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Families of characters for the Ariki-Koike algebras

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1. The complex reflection groups and their associated cyclotomic Hecke algebras appear naturally in the classification of the "cyclotomic Harish-Chandra series" of the characters of the finite reductive groups (cf. [Broué, Malle, Michel, 1993], [Broué, Malle, 1993]).

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2. For some complex reflection groups (non-Coxeter) W, some data have been gathered which seem to indicate that behind the group W, there exists another mysterious object — the *Spets* — that could play the role of the "series of finite reductive groups with Weyl group W" (cf. [Broué, Malle, Michel, 1999]).

Ariki-Koike algebras

The "generic" Ariki-Koike algebra $\mathcal{H}_{d,r}$ is generated over the Laurent polynomial ring in $d+1$ indeterminates

$$\mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, \ldots, u_{d-1}, u_{d-1}^{-1}, x, x^{-1}]$$

by the elements $\mathbf{s}, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{r-1}$ satisfying the relations

- $\mathbf{s}\mathbf{t}_1\mathbf{s}\mathbf{t}_1 = \mathbf{t}_1\mathbf{s}\mathbf{t}_1\mathbf{s}$,
- $\mathbf{s}\mathbf{t}_j = \mathbf{t}_j\mathbf{s}$, for all $j = 2, \ldots, r-1$,
- $\mathbf{t}_{j-1}\mathbf{t}_j\mathbf{t}_{j-1} = \mathbf{t}_j\mathbf{t}_{j-1}\mathbf{t}_j$, for all $j = 2, \ldots, r-1$,
- $\mathbf{t}_i\mathbf{t}_j = \mathbf{t}_j\mathbf{t}_i$, for all $1 \leq i, j \leq r-1$ with $|i-j| > 1$,
- $(\mathbf{s} - u_0)(\mathbf{s} - u_1)\cdots(\mathbf{s} - u_{d-1}) = 0$,
- $(\mathbf{t}_j - x)(\mathbf{t}_j + 1) = 0$, for all $j = 1, \ldots, r-1$.

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We call it "generic", because it can be viewed as the generic Hecke algebra of the complex reflection group $G(d, 1, r)$.

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- The irreducible characters of $G(d,1,r)$ and of $\mathcal{H}_{d,r}$ are parametrized by the d-partitions of r. For every d-partition $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)})$ of r, we denote by χ_λ the corresponding irreducible character.

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The coefficient of s_λ is a unit in $\mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, \ldots, u_{d-1}, u_{d-1}^{-1}, x, x^{-1}]$.

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Corollary

The Schur elements of $(\mathcal{H}_{d,r})_\varphi$ are products of $\mathbb{Q}(\zeta_d)$-cyclotomic polynomials,

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Corollary

The Schur elements of $(\mathcal{H}_{d,r})_\varphi$ are products of $\mathbb{Q}(\zeta_d)$-cyclotomic polynomials, *i.e.*, they are of the form

$$s_{\lambda,\varphi} = \xi_{\lambda,\varphi} \, q^{a_{\lambda,\varphi}} \prod_{\Psi \in C_{\lambda,\varphi}} \Psi(q)$$

where $\xi_{\lambda,\varphi} \in \mathbb{Z}[\zeta_d]$, $a_{\lambda,\varphi} \in \mathbb{Z}$ and $C_{\lambda,\varphi}$ is a family of $\mathbb{Q}(\zeta_d)$-cyclotomic polynomials.

Rouquier blocks

The Rouquier blocks of the cyclotomic Ariki-Koike algebra $(\mathcal{H}_{d,r})_{\varphi}$ are the blocks of the algebra $\mathcal{R} \otimes_{\mathbb{Z}[\zeta_d][q,q^{-1}]} (\mathcal{H}_{d,r})_{\varphi}$, where

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i.e., the minimal subsets B of $\mathrm{Irr}(\mathcal{H}_{d,r})$ with respect to the property:

$$\sum_{\chi_\lambda \in B} \frac{(\chi_\lambda)_\varphi(h)}{s_{\lambda,\varphi}} \in \mathcal{R}, \ \forall h \in (\mathcal{H}_{d,r})_\varphi.$$

Essential hyperplanes

The irreducible factors of s_λ are of the form

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We have

1. $\varphi(\Phi_m(x)) = \Phi_m(q^n)$.
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$$\zeta_d^{s-t} - 1 \text{ is not a unit in } \mathbb{Z}[\zeta_d].$$

Let H be an essential hyperplane for $G(d,1,r)$ and let

$$\varphi_H : \begin{cases} u_j \mapsto \zeta_d^j q^{m_j}, (0 \leq j < d), \\ x \mapsto q^n \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}_{d,r}$ such that the integers $((m_j)_{0 \leq j < d},\ n)$ belong to H and to no other essential hyperplane. Then φ_H is said to be associated with the essential hyperplane H and the Rouquier blocks of $(\mathcal{H}_{d,r})_{\varphi_H}$ are called Rouquier blocks associated with the essential hyperplane H.

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be any cyclotomic specialization of $\mathcal{H}_{d,r}$. If the integers $((m_j)_{0\leq j<d},\, n)$ belong to no essential hyperplane for $G(d,1,r)$, then the Rouquier blocks of $(\mathcal{H}_{d,r})_\varphi$ are trivial. Otherwise, the Rouquier blocks of $(\mathcal{H}_{d,r})_\varphi$ are

1. unions of the Rouquier blocks associated with every essential hyperplane for $G(d,1,r)$ to which the integers $((m_j)_{0\leq j<d},\, n)$ belong,

Let H be an essential hyperplane for $G(d,1,r)$ and let

$$\varphi_H : \begin{cases} u_j \mapsto \zeta_d^j q^{m_j}, (0 \leq j < d), \\ x \mapsto q^n \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}_{d,r}$ such that the integers $((m_j)_{0 \leq j < d},\, n)$ belong to H and to no other essential hyperplane. Then φ_H is said to be associated with the essential hyperplane H and the Rouquier blocks of $(\mathcal{H}_{d,r})_{\varphi_H}$ are called Rouquier blocks associated with the essential hyperplane H.

Theorem (C.)

Let

$$\varphi : \begin{cases} u_j \mapsto \zeta_d^j q^{m_j}, (0 \leq j < d), \\ x \mapsto q^n \end{cases}$$

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1. unions of the Rouquier blocks associated with every essential hyperplane for $G(d,1,r)$ to which the integers $((m_j)_{0 \leq j < d},\, n)$ belong,
2. minimal with respect to the property 1.

Determination of the Rouquier blocks

Proposition (C.)

Let λ, μ be two d-partitions of r. The characters χ_λ and χ_μ are in the same Rouquier block associated with the essential hyperplane $N = 0$ if and only if

$$|\lambda^{(a)}| = |\mu^{(a)}| \text{ for all } a = 0, 1, \ldots, d-1.$$

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Example: The characters corresponding to the multipartitions

$$\lambda = ((1,1),(3,2),\emptyset) \text{ and } \mu = ((2),(2,1,1,1),\emptyset)$$

are in the same Rouquier block associated with the essential hyperplane $N = 0$ for $G(3,1,7)$.

Let H be an essential hyperplane for $G(d,1,r)$ of the form $kN + M_s - M_t = 0$ and let

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Proposition (C.)

Let λ, μ be two d-partitions of r. The irreducible characters χ_λ and χ_μ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\varphi$ if and only if:

1. We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s,t\}$.
2. If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$ and $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$, then the characters $\chi_{\lambda^{st}}$ and $\chi_{\mu^{st}}$ belong to the same Rouquier block of $(\mathcal{H}_{2,l})_\vartheta$, where $l := |\lambda^{st}| = |\mu^{st}|$ and

$$\vartheta : U_0 \mapsto q^{m_s}, U_1 \mapsto -q^{m_t}, X \mapsto q^n.$$

Idea of the proof.
Following [Dipper-Mathas, 2002] we obtain that the algebra $(\mathcal{H}_{d,r})_\varphi$ defined over the Rouquier ring is Morita equivalent to the algebra

$$\bigoplus_{n_1+\ldots+n_{d-1}=r} (\mathcal{H}_{2,n_1})_\varphi \otimes \mathcal{H}(\mathfrak{S}_{n_2})_\varphi \otimes \ldots \otimes \mathcal{H}(\mathfrak{S}_{n_{d-1}})_\varphi .$$

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We deduce that the irreducible characters χ_λ and χ_μ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\varphi$ if and only if:

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$$\varphi^{st} : U_0 \mapsto \zeta_d^s q^{m_s}, U_1 \mapsto \zeta_d^t q^{m_t}, X \mapsto q^n.$$

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$$\varphi^{st} : U_0 \mapsto \zeta_d^s q^{m_s}, U_1 \mapsto \zeta_d^t q^{m_t}, X \mapsto q^n .$$

Using the algorithm for the blocks of the Ariki-Koike algebra over a field given by [Lyle-Mathas, 2007], we obtain that the second condition is equivalent to the second condition of the proposition.

The group $G(de, e, r)$

The group $G(de, e, r)$ is the group of all $r \times r$ monomial matrices with non-zero entries in $\mathbb{Z}/de\mathbb{Z}$ and product of the non-zero entries in $\mathbb{Z}/d\mathbb{Z}$.

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Thanks to [Ariki, 1995], any cyclotomic Hecke algebra of $G(de, e, r)$ (for $r > 2$ or $r = 2$ and e odd) can be viewed as a subalgebra of a cyclotomic Ariki-Koike algebra associated to $G(de, 1, r)$. Then Clifford Theory allows us to obtain the Rouquier blocks of the former from the Rouquier blocks of the latter.