Preconditioning saddle point problems arising from discretizations of partial differential equations

Part I

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Centre of Mathematics for Applications

Outline of the three lectures

- The continuous problems
  1. Iterative methods for pde problems
  2. Preconditioning pde problems
  3. Preconditioning parameter dependent problems

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  2. Preconditioning in $H(\mathrm{div})$ and $H(curl)$
  3. A general approach to the preconditioning of finite element systems
  4. Various examples

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  4. Various examples
- Mixed methods for elasticity and finite element exterior calculus

Krylov space methods

Assume that $\mathcal{A} : X \mapsto X$ is a symmetric isomorphism on a Hilbert space X, i.e.

$$\mathcal{A}, \mathcal{A}^{-1} \in \mathcal{L}(X, X),$$

and consider a linear system of the form

$$\mathcal{A}x = f.$$

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and consider a linear system of the form

$$\mathcal{A}x = f.$$

If the operator $\mathcal{A}$ is positive definite then we can approximate x by the *conjugate gradient method (CG)*, i.e.,

$$\langle \mathcal{A}x_m, v \rangle = \langle f, v \rangle, \quad v \in K_m(f),$$

where the Krylov space K_m is given by

$$K_m = \mathrm{span}\{\, f, \mathcal{A}f, \dots, \mathcal{A}^{m-1}f \,\}.$$

Convergence og CG

We obtain

$$\|x - x_m\|_{\mathcal{A}} = \inf_{v \in K_m} \|x - v\|_{\mathcal{A}},$$

where $\|v\|_{\mathcal{A}}^2 = \langle \mathcal{A}v, v \rangle$.

Theorem The CG-method converges in the enegy norm $|| \cdot ||_{\mathcal{A}}$, with a rate which can be bounded by $\kappa(\mathcal{A}) = ||\mathcal{A}|| \cdot ||\mathcal{A}^{-1}||$.

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Observe that X is allowed to be of infinite dimension, as long as the operator $\mathcal{A}$ can be evaluated.

Indefinite problems

If the operator $\mathcal{A}$ is indefinite, but still a symmetric isomorphism mapping X to itself, then we can use the *minimum residual method* (MINRES), i.e.,

$$\langle \mathcal{A}x_m, \mathcal{A}v \rangle = \langle f, \mathcal{A}v \rangle, \quad v \in K_m(f).$$

and

$$\|\mathcal{A}(x - x_m)\| = \inf_{v \in K_m} \|\mathcal{A}(x - v)\|,$$

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Short recurrence relations!
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Theorem A Krylov space method like MINRES or CGN converges in $||\cdot||_X$, with a rate which can be bounded by $\kappa(\mathcal{A}) = ||\mathcal{A}||_{\mathcal{L}(X,X)} \cdot ||\mathcal{A}^{-1}||_{\mathcal{L}(X,X)}$. ($\dim X = \infty$ allowed).

Example 1, Integral equation

Fredholm equation of the second kind:

$$\mathcal{A}u(x) := u(x) + \int_{\Omega} k(x,y)u(y)\,dy = f(x),$$

where we assume that the kernel k is continuous and symmetric, and that the operator $\mathcal{A} : X \to X$ is one–one, where $X = L^2(\Omega)$.

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In fact, since the operator $\mathcal{A}$ has the form "identity + compact" the convergence is superlinear (W, SINUM 1980).

Example 2, The Laplace operator

Let

$$H_0^1(\Omega) = X \subset L^2(\Omega) \subset X^* = H^{-1}(\Omega),$$

and define the negative Laplacian $\mathcal{A} : X \to X^*$ by

$$\langle \mathcal{A}u, v\rangle = \int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \, dx, \quad u, v \in X.$$

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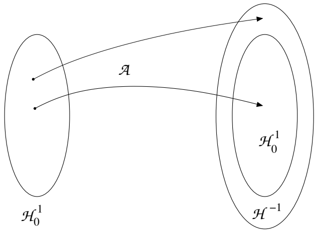

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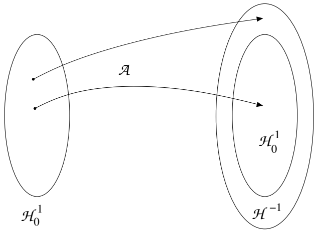

Conclusion: Krylov space methods are not well–defined for the operator $\mathcal{A}$.

PDE–problems

Consider the system

$$\mathcal{A}x = f,$$

where typically, $\mathcal{A}$ is an unbounded operator on X, or alternatively $\mathcal{A} \in \mathcal{L}(X, X^*)$ with X strictly contained in X^*. So the Krylov space methods are not well defined.

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We have:

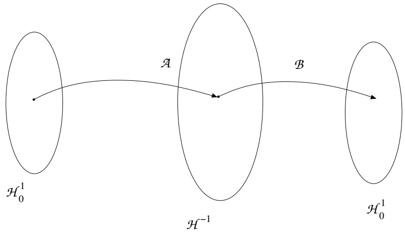
$$X \xrightarrow{\mathcal{A}} X^* \supset X$$

An isomorphism $\mathcal{B} : X^* \mapsto X$, i.e., a Riez operator, is called a *preconditioner for* $\mathcal{A}$. Then

$$\mathcal{B}\mathcal{A} : \quad X \xrightarrow{\mathcal{A}} X^* \xrightarrow{\mathcal{B}} X$$

is an isomorphism mapping X to itself.

Preconditioned Krylov space method



Symmetry: Assume that $\mathcal{A} : X \to X^*$ is symmetric in the sense that

$$\langle \mathcal{A}x, y \rangle = \langle \mathcal{A}y, x \rangle, \quad x, y \in X$$

and $\mathcal{B} : X^* \to X$ is symmetric and positive definite in the sense that $\langle \mathcal{B}\cdot, \cdot \rangle$ is an inner product on X^*. Here $\langle \cdot, \cdot \rangle$ is the duality pairing (L^2–symmetry).

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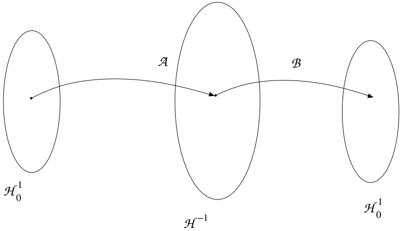

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As a consequence: $\langle \mathcal{B}^{-1} \cdot, \cdot \rangle$ is an inner product on X, and $\mathcal{B}\mathcal{A} : X \to X$ is symmetric in this inner product.

Preconditioned system

Conclusion. The preconditioned system

$$\mathcal{B}\mathcal{A}x = \mathcal{B}f$$

can be solved by a Krylov space method with a convergence rate bounded by $\kappa(\mathcal{B}\mathcal{A}) = ||\mathcal{B}\mathcal{A}||_{\mathcal{L}(X,X)}||(\mathcal{B}\mathcal{A})^{-1}||_{\mathcal{L}(X,X)}$.

Abstract saddle point problem

Abstract variational problem:

$$\min_{v \in V} E(v) \equiv \frac{1}{2} a(v,v) - F(v) \quad \text{subject to } b(v,q) = G(q), \quad q \in Q,$$

where

- V and Q are Hilbert spaces
- $F : V \to \mathbb{R}$ and $G : Q \to \mathbb{R}$ are bounded linear functionals
- $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ are bilinear and bounded
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Associated saddle point system:
Find $(u, p) \in V \times Q$ such that

$$\begin{array}{llll} a(u, v) & + b(v, p) & = F(v) & \qquad v \in V \\ b(u, q) & & = G(q) & \qquad q \in Q. \end{array}$$

Brezzi conditions

$$\begin{array}{llll} a(u,v) & +b(v,p) & = F(v) & \qquad v \in V \\ b(u,q) & & = G(q) & \qquad q \in Q. \end{array}$$

There are constants $c_1, c_2 > 0$ such that

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geq c_1,$$

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where $Z = \{ v \in V \mid b(v,q) = 0, \quad q \in Q \}$.

Then the coefficient operator $\mathcal{A} : V \times Q \to V^* \times Q^*$ is an isomorphism.

Block diagonal preconditioners

The coefficient operator

$$\mathcal{A} = \begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix} : V \times Q \to V^* \times Q^*$$

is an isomorphism, where $A : V \to V^*$ and $B : V \to Q^*$.

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Preconditioner:

$$\mathcal{B} = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$$

where $M : V^* \to V$ and $N : Q^* \to Q$ are positive definite isomorphisms.

So block diagonal preconditioners are natural, cf. Rusten–W (1992) and Silvester and Wathen (1993–1994)

Stokes problem

$$
\begin{aligned}
-\Delta u - \operatorname{grad} p &= f \quad \text{in } \Omega, \\
\operatorname{div} u &= g \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
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Weak formulation: Find $(u, p) \in (H_0^1)^n \times L_0^2$ such that

$$\begin{aligned} \langle \operatorname{grad} u, \operatorname{grad} v\rangle \quad &+ \langle p, \operatorname{div} v\rangle &= \langle f, v\rangle \qquad & v \in (H_0^1)^n \\ \langle \operatorname{div} u, q\rangle \quad & &= \langle g, q\rangle \qquad & q \in L_0^2. \end{aligned}$$

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Note that the verification of the inf–sup condition

$$\sup_{v \in (H_0^1)^n} \frac{\langle q, \operatorname{div} v \rangle}{\|v\|_1} \geq c\|q\|_0 \quad q \in L_0^2,$$

where $c > 0$, is nontrivial (Nečas 1967).

Preconditioning Stokes problem

Coefficient operator:

$$\mathcal{A} = \begin{pmatrix} -\Delta & -\operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix} : (H_0^1)^n \times L_0^2 \to (H^{-1})^n \times L_0^2.$$

The operator

$$\mathcal{B} = \begin{pmatrix} (-\Delta)^{-1} & 0 \\ 0 & I \end{pmatrix} : (H^{-1})^n \times L_0^2 \to (H_0^1)^n \times L_0^2$$

is the canonical preconditioner, where the positive definite operator $(-\Delta)^{-1}$ can be repalaced by a spectral equivalent operator.

Mixed formulation of the Poisson's equation

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\begin{aligned}
u - \operatorname{grad} p &= f \quad \text{in } \Omega,\\
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The problem is formally equivalent to:

$$-\Delta p = \operatorname{div} f - g \quad \text{in } \Omega, \quad \partial p/\partial n = f \cdot n \quad \text{on } \partial\Omega.$$

Weak formulation of the mixed system:
Find $(u, p) \in H_0(\operatorname{div}) \times L^2_0$ such that

$$\begin{aligned} \langle u, v \rangle \quad &+ \langle p, \operatorname{div} v \rangle &= \langle f, v \rangle \qquad & v \in H_0(\operatorname{div}), \\ \langle \operatorname{div} u, q \rangle \quad & &= \langle g, q \rangle \qquad & q \in L^2_0. \end{aligned}$$

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Note, the second Brezzi condition should be verified:

$$\|v\|_0^2 \geq c\,\|v\|_{\operatorname{div}}^2 \equiv c\,(\|v\|_0^2 + \|\operatorname{div} v\|^2) \quad v \in Z,$$

where $Z = \{v \in H_0(\operatorname{div}) \,|\, \langle \operatorname{div} v, q\rangle = 0,\ q \in L^2_0\}$. This is trivial in the continuous case, but not in the discrete case.

The preconditioner

Coefficient operator:

$$\mathcal{A} = \begin{pmatrix} I & -\operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix} : H_0(\operatorname{div}) \times L_0^2 \to H_0(\operatorname{div})^* \times L_0^2,$$

where $H_0(\operatorname{div})^*$ can be identified with

$$H^{-1}(\operatorname{curl}) = \{ f \in H^{-1} \mid \operatorname{curl} f \in H^{-1} \}.$$

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Preconditioner:

$$\mathcal{B} = \mathcal{B}_1 = \begin{pmatrix} (I - \operatorname{grad} \operatorname{div})^{-1} & 0 \\ 0 & I \end{pmatrix}$$

Here the operator $I - \operatorname{grad} \operatorname{div}$ is the operator corresponding to the $H_0(\operatorname{div})$–inner product

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Note that this operator is special in the sense that it acts as a second order elliptic operator on gradient–fields, but degenerates to the identity on curl–fields.

Alternative formulation of the Poisson's equation

Formulation 1:
Find $(u, p) \in H_0(\mathrm{div}) \times L^2_0$ such that

$$\begin{aligned} \langle u, v\rangle \quad &+\langle p, \mathrm{div}\, v\rangle &= \langle f, v\rangle \qquad & v \in H_0(\mathrm{div}),\\ \langle \mathrm{div}\, u, q\rangle \quad & &= \langle g, q\rangle \qquad & q \in L^2_0. \end{aligned}$$

Formulation 2:
Find $(u, p) \in (L^2)^n \times H^1 \cap L^2_0$ such that

$$\begin{aligned} \langle u, v\rangle \quad &-\langle \mathrm{grad}\, p, v\rangle &= \langle f, v\rangle \qquad & v \in (L^2)^n,\\ -\langle u, \mathrm{grad}\, q\rangle \quad & &= \langle g, q\rangle \qquad & q \in H^1 \cap L^2_0. \end{aligned}$$

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which in fact is equivalent to:

$$\langle \operatorname{grad} p, \operatorname{grad} q\rangle = \\ - \langle f, \operatorname{grad} q\rangle - \langle g, q\rangle \quad \forall q \in H^1 \cap L_0^2.$$

Alternative preconditioners

The coefficient operator

$$\mathcal{A} = \begin{pmatrix} I & -\operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix}.$$

is well defined on two different spaces. Either

$$X = H_0(\operatorname{div}) \times L_0^2 \quad \text{or} \quad Y = (L^2)^n \times (H^1 \cap L_0^2),$$

and $\mathcal{A}$ maps these spaces isomorphically into their dual spaces $X^* \supset L^2$ or $Y^* \supset L^2$.

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This leads to two corresponding preconditioners:

$$\mathcal{B}_1 = \begin{pmatrix} (I - \operatorname{grad}\operatorname{div})^{-1} & 0 \\ 0 & I \end{pmatrix} \quad \text{or} \quad \mathcal{B}_2 = \begin{pmatrix} I & 0 \\ 0 & (-\Delta)^{-1} \end{pmatrix},$$

where the operators $(I - \operatorname{grad}\operatorname{div})^{-1}$ and $(-\Delta)^{-1}$ can be replaced by spectral equivalent operators.

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where the operators $(I - \operatorname{grad}\operatorname{div})^{-1}$ and $(-\Delta)^{-1}$ can be replaced by spectral equivalent operators. Note that the preconditioners B_1 and B_2 are *not similar*.

One operator, two nonsimilar preconditioners

Let $X = \ell_2(\mathbb{R}^2)$ and define an unbounded block diagonal operator $\mathcal{A}$ on X by

$$\mathcal{A} = \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \end{pmatrix}$$

where each A_j is a 2×2 matrix of the form

$$A_j = \begin{pmatrix} 1 & a_j \\ a_j & 0 \end{pmatrix}$$

Here a_j are positive real numbers such that $1 = a_1 \leq a_2 \leq \ldots$ and $\lim_{j\to\infty} a_j = \infty$.

One operator, two nonsimilar preconditioners

Let $X = \ell_2(\mathbb{R}^2)$ and define an unbounded block diagonal operator $\mathcal{A}$ on X by

$$\mathcal{A} = \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \end{pmatrix}$$

where each A_j is a 2×2 matrix of the form

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Here a_j are positive real numbers such that $1 = a_1 \leq a_2 \leq \ldots$ and $\lim_{j \to \infty} a_j = \infty$.

The eigenvalues of $\mathcal{A}$ are

$$\lambda_j = \frac{1 \pm \sqrt{1 + 4a_j^2}}{2} \longrightarrow \pm\infty \quad \text{as } a_j \to \infty.$$

Preconditioning

Let $\mathcal{B} = \text{diag}(B_j)$ where

$$B_j = \begin{pmatrix} \beta_j & 0 \\ 0 & \gamma_j \end{pmatrix}, \quad \beta_j, \gamma_j > 0.$$

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So study:

$$BA = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & a \\ a & 0 \end{pmatrix} = \begin{pmatrix} \beta & \beta a \\ \gamma a & 0 \end{pmatrix}$$

for $a \in [1, \infty)$.

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for $a \in [1, \infty)$.

Alternative 1: $\beta = \frac{1}{1+a^2}$ and $\gamma = 1$ gives

$$BA = \begin{pmatrix} \frac{1}{1+a^2} & \frac{a}{1+a^2} \\ a & 0 \end{pmatrix} \quad \text{with eigenvalues } \lambda(a) \to \pm 1 \quad \text{as } a \to \infty.$$

Alternative preconditioner

Take $\beta = 1$ and $\gamma = \frac{1}{1+a^2}$, which gives

$$B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+a^2} \end{pmatrix} \quad \text{and } BA = \begin{pmatrix} 1 & a \\ \frac{a}{1+a^2} & 0 \end{pmatrix}.$$

Hence, the eigenvalues satisfy

$$\lambda(a) = \frac{1 \pm \sqrt{1 + \frac{4a^2}{1+a^2}}}{2} \longrightarrow \frac{1 \pm \sqrt{5}}{2} \quad \text{as } a \to \infty.$$

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Extension of this example: If each block A_j is a 3×3 matrix on the form

$$A_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a_j \\ 0 & a_j & 0 \end{pmatrix}$$

then we will also include the effect that constraint $(0\, a_j)$ has a large kernel.

Robustness with respect to parameters

We will consider two model problems which depend on small parameters, namely

- The time dependent Stokes problem
- The Reissner–Mindlin plate model

The goal is to produce preconditioners which results in iterations which converge uniformly with respect to the parameters, i.e. $\kappa(\mathcal{B}\mathcal{A}) = ||\mathcal{B}\mathcal{A}||_{\mathcal{L}(X,X)}||(\mathcal{B}\mathcal{A})^{-1}||_{\mathcal{L}(X,X)}$ should be uniformly bounded.

Example, reaction–diffusion problem

Consider the problem

$$-\epsilon^2 \Delta u + u = f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

From energy estimates we see that a natural norm for the solution u is

$$\|u\|_{L^2 \cap \epsilon H_0^1} \equiv (\|u\|_0^2 + \epsilon^2 \|\operatorname{grad} u\|_0^2)^{1/2}.$$

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What is the correct norm to put on f such that we get a sharp bound of the form

$$\|u\|_{L^2 \cap \epsilon H_0^1} \leq c\|f\|_?$$

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$$\|u\|_{L^2 \cap \epsilon H_0^1} \le c\|f\|_?$$

where c is independent of ϵ.

Note that we formally have

$$u = (I - \epsilon^2 \Delta)^{-1} f \quad \text{and that} \quad \|u\|^2_{L^2 \cap \epsilon H_0^1} = \langle (I - \epsilon^2 \Delta) u, u \rangle.$$

The norm on f

We formally have

$$\begin{aligned}
\|u\|^2_{L^2\cap\epsilon H^1_0} &= \langle (I-\epsilon^2\Delta)u, u\rangle \\
&= \langle (I-\epsilon^2\Delta)^{-1}f, f\rangle \\
&= \langle (I-\epsilon^2\Delta)^{-1}f, (I-\epsilon^2\Delta)^{-1}f\rangle \\
&\quad + \epsilon^2\langle (-\Delta)(I-\epsilon^2\Delta)^{-1}f, (I-\epsilon^2\Delta)^{-1}f\rangle \\
&= \|f_0\|_0^2 + \epsilon^{-2}\langle (-\Delta)^{-1}f_1, f_1\rangle \\
&= \|f_0\|_0^2 + \epsilon^{-2}\,\|f_1\|_{-1}^2,
\end{aligned}$$

where

$$f_0 = (I-\epsilon^2\Delta)^{-1}f \quad \text{and} \quad f_1 = -\epsilon^2\Delta(I-\epsilon^2\Delta)^{-1}f.$$

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Note that $f_0 + f_1 = f$.

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$$

Note that $f_0 + f_1 = f$. In fact,

$$
\langle (I-\epsilon^2\,\Delta)^{-1} f, f\rangle = \inf_{\substack{f=f_0+f_1 \\ f_0\in L^2,\, f_1\in H^{-1}}} \|f_0\|_0^2 + \epsilon^{-2}\,\|f_1\|_{-1}^2.
$$

Intersection and sum of Hilbert spaces.

If X and Y are Hilbert spaces, then $X \cap Y$ and $X + Y$ are themselves Hilbert spaces with the norms

$$\|z\|_{X\cap Y} = (\|z\|_X^2 + \|z\|_Y^2)^{1/2}$$

and

$$\|z\|_{X+Y} = \inf_{\substack{z=x+y \\ x\in X,\, y\in Y}} (\|x\|_X^2 + \|y\|_Y^2)^{1/2}.$$

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Furthermore, if $X \cap Y$ is dense in both X and Y then $(X \cap Y)^* = X^* + Y^*$ and $(X + Y)^* = X^* \cap Y^*$.

Examples

Example 1: The norm of the space $L^2 \cap \epsilon \cdot H^1$, $\epsilon > 0$, is given by

$$\|f\|_{L^2 \cap \epsilon \cdot H^1}^2 = \|f\|_0^2 + \epsilon^2 \, \|f\|_1^2.$$

The space $L^2 \cap \epsilon \cdot H^1$ is equal to H^1 as a set. However, the norm approaches the L^2–norm as ϵ tends to zero.

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The space $L^2 \cap \epsilon \cdot H^1$ is equal to H^1 as a set. However, the norm approaches the L^2–norm as ϵ tends to zero.

Example 2: Consider the space $H^1 + \epsilon^{-1} L^2$ with norm given by

$$\|f\|^2_{H^1+\epsilon^{-1} L^2} = \inf_{\substack{f=f_1+f_2 \\ f_1\in H^1,\, f_0\in L^2}} \|f_1\|^2_1 + \epsilon^{-2} \|f_0\|^2_0.$$

This space is equal to L^2 as a set, but the corresponding norm appraches the H^1–norm as ϵ tends to zero.

Example 3: Let $X = X_\epsilon = L^2 \cap \epsilon \cdot H_0^1$. If the duality pairing is an extension of the L^2 inner product, then $X^* = L^2 + \epsilon^{-1} H^{-1}$.

Furthermore,

$$\|f\|_{X^*}^2 \sim \langle (I - \epsilon^2 \Delta)^{-1} f, f \rangle$$

The time dependent Stokes problem

$$
\begin{aligned}
u_t - \Delta u - \operatorname{grad} p &= f \quad \text{in } \Omega \times \mathbb{R}^+, \\
\operatorname{div} u &= g \quad \text{in } \Omega \times \mathbb{R}^+, \\
u &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\
u &= 0 \quad \text{on } \Omega \times \{t = 0\}.
\end{aligned}
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Various implicit time stepping schemes leads to systems of the form

$$\begin{aligned} (I - \epsilon^2 \Delta) u - \operatorname{grad} p &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= g \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\epsilon > 0$ is the square root of the time step.
Coefficient operator:

$$\mathcal{A}_\epsilon = \begin{pmatrix} I - \epsilon^2 \Delta & -\operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix}$$

A preconditioner for time dependent Stokes

Recall that for $\epsilon = 0$ the operator

$$\mathcal{A}_0 = \begin{pmatrix} I & -\operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix}$$

is bounded from $H_0(\operatorname{div}) \times L_0^2$ into $H^{-1}(\operatorname{curl}) \times L_0^2$.

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In fact, since

$$\sup_{v \in H^1_0} \frac{\langle q, \operatorname{div} v \rangle}{\|v\|_{H(\operatorname{div}) \cap \epsilon \cdot H^1}} \geq \sup_{v \in H^1_0} \frac{\langle q, \operatorname{div} v \rangle}{2\|v\|_1} \geq c_0 \|q\|_0$$

we can conclude that

$$\mathcal{A}_\epsilon = \begin{pmatrix} I - \epsilon^2 \Delta & -\operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix}$$

is an isomorphism mapping $(H_0(\operatorname{div}) \cap \epsilon\, H^1_0) \times L^2_0$ onto its L^2 dual $(H_0(\operatorname{div})^* + \epsilon^{-1} H^{-1}) \times L^2_0$, and with *operator norm independent of* ϵ.

We conclude that a uniform preconditioner therefore should be a positive definte mapping $\mathcal{B}_\epsilon$ mapping $(H_0(\mathrm{div})^* + \epsilon^{-1} H^{-1}) \times L_0^2$ isomorphically onto $(H_0(\mathrm{div}) \cap \epsilon H_0^1) \times L_0^2$.

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Hence, we can choose

$$\mathcal{B}_\epsilon = \begin{pmatrix} (I - \operatorname{grad}\operatorname{div} - \epsilon^2 \Delta)^{-1} & 0 \\ 0 & I \end{pmatrix},$$

or any spectrally equivalent operator.

A preliminary on discretizations

If we want to be able to reach a similar conclusion for the corresponding preconditioner for a finite element discretization of the time dependent Stokes problem, it will be necessary that the finite element method is uniformly stable in the appropriate ϵ dependent norm. However, this is usually not the case. In fact, *most common stable Stokes elements are not stable for the mixed Poisson problem.* These elements fail to satisfy the coercivity condition (*the second Brezzi condition*)

$$\|v\|_0^2 + \epsilon^2 \|\operatorname{grad} v\|_0^2 \geq c_0 \|v\|_{\mathrm{div}}^2 \quad v \in Z_h,$$

where $Z_h \subset V_h$ denotes the set of discrete weakly divergence free vector fields.

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Therefore, the discussion above will not carry over to the corresponding discrete systems.
Furthermore, I do not know any study of the construction of uniform preconditioners for the operator $I - \operatorname{grad}\operatorname{div} - \epsilon^2 \Delta$.

Alternative function space

Recall that the operator $\mathcal{A}_0$ also is bounded from $(L^2)^n \times (H^1 \cap L^2_0)$. Hence, it seems that

$$\mathcal{A}_\epsilon = \begin{pmatrix} I - \epsilon^2 \Delta & -\,\mathrm{grad} \\ \mathrm{div} & 0 \end{pmatrix}$$

can be defined on a larger space, where the velocity is allowed to be in $(L^2 \cap \epsilon\, H^1_0)^n$.

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can be defined on a larger space, where the velocity is allowed to be in $(L^2 \cap \epsilon\, H^1_0)^n$. Then the proper norm for the scalar variable should be

$$\sup_{v \in (H^1_0)^n} \frac{\langle q, \operatorname{div} v \rangle}{\|v\|_{L^2 \cap \epsilon\, H^1}} \sim \|\operatorname{grad} q\|_{L^2 + \epsilon^{-1} H^{-1}} \sim \|q\|_{H^1 + \epsilon^{-1} L^2_0}.$$

Note that for each fixed $\epsilon > 0$ this is equivalent to the L^2 norm, but it approaches the H^1 norm as ϵ tends to zero.

Alternative uniform preconditioner

The solution space indicated above will lead to a uniform preconditioner of the form

$$\mathcal{B}_\epsilon = \begin{pmatrix} (I - \epsilon^2 \Delta)^{-1} & 0 \\ 0 & (-\Delta)^{-1} + \epsilon^2 I \end{pmatrix}$$

In fact, this result is correct, at least in the case of convex domains. Discrete preconditioners along these lines have been suggested by Cahouert and Chabard (1988), Bramble and Pasciak (1997), Turek (1999), Elman (2002), Elman, Silvester and Wathen (2002), Loghin and Wathen (2002), Mardal and W (2004), Olshanskii, Peters and Reusken (2005).

A technical difficulty

The problem in the nonconvex case is to establish the proper uniform inf–sup condition given by

$$\sup_{v\in(H^1_0)^n} \frac{\langle q, \operatorname{div} v\rangle}{\|v\|_{L^2\cap\epsilon H^1}} \geq c_0\|q\|_{H^1+\epsilon^{-1}L^2} \quad q \in L^2_0,$$

or equivalently,

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Note that for $\epsilon = 0$ this final condition reduces to

$$\|q\|_1 \leq c_0^{-1} \| \operatorname{grad} q\|_0 \quad q \in H^1 \cap L^2_0,$$

which is just the Poincaré inequality, while for ϵ bounded away from zero it is equivalent to

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However, this is equivalent to the proper inf–sup condition for the Stokes problem.

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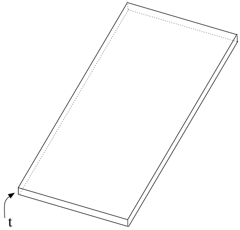
However, this is equivalent to the proper inf–sup condition for the Stokes problem. So both the extreme cases hold, but ....

Reissner-Mindlin plate



We study a clamped plate exposed to a trasnsverse load.

The Reissner-Mindlin plate model

based on: Arnold, Falk and Winther (Math. Mod. Num. Anal., 1997)

$$\min_{(\phi,u)\in(H_0^1)^2\times H_0^1} E(\phi,u),$$

where

$$E(\phi,u)=\frac{1}{2}\int_\Omega\{(\mathcal{C}\mathcal{E}\phi):(\mathcal{E}\phi)+t^{-2}|\phi-\operatorname{grad}u|^2\}dx \\ -\int_\Omega gu\,dx.$$

Here $\mathcal{E}\phi$ denotes the symmetric part of $\operatorname{grad}\phi$. The thickness parameter $t\in[0,1]$.

Equilibrium system

$$
\begin{aligned}
-\operatorname{div}\mathcal{C}\mathcal{E}\phi + t^{-2}(\phi - \operatorname{grad} u) &= 0,\\
t^{-2}(-\Delta u + \operatorname{div}\phi) &= g,\\
\phi|_{\partial\Omega} = 0, \quad u|_{\partial\Omega} &= 0.
\end{aligned}
$$

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\end{aligned}
$$

Alternatively, with $\zeta = -t^{-2}(\phi - \operatorname{grad} u)$:

$$
\begin{aligned}
-\operatorname{div} \mathcal{C}\mathcal{E}\phi - \zeta &= 0, \\
-\operatorname{div} \zeta &= g, \\
-\phi + \operatorname{grad} u - t^2\zeta &= 0.
\end{aligned}
$$

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-\operatorname{div} \zeta &= g, \\
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\end{aligned}
$$

or in weak form: Find $(\phi, u, \zeta) \in (H^1_0)^2 \times H^1_0 \times (L^2)^2$ such that

$$
\begin{aligned}
\langle \mathcal{C}\mathcal{E}\phi, \mathcal{E}\psi \rangle - \langle \zeta, \psi - \operatorname{grad} v \rangle &= \langle g, v \rangle \quad (\psi, v) \in (H^1_0)^2 \times H^1_0 \\
-\langle \phi - \operatorname{grad} u, \eta \rangle - t^2 \langle \zeta, \eta \rangle &= 0 \quad \eta \in (L^2)^2
\end{aligned}
$$

Saddle point structure

The system can formally be written in the form

$$\mathcal{A}_t \begin{pmatrix} \phi \\ u \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ g \\ 0 \end{pmatrix},$$

where the coefficient operator, $\mathcal{A}_t$, is given by

$$\mathcal{A}_t = \begin{pmatrix} -\operatorname{div}\mathcal{C}\mathcal{E} & 0 & -I \\ 0 & 0 & -\operatorname{div} \\ -I & \operatorname{grad} & -t^2 I \end{pmatrix}.$$

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The two first variables will always be taken in $(H_0^1)^2 \times H_0^1$. Note that for $t = 0$ the norm on the multiplier space should be

$$\sup_{(\psi,v)\in(H_0^1)^2\times H_0^1} \frac{\langle \eta, \psi - \operatorname{grad} v\rangle}{\|\psi\|_1 + \|v\|_1} \sim \|\eta\|_{H^{-1}(\operatorname{div})},$$

where

$$H^{-1}(\operatorname{div}) = \{\eta \in (H^{-1})^2 \mid \operatorname{div}\eta \in H^{-1}\}.$$

Biharmonic problem (t=0)

In particular, for $t = 0$ the operator $\mathcal{A}_0$ is well defined from $(H_0^1)^2 \times H_0^1 \times H^{-1}(\mathrm{div})$ into its L^2 dual given as $(H^{-1})^2 \times H^{-1} \times (H_0(\mathrm{rot})$. Here $\mathrm{rot}\,\eta = (\eta_1)_y - (\eta_2)_x$.

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The canonical operator mapping $H^{-1}(\mathrm{div})^* = H_0(\mathrm{rot})$ onto $H^{-1}(\mathrm{div})$ is the differential operator

$$D_0 = I + \mathrm{curl}\,\mathrm{rot}\,.$$

Therefore, the canonical preconditioner for the biharmonic system is of the form

$$\mathcal{B}_0 = \begin{pmatrix} (-\Delta)^{-1} & 0 & 0 \\ 0 & (-\Delta)^{-1} & 0 \\ 0 & 0 & D_0 \end{pmatrix}.$$

Here curl is the formal adjoint of rot, i.e. $\mathrm{curl}\, v = (-v_y, v_x)$.

Function spaces, general case

For $t > 0$ the variational problem takes the form

$$\langle \mathcal{C}\mathcal{E}\phi, \mathcal{E}\psi\rangle - \langle \zeta, \psi - \operatorname{grad} v\rangle - \langle \phi - \operatorname{grad} u, \eta, \rangle \\ + t^2\langle \zeta, \eta\rangle = \langle g, v\rangle.$$

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Let

$$X_t = (H_0^1)^2 \times H_0^1 \times (H^{-1}(\operatorname{div}) \cap t \cdot (L^2)^2)$$

and X_t^* its L^2–dual

$$X_t^* = (H^{-1})^2 \times H^{-1} \times (H_0(\operatorname{rot}) + t^{-1} \cdot (L^2)^2)$$

Theorem. $\mathcal{A}_t \in \mathcal{L}(X_t, X_t^*)$ and $\mathcal{A}_t^{-1} \in \mathcal{L}(X_t^*, X_t)$ with associated operator norms independent of t.

Preconditioner; The general case

The canonical preconditioner for the operator

$$\mathcal{A}_t = \begin{pmatrix} -\operatorname{div}\mathcal{C}\mathcal{E} & 0 & -I \\ 0 & 0 & -\operatorname{div} \\ -I & \operatorname{grad} & -t^2 I \end{pmatrix}.$$

is therefore of the form

$$\mathcal{B}_t = \begin{pmatrix} (-\Delta)^{-1} & 0 & 0 \\ 0 & (-\Delta)^{-1} & 0 \\ 0 & 0 & D_t \end{pmatrix},$$

where

$$D_t = I + (1 - t^2)\operatorname{curl}(I - t^2\Delta)^{-1}\operatorname{rot}.$$