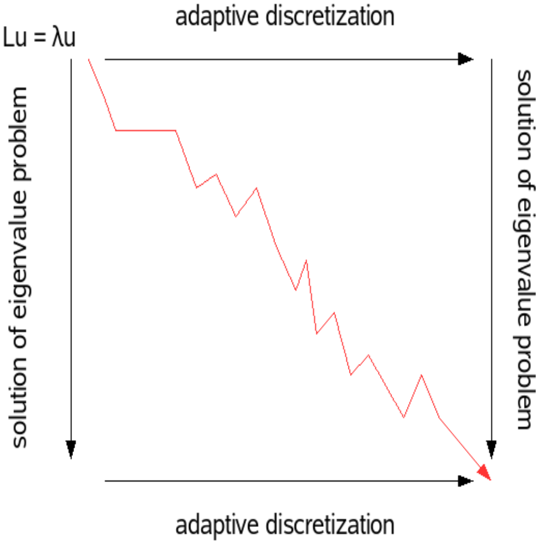


Figure: (a) spectralize-discretise, (b) discretise-spectralize.


adaptive discretization
$Lu = \lambda u$
solution of eigenvalue problem
solution of eigenvalue problem
adaptive discretization

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}$$

Goals

- decision about refinement should be based only on the solution calculated on the coarse grid,
- we should use relation between coarse and fine grid,
- solving the problem using Krylov subspaces reduce computational effort,
- expressing the solution computed on the coarse grid in terms of basis functions from the fine grid can be used to mark elements,
- marking strategy should be based on reliable and efficient a posteriori estimators.

Algorithm 1 AFEM for eigenvalue problem

Input: An initial regular triangulation $\mathcal{T}_0$, k

Output: Smallest eigenvalue λ_1 with corresponding eigenvector

Solve: compute eigenpair (λ_H, x_H) for the coarse mesh $\mathcal{T}_H$ using Arnoldi method with k to compute (λ_H, x_H)

if $\|r_H\| < \epsilon$ **then**

  return (λ_H, x_H)

else

  Express x_H using basis functions from the fine mesh $\mathcal{T}_h$

  P = projection matrix from coarse mesh $\mathcal{T}_H$ to fine mesh $\mathcal{T}_h$

  $\tilde{x}_h = P \cdot x_H$

  Estimate: compute $\tilde{r}_h = A_h \tilde{x}_h - \tilde{\lambda}_h \tilde{x}_h$ and indentify all large coefficients in $\tilde{r}_h$ and corresponding basis functions (nodes)

  Mark: mark all edges that contains identified nodes and apply closure algorithm

  Refine: refine coarse mesh $\mathcal{T}_H$ using RedGreenBlue refinement to get $\tilde{\mathcal{T}}_h$

  Start Algorithm 1 with $\tilde{\mathcal{T}}_h$

end if

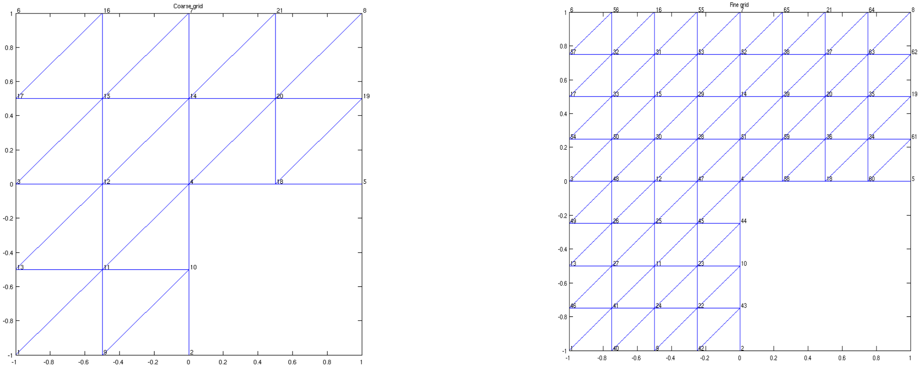


Figure: (a) coarse mesh $\mathcal{T}_H$, (b) fine mesh $\mathcal{T}_h$.

$$u_H = x_{11}\varphi_{11}^H + x_{12}\varphi_{12}^H + x_{14}\varphi_{14}^H + x_{15}\varphi_{15}^H + x_{20}\varphi_{20}^H.$$

Prolongation to fine mesh

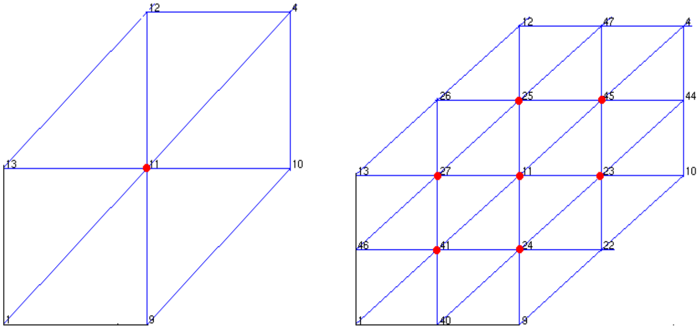
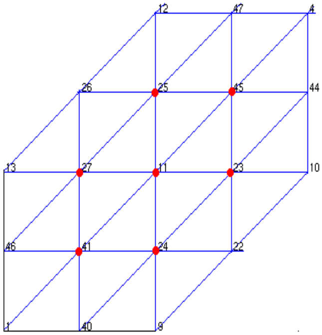


Figure: (a) coarse mesh $\mathcal{T}_H$, (b) fine mesh $\mathcal{T}_h$.

$$\begin{aligned}
\varphi_{11}^H &= \alpha_{41}\varphi_{41}^h + \alpha_{24}\varphi_{24}^h + \alpha_{11}\varphi_{11}^h + \alpha_{23}\varphi_{23}^h + \alpha_{27}\varphi_{27}^h + \alpha_{25}\varphi_{25}^h + \alpha_{45}\varphi_{45}^h \\
&= \frac{1}{2}\varphi_{41}^h + \frac{1}{2}\varphi_{24}^h + \varphi_{11}^h + \frac{1}{2}\varphi_{23}^h + \frac{1}{2}\varphi_{27}^h + \frac{1}{2}\varphi_{25}^h + \frac{1}{2}\varphi_{45}^h
\end{aligned}$$

$$
\begin{aligned}
u_H &= x_{11}\varphi^H_{11} + x_{12}\varphi^H_{12} + x_{14}\varphi^H_{14} + x_{15}\varphi^H_{15} + x_{20}\varphi^H_{20} \\
&= x_{11}\varphi^h_{11} + x_{12}\varphi^h_{12} + x_{14}\varphi^h_{14} + x_{15}\varphi^h_{15} + x_{20}\varphi^h_{20} + \frac{1}{2}x_{11}\varphi^h_{23} + \frac{1}{2}x_{11}\varphi^h_{24} + \\
&\quad (\frac{1}{2}x_{11} + \frac{1}{2}x_{12})\varphi^h_{25} + \frac{1}{2}x_{12}\varphi^h_{26} + \frac{1}{2}x_{11}\varphi^h_{27} + (\frac{1}{2}x_{12} + \frac{1}{2}x_{14})\varphi^h_{28} + \\
&\quad (\frac{1}{2}x_{14} + \frac{1}{2}x_{15})\varphi^h_{29} + (\frac{1}{2}x_{12} + \frac{1}{2}x_{15})\varphi^h_{30} + \frac{1}{2}x_{15}\varphi^h_{31} + \frac{1}{2}x_{15}\varphi^h_{33} + \frac{1}{2}x_{20}\varphi^h_{35} + \\
&\quad \frac{1}{2}x_{20}\varphi^h_{36} + \frac{1}{2}x_{20}\varphi^h_{37} + \frac{1}{2}x_{14}\varphi^h_{38} + (\frac{1}{2}x_{14} + \frac{1}{2}x_{20})\varphi^h_{39} + \frac{1}{2}x_{11}\varphi^h_{41} + \frac{1}{2}x_{11}\varphi^h_{45} + \\
&\quad \frac{1}{2}x_{12}\varphi^h_{47} + \frac{1}{2}x_{12}\varphi^h_{48} + \frac{1}{2}x_{15}\varphi^h_{50} + \frac{1}{2}x_{14}\varphi^h_{51} + \frac{1}{2}x_{14}\varphi^h_{52} + \frac{1}{2}x_{15}\varphi^h_{53} + \frac{1}{2}x_{20}\varphi^h_{59} + \frac{1}{2}x_{20}\varphi^h_{63}
\end{aligned}
$$

$$
\begin{aligned}
u_h &= y_{11}\varphi^h_{11} + y_{12}\varphi^h_{12} + y_{14}\varphi^h_{14} + y_{15}\varphi^h_{15} + y_{20}\varphi^h_{20} + y_{22}\varphi^h_{22} + y_{23}\varphi^h_{23} + y_{24}\varphi^h_{24} + \\
&\quad y_{25}\varphi^h_{25} + y_{26}\varphi^h_{26} + y_{27}\varphi^h_{27} + y_{28}\varphi^h_{28} + y_{29}\varphi^h_{29} + y_{30}\varphi^h_{30} + y_{31}\varphi^h_{31} + y_{32}\varphi^h_{32} + \\
&\quad y_{33}\varphi^h_{33} + y_{34}\varphi^h_{34} + y_{35}\varphi^h_{35} + y_{36}\varphi^h_{36} + y_{37}\varphi^h_{37} + y_{38}\varphi^h_{38} + y_{39}\varphi^h_{39} + y_{41}\varphi^h_{41} + \\
&\quad y_{45}\varphi^h_{45} + y_{47}\varphi^h_{47} + y_{48}\varphi^h_{48} + y_{50}\varphi^h_{50} + y_{51}\varphi^h_{51} + y_{52}\varphi^h_{52} + y_{53}\varphi^h_{53} + y_{59}\varphi^h_{59} + y_{63}\varphi^h_{63}
\end{aligned}
$$



Prolongation matrix

Let $x \in \mathbb{R}^n$ be a vector of coefficients on the coarse grid such that $x = [x_1, \ldots, x_n]^T$ and $y \in \mathbb{R}^m$ be a vector of coefficients on the fine grid $y = [x_1, \ldots, x_m]^T$, $m > n$. Then there exist a matrix $P \in \mathbb{R}^{m \times n}$ (*prolongation matrix*) such that

$$Px = y.$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 11 & 12 & 21 \end{array} \\
\begin{array}{c} 1 \\ 11 \\ \\ 24 \\ 25 \\ 64 \end{array} &
\begin{bmatrix}
\ddots & \vdots & \vdots & \ddots \\
\cdots & 1 & \cdot & \cdots \\
\ddots & \vdots & \vdots & \ddots \\
\cdots & \frac{1}{2} & \cdot & \cdots \\
\cdots & \frac{1}{2} & \frac{1}{2} & \cdots \\
\ddots & \vdots & \vdots & \ddots
\end{bmatrix}
\end{array}
\begin{bmatrix} \vdots \\ x_{11} \\ x_{12} \\ \vdots \end{bmatrix}
=
\begin{bmatrix} \vdots \\ y_{11} \\ \vdots \\ y_{24} \\ y_{25} \\ \vdots \end{bmatrix}
$$

$$P \qquad x = y$$

A priori error estimate for eigenvalue

$$\begin{aligned} |\lambda_h - \tilde{\lambda}_h| &= \|u_h^H A_h u_h - \tilde{u}_h^H A_h \tilde{u}_h\| \leqslant spr(A_h) \cdot \sin^2 \Theta(u_h, \tilde{u}_h), \\ |\lambda - \tilde{\lambda}_h| &\leqslant |\lambda - \lambda_h| + spr(A_h) \cdot \sin^2 \Theta(u_h, \tilde{u}_h). \end{aligned}$$

A priori error estimates for eigenvectors

$$\begin{aligned} \|u - \tilde{u}_h\| &\leqslant \|u - u_H\| + \|u_H - u_H^k\| + \|u_H^k - \tilde{u}_h\| \\ &\leqslant \|u - u_H\| + \|u_H - u_H^k\| + \|u_H^k - P u_H^k\| \end{aligned}$$

Residual error

$$\begin{aligned}\|A_h\tilde{u}_h - \tilde{\lambda}_h\tilde{u}_h\| &\leqslant \|A_h\tilde{u}_h - \lambda_h\tilde{u}_h\| + \|\lambda_h\tilde{u}_h - \tilde{\lambda}_h\tilde{u}_h\| \\ &\leqslant \|A_h - \lambda_h I\|\|\tilde{u}_h\| + |\lambda_h - \tilde{\lambda}_h|\|\tilde{u}_h\| \\ &\leqslant spr(A_h) + spr(A_h)\cdot\sin^2\Theta(u_h,\tilde{u}_h).\end{aligned}$$

A posteriori error estimate

$$\|A\tilde{u}_h - \tilde{\lambda}_h\tilde{u}_h\| \leqslant \epsilon \Longrightarrow \text{small error}$$

Marking strategies

- max strategy:

$$\eta_T \geqslant \Theta \max_{T \in \mathcal{T}} \eta_T$$

- bulk strategy:

$$\sum_{T \in \mathcal{M}} \eta_T \geqslant \Theta \sum_{T \in \mathcal{T}} \eta_T$$

$$0 \leqslant \Theta \leqslant 1$$

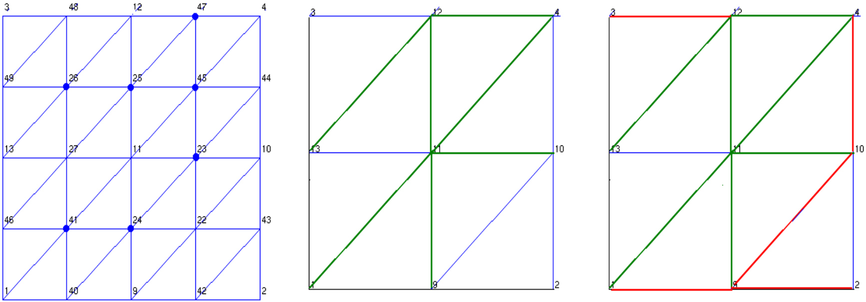


Figure: Marking strategy.

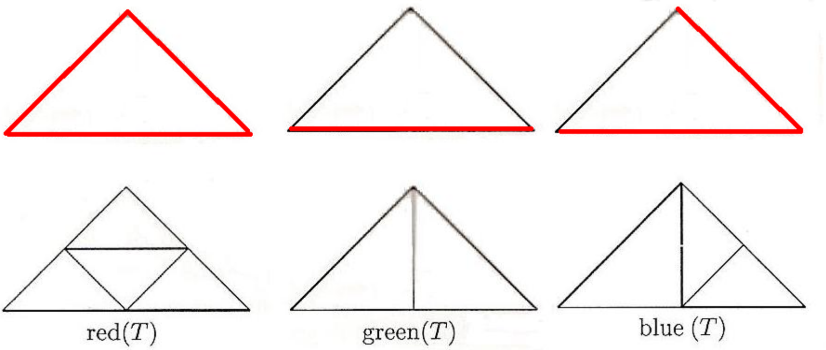


Figure: Refinement strategies.

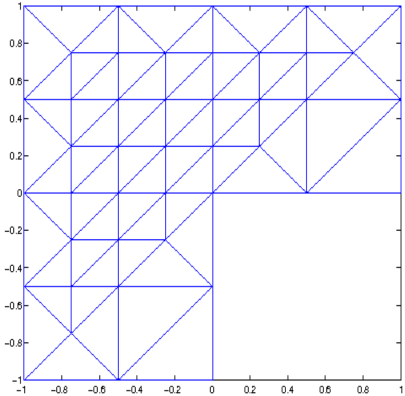


Figure: Adaptively refined mesh after 1 step.

$\lambda \approx 9.6397$

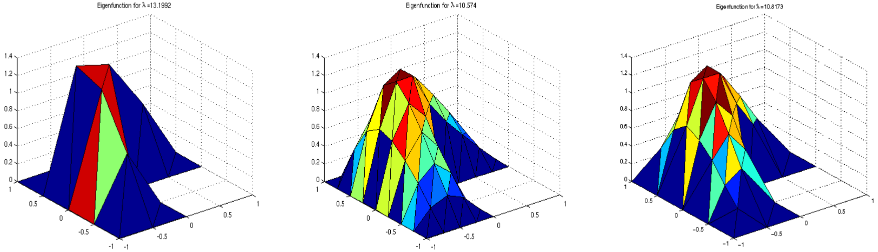


Figure: First eigenfunction for (a) coarse mesh, (b) uniformly refined mesh, (c) adaptively refined mesh.

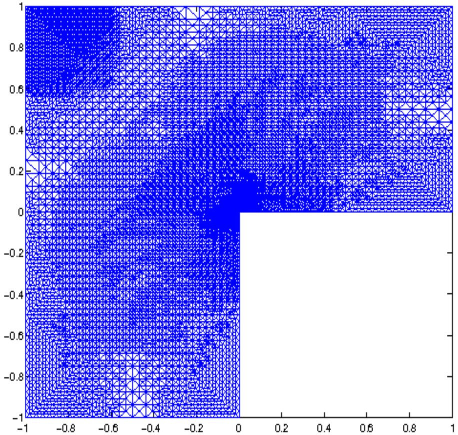


Figure: Adaptively refined mesh after 8 steps.

Approximations for the smallest eigenvalue

$\lambda \approx 9.6397$

step	#DOF	λ
1	5	13.1992
2	27	10.8173
3	99	9.9982
4	306	9.7721
5	641	9.6982
6	1461	9.6652
7	2745	9.6528
8	5961	9.6455

Table: Smallest eigenvalue approximation.

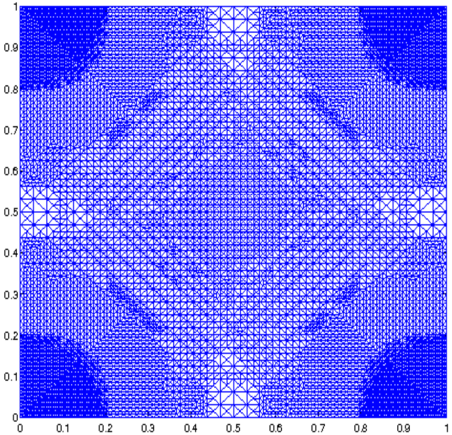


Figure: Adaptively refined mesh after 8 steps.

$\lambda = 2\pi^2 \approx 19.7392$

step	#DOF	λ
1	1	32.0000
2	9	23.0695
3	37	20.6068
4	121	19.9673
5	439	19.7998
6	1321	19.7613
7	2449	19.7524
8	5353	19.7448

Table: Smallest eigenvalue approximation.

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Thank you very much for your attention !