Preconditioning saddle point problems arising from discretizations of partial differential equations

Part II, Discrete problems

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Outline of Part II

1. Finite elements and representation of operators
2. Preconditioning in $H(\mathrm{div})$ and $H(curl)$
3. A general approach to preconditioning finite element systems
4. Mixed Poisson problem, Stokes problem and Reissner–Mindlin problem revisited
5. Other examples

Remarks on represetation of finite element operators

Let $a = a(u, v)$ be the bilinear form, i.e.,

$$a(u, v) = \int_\Omega uv + \operatorname{grad} u \cdot \operatorname{grad} v \, dx.$$

Consider a finite element method for the corresponding problem

$$-\Delta u + u = f, \quad \partial_n u = 0,$$

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Frequently, in the finite element literature this problem is written $A_h u_h = f_h$, where the operator $A_h : V_h \to V_h$ is defined by

$$\langle A_h u, v \rangle = a(u, v), \quad u, v \in V_h.$$

This operator depends on the finite element space V_h, but is independent of any basis of the this space.

Stiffness matrix

Recall that the corresponding stiffness matrix,

$$\mathbb{A}_h : \mathbb{R}^n \to \mathbb{R}^n,$$

is given by $(\mathbb{A}_h)_{i,j} = a(\phi_j, \phi_i)$, where $\{\phi_j\}_{j=1}^n$ is a basis for the finite element space V_h.

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We define two "representation operators" $\pi_h, \mu_h : V_h \to \mathbb{R}^n$ by

$$v = \sum_j (\pi_h v)_j \phi_j, \quad (\mu_h v)_j = \langle v, \phi_j \rangle.$$

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Note that

$$(\pi_h u) \cdot (\mu_h v) = \sum_j (\pi_h u)_j \langle v, \phi_j \rangle = \langle u, v \rangle$$

so $\pi_h^{-1} = \mu_h^*$ and $\mu_h^{-1} = \pi_h^*$.

From A_h to $\mathbb{A}_h$

For any $v \in V_h$ we have

$$(\mu_h(A_h v))_i = \langle A_h v, \phi_i \rangle = a(v, \phi_i) = \sum_j (\pi_h v)_j a(\phi_j, \phi_i) = (\mathbb{A}_h \pi_h v)_i.$$

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Hence, $\mu_h A_h = \mathbb{A}_h \pi_h$, or the diagram

$$\begin{array}{ccc} V_h & \xrightarrow{A_h} & V_h \\ \Big\downarrow \pi_h & & \Big\downarrow \mu_h \\ \mathbb{R}^n & \xrightarrow{\mathbb{A}_h} & \mathbb{R}^n \end{array}$$

commutes. Alternatively, the sparse stiffness matrix is given by

$$\mathbb{A}_h = \mu_h A_h \pi_h^{-1} = \mu_h A_h \mu_h^*.$$

Squaring the operators

Note that the matrix

$$\mathbb{A}_h^2 = \mu_h A_h \pi_h^{-1} \mu_h A_h \pi_h^{-1} = \mu_h A_h B_h A_h \pi_h \neq \mu_h A_h^2 \pi_h^{-1},$$

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where $B_h = \pi_h^{-1} \mu_h$. Hence, $\mathbb{A}_h^2$ is not a sparse representation of A_h^2. Instead the matrix

$$\mathbb{A}_h \pi_h \mu_h^{-1} \mathbb{A}_h = \mu_h A_h \pi_h^{-1} (\pi_h \mu_h^{-1}) \mu_h A_h \pi_h^{-1} = \mu_h A_h^2 \pi_h^{-1}$$

represents A_h^2, but the matrix $\pi_h \mu_h^{-1}$ (the inverse mass matrix) is not sparse.

Representation of finite element operators

Conclusion: The stiffness matrix $\mathbb{A}_h$ is a sparse matrix representation of $\mu_h A_h \pi_h^{-1}$,
(while for example the matrix representation of $\pi_h A_h \mu_h^{-1}$ is not sparse.)

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When we do preconditioned iterative methods we usually rely on representations of the form the form $\mu_h A_h \pi_h^{-1}$ for the differential operator and $\pi_h B_h \mu_h^{-1}$ for the preconditioner, i.e.

$$B_h A_h : \quad \text{primal repr.} \xrightarrow{A_h} \text{dual repr.} \xrightarrow{B_h} \text{primal repr.}$$

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This should be compared to the corresponding diagram for the continuous problem

$$BA : \quad X \xrightarrow{A} X^* \xrightarrow{B} X$$

Preconditioning in $H(\mathrm{div})$ and $H(\mathrm{curl})$

The function spaces H^1, $H(\mathrm{curl})$, $H(\mathrm{div})$ and also L^2 appears naturally in weak formulations of various systems of partial differential equations. As a consequence, we will need preconditioners for the corresponding Riez operators, i.e., the for the operators

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corresponding to the inner products

$$\langle u, v\rangle + \langle \mathrm{grad}\, u, \mathrm{grad}\, v\rangle \qquad \langle u, v\rangle + \langle \mathrm{curl}\, u, \mathrm{curl}\, v\rangle \qquad \langle u, v\rangle + \langle \mathrm{div}\, u, \mathrm{div}\, v\rangle$$

and $\langle u, v\rangle$.

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and $\langle u, v\rangle$.

Note that the last inner product gives rise to the *mass matrix* $\mu_h \pi_h^{-1}$ which can be expensive to invert exactly on finite element spaces with continuity constraints.

Non strongly elliptic operators

Of course, there is a the large collection of litterature on how to construct efficient preconditioners for discrete versions of the operator

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and we will not give a detailed overview of such techniques here. However, the operators

$$I + \operatorname{curl}\operatorname{curl} \qquad \text{and} \qquad I - \operatorname{grad}\operatorname{div}$$

represents additional difficulties. These difficulties are basically caused by the fact that these operators are not *strongly elliptic*, since the operators curl and div has a large null–space.

Here we shall outline how to construct multigrid preconditioners for such operators.

The V–cycle multigrid algorithm

Let $X \subset Y$ be Hilbert spaces, the bilinear form $a : X \times X \to \mathbb{R}$ is an inner product on X, and $\langle \cdot, \cdot \rangle$ the inner product on Y. Furthermore, let

$$X_1 \subset X_2 \subset \cdots \subset X_J$$

be finite dimensional subspaces of X. The *finite element operator* $A_j : X_j \to X_j$ is then defined by

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Let $R_j : X_j \to X_j$ be Y symmetric and positive definite smoothing operators, and $B = B_J : X_J \to X_J$ the corresponding multigrid preconditioner obtained from the standard V–cycle algorithm with m smoothing steps.

Finally, let $P_j : X \to X_j$ be the projection (with respect to the inner product a).

The Braess–Hackbusch Theorem (1983)

Theorem: Suppose that the smoothers satisfy the conditions

$$a((I - R_j A_j)x, x) \geq 0 \quad x \in X_j,$$

and

$$\langle R_j^{-1} x, x \rangle \leq \alpha a(x, x) \quad x \in (I - P_{j-1}) X_j.$$

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Then

$$0 \leq a((I - BA)x, x) \leq \frac{\alpha}{\alpha + m} a(x, x), \quad x \in X_J$$

and, as a consequence, $\kappa(BA) \leq 1 + \frac{\alpha}{2m}$.

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The first condition states that the smoothers are properly bounded. This condition will automatically be satisfied for a multiplicative smoother, i.e., a Gauss–Seidel type smoother.

Construction of smoothers

The basic challenge is to construct smoothers which satisfy the second condition of the theorem, i.e.,

$$\langle R_j^{-1} x, x\rangle \le \alpha a(x,x) \quad x \in (I - P_{j-1})X_j$$

or with simplified notation

$$\langle R^{-1} x, x\rangle \le \alpha a(x,x) \quad x \in (I - P_0)X,$$

where $X_0 \subset X$ are the finite element spaces, replacing $X_{j-1} \subset X_j$.

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where $X_0 \subset X$ are the finite element spaces, replacing $X_{j-1} \subset X_j$.

Note that this condition states that the smoother satisfy a proper lower bound when restricted to "the high frequency components."

Domain decomposition smoothers

If the finite element space X can be decomposed as a sum of spaces (not necessarily direct) of the form

$$X = \sum_i X^i,$$

then we can define an additive (Jacobi) or a multiplicative (Gauss–Seidel) smoother with respect to this decomposition, based on solving "local" problems of the form

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Furthermore, the verification of the second condition is then a consequence of the *stable decomposition property*, i.e.,

$$\inf_{\substack{x^i \in X^i \\ x = \sum x^i}} \sum_i a(x^i, x^i) \leq \gamma a(x, x) \quad x \in (I - P_0)X$$

with α proportional to γ.

Preconditioning in H^1

Let the domain Ω be decomposed into a finite union of overlapping subdomains, $\Omega = \cup\Omega^i$, with a finite number of overlaps.
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Verification, $v \in (I - P_0)V_h$, $v = \sum_i \pi_h(v)_i \phi_i$. Take $v^i = \pi_h(v)\phi_i$:

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$\leq \gamma a(v, v)$.

The role of duality and improved estimates

The critical estimate, which will not hold for $H(\text{curl})$ or $H(\text{div})$, is the "high frequency estimate"

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Example: $H(\text{div} : \Omega)$, where $\Omega \subset \mathbb{R}^2$ is simply connected. So we have

$$a(u,v) = \langle u, v \rangle + \langle \operatorname{div} u, \operatorname{div} v \rangle$$

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such that $a(v,v) = \|v\|_0$ if $v = \mathrm{curl}\, \phi = (-\phi_y, \phi_x)^t$.
In this case the de–Rham complex takes the form:

$$H^1(\Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)$$

implying that $v \in H(\mathrm{div};\Omega)$ has an orthogonal Helmholtz decomposition of the form

$$H(\mathrm{div}) = \mathrm{curl}(H^1) \oplus Z^\perp, \quad \text{where} \quad Z^\perp = \ker(\mathrm{div})^\perp.$$

Discrete Helholtz decomposition

The proper "high frequency estimates" in the $H(\mathrm{div})$ case is derived from a proper discrete Helmholz decomposition. Assume we have a correspondig discrete complex

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\begin{array}{ccccccccc}
\mathbb{R} \hookrightarrow & H^1(\Omega) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & 0 \\
& \Big\downarrow \subset & & \Big\downarrow \subset & & \Big\downarrow \subset & & \\
\mathbb{R} \hookrightarrow & W_h & \xrightarrow{\mathrm{curl}} & V_h & \xrightarrow{\mathrm{div}} & Q_h & \longrightarrow & 0.
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leading to a discrete Helholtz decomposition:

$$V_h = \mathrm{curl}(H^1) \oplus Z_h^{\perp}, \quad \text{where} \quad Z_h^{\perp} \subset V_h,\ Z_h^{\perp} = \ker(\mathrm{div})^{\perp}.$$

Improved estimates in $H(\mathrm{div})$

The estimate

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which holds in the H^1 case should now be replaced by the following:
Assume $v \in (I - P_0)V_h$, with a corresponding decomposition

$$v = \operatorname{curl} w + z, \quad w \in W_h, z \in Z_h^{\perp}.$$

Then

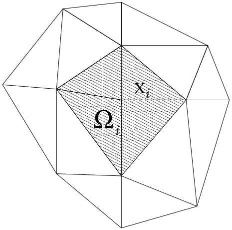
$$\|w\|_0 \leq c\, h_0 \|v\|_0 \quad \text{and} \quad \|z\|_0 \leq c\, h_0 \|v\|_{\mathrm{div}}.$$

Consequences for $H(\mathrm{div})$ smoothers

If $\Omega = \sum_i \Omega_i$ and we want to use a corresponding additive or multiplicative smoother derived from the decomposition

$$V_h = \sum_i V_h^i, \quad V_h^i = V_h(\Omega_i)$$

then the domains Ω_i have to be chosen so large that the space V_h^i itself admits a Helmholtz decomposition.

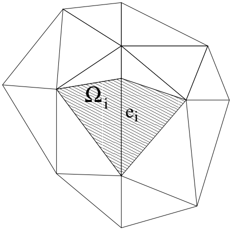


Smoothers of this form were proposed by Arnold, Falk, W (1997, 2000).

Explicit use of the Helholz decomposition

Alternatively, we can explicitly use a decomposition of the form

$$V_h = \sum_i V_h^i + \sum_j \operatorname{curl} W_h^j.$$



Smoothers of this form has been proposed by Vassilevski and Wang (1992), Hiptmair (1997, 1999), Hiptmair and Toselli (2000).

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then we also know the basic structure of

- the mesh independent preconditioner for the corresponding discrete problem.

Abstract variational problem, Babuska-Aziz 1972

This theory is in some sense more general than the Brezzi theory. On the other hand, the Brezzi theory is more useful for saddle point problems since we obtain conditions which are more easily checked. Let X be a Hilbert space and $a : X \times X \to \mathbb{R}$ a bounded and symmetric (but not necessary coercive) bilinear form satifying

$$\inf_{x \in X} \sup_{y \in X} \frac{a(x,y)}{\|x\|_X \|y\|_X} = \inf_{x \in X} \frac{|a(x,x)|}{\|x\|_X^2} \geq c_0 > 0.$$

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For a given $f \in X^*$ consider the variational problem: Find $x \in X$ such that

$$a(x,y) = \langle f, y \rangle \quad y \in X \quad \text{or equivalently} \quad \mathcal{A}x = f,$$

where $\mathcal{A} : X \to X^*$ is given by

$$\langle \mathcal{A}x, y \rangle = a(x,y) \quad x, y \in X.$$

Abstract variational problem

The linear system $\mathcal{A}x = f$ has a unique solution and

$$\|\mathcal{A}^{-1}\|_{\mathcal{L}(X^*,X)} \leq c_0^{-1} \quad \text{and} \quad \|\mathcal{A}\|_{\mathcal{L}(X,X^*)} \leq C_1,$$

where $C_1 > 0$ is the bound for a, i.e.,

$$a(x,y) \leq C_1 \|x\|_X \|y\|_X.$$

Example, Mixed Poisson problem

Find $(u, p) \in H_0(\text{div}) \times L_0^2$ such that

$$\begin{aligned} \langle u, v\rangle + \langle p, \operatorname{div} v\rangle &= \langle f, v\rangle && \forall v \in H_0(\text{div}), \\ \langle \operatorname{div} u, q\rangle &= \langle g, q\rangle && \forall q \in L_0^2. \end{aligned}$$

Hence, we have $X = H_0(\text{div}) \times L_0^2$ and

$$a(x, y) = \langle u, v\rangle + \langle p, \operatorname{div} v\rangle + \langle \operatorname{div} u, q\rangle,$$

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with $x = (u, p)$ and $y = (v, q)$. The inner product on X is given by

$$\langle x, y\rangle_X = \langle u, v\rangle + \langle \mathrm{div}\, u, \mathrm{div}\, v\rangle + \langle p, q\rangle.$$

From Brezzi to Babuska-Aziz

Consider an abstract saddle point problem of the form:
Find $(u, p) \in V \times Q$ such that

$$\begin{array}{llll} a_0(u, v) & +b(v, p) & = F(v) & \quad v \in V \\ b(u, q) & & = G(q) & \quad q \in Q. \end{array}$$

and let

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with $x = (u, p)$ and $y = (v, q)$.

If the Brezzi conditions holds for the bilinear forms a_0 and b, then the bilinear form $a = a(x, y)$ satisfies the Babuska-Aziz conditions.

The abstract preconditioner, continuous case

Define $\mathcal{B} : X^* \to X$ by

$$\langle \mathcal{B}f, y \rangle_X = \langle f, y \rangle \quad y \in X.$$

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Furthermore,

$$\|\mathcal{B}\mathcal{A}\|_{\mathcal{L}(X,X)} = \sup_{x\in X} \frac{|\langle \mathcal{B}\mathcal{A}x, x\rangle_X|}{\|x\|_X^2} = \sup_{x\in X} \frac{|a(x, x)|}{\|x\|_X^2} \le C_1$$

and

$$\|(\mathcal{B}\mathcal{A})^{-1}\|^{-1}_{\mathcal{L}(X,X)} = \inf_{x\in X} \frac{|\langle \mathcal{B}\mathcal{A}x, x\rangle_X|}{\|x\|_X^2} = \inf_{x\in X} \sup_{y\in Y} \frac{a(x, y)}{\|x\|_X\|y\|_X} \ge c_0 > 0.$$

Stable finite element discretization

Let $X_h \subset X$ and consider the corresponding discrete variational problem: Find $x_h \in X_h$ such that

$$a(x_h, y) = \langle f, y \rangle \quad y \in X_h \quad \text{or equivalently} \quad \mathcal{A}_h x_h = f_h,$$

where $\mathcal{A} : X_h \to X_h^*$ is given by

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Since, a is not positive definite it is not clear that this discretization is stable, in fact the system can even be singular. The stable discretizations are characterized by the a corresponding discrete inf–sup condition of the form

$$\inf_{x \in X_h} \sup_{y \in X_h} \frac{a(x, y)}{\|x\|_X \|y\|_X} \geq c > 0,$$

where the constant c is independent of the mesh parameter h. This condition does not follow from the corresponding condition for the continuous case.

The abstract preconditioner, discrete case

As in the continuous case we define the preconditioner $\mathcal{B}_h : X_h \to X_h$ by

$$\langle \mathcal{B}_h f, y \rangle_X = \langle f, y \rangle \quad y \in X_h.$$

The same arguments as in the continuous case shows that $\mathcal{B}_h \mathcal{A}_h : X_h \to X_h$ is symmetric (with respect to $\langle \cdot, \cdot \rangle_X$) and that

$$\|\mathcal{B}_h \mathcal{A}_h\|_{\mathcal{L}(X_h, X_h)} \le C_1, \quad \text{and} \quad \|(\mathcal{B}_h \mathcal{A}_h)^{-1}\|_{\mathcal{L}(X_h, X_h)} \le c^{-1}.$$

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So we have confirmed the claim that for stable discretizations *the structure of the preconditioner for the discrete problems* follows from the structure of the preconditioner in the continuous case. Furthermore, the inner product, $\langle \cdot, \cdot \rangle_X$ on X_h is only determined up to equivalence of norms.

The mixed Poisson problem

Recall that we consider the operator

$$\mathcal{A} = \begin{pmatrix} I & -\operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix}$$

or more precisely the weak system:
Find $(u, p) \in H_0(\operatorname{div}) \times L_0^2$ such that

$$\begin{aligned} \langle u, v\rangle + \langle p, \operatorname{div} v\rangle &= \langle f, v\rangle && \forall v \in H_0(\operatorname{div}), \\ \langle \operatorname{div} u, q\rangle &= \langle g, q\rangle && \forall q \in L_0^2. \end{aligned}$$

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Discretization: Find $(u_h, p_h) \in V_h \times Q_h \subset H_0(\operatorname{div}) \times L_0^2$ such that

$$\begin{aligned} \langle u, v \rangle + \langle p, \operatorname{div} v \rangle &= \langle f, v \rangle \qquad && \forall v \in V_h, \\ \langle \operatorname{div} u, q \rangle &= \langle g, q \rangle \qquad && \forall q \in Q_h. \end{aligned}$$

Stable discretization of the mixed Poisson problem

For numerical stability the pair of spaces (V_h, Q_h) has to satisfy the two Brezzi conditions, i.e., $c_1, c_1 > 0$, independent of h, such that

$$\sup_{v \in V_h} \frac{\langle q, \operatorname{div} v \rangle}{\|v\|_{\mathrm{div}}} \geq c_1 \|q\|_0 \quad q \in Q_h,$$

and

$$\|v\|_0^2 \geq c_2 \|v\|_{\mathrm{div}}^2 = c_2(\|v\|_0^2 + \|\operatorname{div} v\|_0^2) \quad v \in Z_h,$$

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Note that any stable Stokes element satisfies the first condition, but not the second. In fact, all the standard Stokes elements fails to satisfy this condition.

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Note that any stable Stokes element satisfies the first condition, but not the second. In fact, all the standard Stokes elements fails to satisfy this condition.
The second condition is usually fulfilled by requiring that $\operatorname{div} V_h \subset Q_h$.

Commuting diagram

When $\operatorname{div} V_h \subset Q_h$ then the first Brezzi condition

$$\sup_{v \in V_h} \frac{\langle q, \operatorname{div} v \rangle}{\|v\|_{\operatorname{div}}} \geq c\|q\|_0 \quad q \in Q_h,$$

will follow from the existence of a uniformly bounded operator $\Pi_h : H_0(\operatorname{div}) \to V_h$ such that

$$\operatorname{div}(\Pi_h v) = P_h \operatorname{div} v,$$

where P_h is the L^2–projecton onto Q_h.

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This relation can be expressed in a commuting diagram of the form

$$\begin{array}{ccc}
H_0(\operatorname{div}) & \xrightarrow{\operatorname{div}} & L_0^2 \\
\Big\downarrow \mathit{\Pi}_h & & \Big\downarrow P_h \\
V_h & \xrightarrow{\operatorname{div}} & Q_h
\end{array}$$

The mixed Poisson problem, preconditioning

Preconditioner, continuous case:

$$\mathcal{B} = \mathcal{B}_1 = \begin{pmatrix} (I - \operatorname{grad}\operatorname{div})^{-1} & 0 \\ 0 & I \end{pmatrix}.$$

Preconditioner, discrete case:

$$\mathcal{B}_{1,h} = \begin{pmatrix} M_h & 0 \\ 0 & I \end{pmatrix}.$$

where $M_h \approx (I - \operatorname{grad}\operatorname{div})^{-1}$.

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Numerical results (Lowest order Raviart-Thomas $= \mathcal{P}_1^- - \mathcal{P}_0$):

h	1	2^{-1}	2^{-2}	2^{-3}	2^{-4}
$\kappa(\mathcal{A}_h)$	8.25	15.0	29.7	59.6	119
$\kappa(\mathcal{B}_h\mathcal{A}_h)$	1.04	1.32	1.68	2.18	2.34

Alternative preconditioner

Consider the same problem as above with the $\mathcal{P}_1^- - \mathcal{P}_0$ element, or more generally the $\mathcal{P}_r^- - \mathcal{P}_{r-1}$ element. From the continuous discussion we should also be able to use a preconditione rof the form

$$\mathcal{B}_{2,h} = \begin{pmatrix} I & 0 \\ 0 & N_h \end{pmatrix},$$

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A technical problem is that the operator N_h must be defined on the discontinuous pressure space. However, in Rusten, Vassilevski and W (Math. Comp 1996) we showed that we should use a preconditioner for the bilinear form

$$a(p,q) = \sum_T \int_T \operatorname{grad} p \cdot \operatorname{grad} q \, dx + \sum_e h_e^{-1} \int_e [p][q] \, ds.$$

The time dependent Stokes problem

Recall that we consider the operator

$$\mathcal{A}_\epsilon = \begin{pmatrix} I - \epsilon^2 \Delta & -\operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix}.$$

For ϵ positive, and bounded away from zero, this operator is well defined on $(H_0^1)^n \times L_0^2$. In order to obtain norm estimates uniformly in ϵ, then one possible function space is $(L^2 \cap \epsilon \, H_0^1)^n \times ((H^1 \cap L_0^2) + \epsilon^{-1} \, L_0^2)$ into its L^2–dual.

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This leads to a uniform preconditioner of the form

$$\mathcal{B}_\epsilon = \begin{pmatrix} (I - \epsilon^2 \Delta)^{-1} & 0 \\ 0 & (-\Delta)^{-1} + \epsilon^2 I \end{pmatrix}.$$

Time dependent Stokes, discrete problems

$$\mathcal{B}_{\epsilon,h} = \begin{pmatrix} M_{\epsilon,h} & 0 \\ 0 & \epsilon^2 I_h + N_h \end{pmatrix},$$

where $M_{\epsilon,h} \approx (I - \epsilon^2 \Delta)^{-1}$, $N_h \approx (-\Delta)^{-1}$ and $I_h \approx I$.

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In the experiments below the operators $M_{\epsilon,h}$ and N_h are generated by a V–cycle multigrid procedure.

The Taylor–Hood element

So we use continuous $\mathcal{P}_2$ velocities and continuous $\mathcal{P}_1$ pressures. The operator I_h, approximating the inverse mass matrix $\pi_h I \mu_h^{-1}$, is derived by a simple symmetric Gauss–Seidel iterartion.

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
$\kappa(N_h(-\Delta_h))$	1.71	1.50	1.47	1.47	1.47	1.47
$\kappa(I_h)$	1.66	1.62	1.61	1.60	1.60	1.60

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$h \backslash \epsilon$	0	0.001	0.01	0.1	0.5	1.0
2^{-3}	1.11	1.11	1.03	1.14	1.22	1.22
2^{-5}	1.11	1.09	1.03	1.23	1.24	1.24
2^{-7}	1.11	1.02	1.20	1.24	1.24	1.24

Table: $\kappa(M_{\epsilon,h}(I - \epsilon^2 \Delta_h))$

The Taylor–Hood element

$h \backslash \epsilon$	0	0.001	0.01	0.1	0.5	1.0
2^{-3}	6.03	6.05	6.92	13.42	15.25	15.32
2^{-5}	6.07	6.23	10.62	15.14	15.59	15.61
2^{-7}	6.08	7.81	14.18	15.55	15.64	15.65

Table: $\kappa(\mathcal{B}_{\epsilon,h}\mathcal{A}_{\epsilon,h})$, Taylor–Hood element.

Alternative preconditioner

Following the continuous theory we should also be able to find a uniform preconditioner of the form

$$\mathcal{B}_{\epsilon,h} = \begin{pmatrix} M_{\epsilon,h} & 0 \\ 0 & I \end{pmatrix},$$

where $M_{eps,h} \approx (I - \operatorname{grad}\operatorname{div} - \epsilon^2 \Delta)^{-1}$. However, the Taylor-Hood method, like most other Stokes elements, is not stable in $(H_0(\operatorname{div}) \cap \epsilon H_0^1) \times L_0^2$ uniformly in ϵ, cf. Mardal, Tai and W (SINUM 2002).

Actually, in that paper we constructed a stable 2d–element, with velocities taken as a nine dimensional space of reduced cubic vector fields, and with piecewise constant pressures. A corresponding 3–d element was presented by Tai and W (Calcolo 2006).

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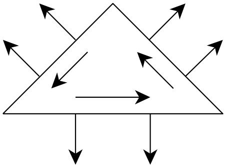

Mini–element

So we use continuous velocities of the form $\mathcal{P}_1+$ "cubic bubbles" and as above continuous $\mathcal{P}_1$ pressures.

$h \backslash \epsilon$	0	0.001	0.01	0.1	0.5	1.0
2^{-3}	2.79	2.73	1.35	1.05	1.14	1.16
2^{-5}	2.94	2.22	1.02	1.15	1.20	1.21
2^{-7}	2.95	1.14	1.11	1.20	1.23	1.23

Table: $\kappa(M_{\epsilon,h}(I - \epsilon^2 \Delta_h))$

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Table: $\kappa(M_{\epsilon,h}(I - \epsilon^2 \Delta_h))$

$h \backslash \epsilon$	0	0.001	0.01	0.1	0.5	1.0
2^{-3}	4.32	4.23	3.59	13.87	19.18	19.43
2^{-5}	4.65	3.50	8.56	18.53	19.83	19.88
2^{-7}	4.67	4.52	15.74	19.72	19.93	19.93

Table: $\kappa(\mathcal{B}_{\epsilon,h}\mathcal{A}_{\epsilon,h})$

The $\mathcal{P}_2$–$\mathcal{P}_0$ element

We use continuous $\mathcal{P}_2$ velocities and discontinuous $\mathcal{P}_0$ pressures. Note that for discontinuous pressures it is not obvious how to generate $N_h \approx (-\Delta)^{-1}$. We utilize Xu's auxiliary space approach with a mapping to continuous piecewise linears.

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2^{-3}	4.96	4.95	4.48	5.56	7.85	7.79
2^{-5}	5.22	5.07	4.46	7.12	8.72	8.74
2^{-7}	5.28	4.30	5.93	8.27	9.24	9.28

Table: $\kappa(\mathcal{B}_{\epsilon,h}\mathcal{A}_{\epsilon,h})$ using the $P_2 - P_0$ element.

The Reissner–Mindlin plate model

We recall that

$$\mathcal{A}_t = \begin{pmatrix} -\operatorname{div}\mathcal{C}\mathcal{E} & 0 & -I \\ 0 & 0 & -\operatorname{div} \\ -I & \operatorname{grad} & -t^2 I \end{pmatrix}.$$

and that the continuous preconditioner takes the form

$$\mathcal{B}_t = \begin{pmatrix} (-\Delta)^{-1} & 0 & 0 \\ 0 & (-\Delta)^{-1} & 0 \\ 0 & 0 & D_t \end{pmatrix},$$

where

$$D_t = I + (1 - t^2)\operatorname{curl}(I - t^2\Delta)^{-1}\operatorname{rot}.$$

The Arnold–Falk method

For a general overview of stable elements for the Reissner–Mindlin model, see Falk 2008 (C.I.M.E. summer school 2006).
The examples presented here are taken from A-F-W97 and uses the Arnold –Falk element, i.e.,

- continuous $\mathcal{P}_1+$ "cubic bubbles" for the rotation ϕ
- $\mathcal{P}_1$ with continuity at midpoints of edges for displacement u
- discontinuous $\mathcal{P}_0$ for the shear stress ζ

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To evaluate the operator $D_t = I + (1 - t^2)\,\mathrm{curl}(I - t^2\Delta)^{-1}\,\mathrm{rot}$ on the latter space, we introduce an operator rot_h into continuous piecewise linears.
The elliptic preconditioners needed are generated by a combination of V–cycle multigrid procedures and the auxiliary space approach.
In particular, we like to show that we obtain condition numbers independent of h and the thickness parameter t.

Experiments, Reissner–Mindlin

Preconditioner for $\mathcal{A}_{t,h}$:

$$\mathcal{B}_t = \begin{pmatrix} L_h & 0 & 0 \\ 0 & M_h & 0 \\ 0 & 0 & N_{t,h} \end{pmatrix},$$

where $L_h \approx (-\Delta_h)^{-1})$, $M_h \approx (-\Delta_h)^{-1}$ and $N_{t,h} \approx D_{t,h}$ where

$$D_{t,h} = I + (1 - t^2)\,\mathrm{curl}(I - t^2\Delta_h)^{-1}\,\mathrm{rot}_h\,.$$

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$$D_{t,h} = I + (1 - t^2)\,\mathrm{curl}(I - t^2\Delta_h)^{-1}\,\mathrm{rot}_h\,.$$

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
iterations–MR	41	41	35	29	24
iterations–CGN	48	50	48	40	34
$\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$	8.17	10.7	11.1	10.6	9.62

Table: $t = 0$

Experiments, Reissner–Mindlin, $t = 0.01$

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
iterations–MR	39	35	28	40	72
iterations–CGN	48	50	48	108	360
$\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$	8.15	10.7	11.4	32.9	113

Table: $t = 0.01$, $N_{t,h} = D_{0,h}$

Experiments, Reissner–Mindlin, $t = 0.01$

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
iterations–MR	39	35	28	40	72
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Table: $t = 0.01$, $N_{t,h} = D_{0,h}$

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
iterations–MR	39	36	28	25	24
iterations–CGN	48	50	48	42	36
$\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$	8.15	10.7	11.2	11.1	9.68

Table: $t = 0.01$, $N_{t,h} = D_{t,h}$

Experiments, Reissner–Mindlin, $t = 1$

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
iterations–MR	22	22	20	20	20
iterations–CGN	102	104	106	104	102
$\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$	17.5	18.4	19.0	19.0	18.9

Table: $t = 1$, $N_{t,h} = D_{t,h} = I$

Experiments, Reissner–Mindlin, $t = 0.1$

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
iterations–MR	78	80	80	78	78
iterations–CGN	226	214	198	190	188
$\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$	90.2	78.5	72.7	70.1	70.7

Table: $t = 0.1$, $N_{t,h} = I$

Experiments, Reissner–Mindlin, $t = 0.1$

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
iterations–MR	78	80	80	78	78
iterations–CGN	226	214	198	190	188
$\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$	90.2	78.5	72.7	70.1	70.7

Table: $t = 0.1$, $N_{t,h} = I$

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
iterations–MR	28	28	27	26	26
iterations–CGN	48	54	52	50	50
$\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$	8.64	10.8	11.1	11.2	11.1

Table: $t = 0.1$, $N_{t,h} = D_{t,h}$

Other examples I, Scatterd data interpolation

In a paper in Adv. Comp. Math. 2002 by Lyche, Nilssen and W we study a scatterd data interpolation problem of the form

$$\min_{v \in H_0^2} E(v) \equiv \sum_{|\alpha|=2} \int_\Omega |D^\alpha v|^2 \, dx \quad \text{subject to} \;\; v(x_i) = g_i.$$

where $\Omega \subset \mathbb{R}^2$. This is a generalization of the classical problem of interpolating cubic splines. We study preconditioning of the obtained saddle point problem.

Scattered data interpolation

The saddle point system has the structure

$$\begin{pmatrix} \Delta^2 & \pi^* \\ \pi & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

where the operator $\pi : H_0^2(\Omega) \to \mathbb{R}^m$ is defined by interpolation at the points $\{x_i\}$.

Other examples II, A nonlinear problem

In a recent preprint by Hu, Tai and W we study nonlinear so–called harmonic map problems, i.e., we consider a problem of the form

$$\min_{v \in H^1_g(\Omega,\mathcal{M})} E(v) \equiv \int_\Omega |\operatorname{grad} v|^2 \, dx.$$

Here the space $H^1_g(\Omega, \mathcal{M})$ is the set of H^1 vector fields with values in a smooth manifold $\mathcal{M}$, given on the form

$$\mathcal{M} = \{ v \in \mathbb{R}^d \mid F(v) = 0 \}$$

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The most classical example is

$$\mathcal{M} = \{ v \in \mathbb{R}^d \,|\, |v| = 1 \}.$$

In particular, we study preconditioning of the linear systems occuring through the Newton process.

Saddle point problem, harmonic map

We consider the following nonlinear saddle point problem:
Find u and λ such that

$$\begin{array}{rlll} \langle \operatorname{grad} u, \operatorname{grad} v\rangle & +\langle DF(u)\cdot v, \lambda\rangle & = 0 & v \in H_0^1 \\ \langle F(u), \mu\rangle & & = 0 & \mu \in H^{-1} \end{array}$$

By solving this system by Newton's method we end up with linear saddle point problems at each step of the iteration, and the prposed preconditioners for these systems will be of the form

$$\mathcal{B} = \begin{pmatrix} -\Delta^{-1} & 0 \\ 0 & -\Delta \end{pmatrix}$$

Other examples III

- For a study of an inverse problem, where preconditioning with respect to the regularization parameter is studied, see Nielsen and Mardal (SIMULA 2008).
- For a study of preconditioners for the Babuska Langrange multiplier method, see Haug and W, Math. Comp. 1999.