PARITY SHEAVES

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ABSTRACT. Given a stratified variety X with strata satisfying a cohomological parity-vanishing condition, we define and show the uniqueness of "parity sheaves", which are objects in the constructible derived category of sheaves with coefficients in an arbitrary field or complete discrete valuation ring.

If X admits a resolution also satisfying a parity condition, then the direct image of the constant sheaf decomposes as a direct sum of parity sheaves. If moreover the resolution is semi-small, then the multiplicities of the indecomposable summands are encoded in certain intersection forms appearing in the work of de Cataldo and Migliorini. We give a criterion for the Decomposition Theorem to hold.

Our framework applies in many situations arising in representation theory. We give examples in generalised flag varieties (in which case we recover a class of sheaves considered by Soergel), toric varieties, and nilpotent cones. Finally, we show that tilting modules and parity sheaves on the affine Grassmannian are related through the geometric Satake correspondence, when the characteristic is bigger than an explicit bound.

CONTENTS

1. Introduction

1.1. **Overview.** Perverse sheaves often encode information about representation theory. One may associate complex algebraic varieties to many important objects in Lie theory (for example the flag variety or nilpotent cone) and the study of perverse sheaves on these spaces has led to breakthroughs in representation theory — the first example being the proof of the Kazhdan-Lusztig conjecture [BB81, BK81].

Generally, the coefficients of the perverse sheaves are chosen to be a field of characteristic zero and a key role is played by the Decomposition Theorem of [BBD82]. The Decomposition Theorem states that a distinguished class of objects, consisting of direct sums of shifts of simple perverse sheaves, is preserved under push-forward along proper maps.

Over the last ten years, several authors (e.g. [Soe00, MV07, Jut07]) have noticed that questions in modular representation theory can be rephrased in terms of questions about "modular perverse sheaves"; here the variety is still complex algebraic, however the coefficients lie in a field of positive characteristic. The study of modular perverse sheaves is considerably more difficult than its characteristic zero counterpart. One reason for this difficulty is that the Decomposition Theorem does not apply in the modular context. In other words, there are proper maps for which the push-forward of a simple perverse sheaf is not a direct sum of shifts of simple perverse sheaves. When this happens, we will say that "the Decomposition Theorem fails". In [JMW09], we

explain a number of examples of such "failures" and survey applications to representation theory.

In this article we introduce a new class of indecomposable complexes which we call **parity sheaves**. First, we say that a complex of sheaves is a **parity complex** is it has the property that its stalks and costalks both vanish in either even or odd degree. The indecomposable parity complexes are shifts of certain "normalised" indecomposable parity complexes that we chose to call **parity sheaves**. A remark on terminology: as for "perverse sheaves", we allow ourselves to call "sheaves" some objects which are actually complexes. We prove that, under certain assumptions on the stratification of the underlying complex algebraic variety, parity sheaves share many features with intersection cohomology sheaves with coefficients in characteristic zero. Some examples of their properties include:

- parity sheaves are always an extension of a local system on a stratum, and there exists up to isomorphism at most one such extension;
- in many examples, if the coefficients are a field of characteristic zero, the parity sheaves are the intersection cohomology sheaves;
- the class of parity complexes is preserved by certain "even" proper maps whose fibres satisfy a form of parity vanishing.

Moreover, many stratified complex algebraic varieties arising from representation theory satisfy our assumptions, and the parity sheaves correspond to important objects in representation theory. For example:

- If the characteristic of the field of coefficients is larger than explicit bounds, the parity sheaves on the affine Grassmannian correspond, under the geometric Satake equivalence, to indecomposable tilting modules.
- Using equivalences proved recently by Fiebig [Fie07] one may show that the parity sheaves on a subvariety of the affine flag variety correspond to projective modules for the restricted enveloping algebra of the dual group (the connection between parity sheaves and Fiebig's work will be explained in [FW]). This allows a reformulation of Lusztig's conjecture in terms of the stalks of parity sheaves.
- On the nilpotent cone for the general linear group, the parity sheaves correspond to Young modules for the symmetric group. This will be explained in [Mau].

Thus we are able to translate several important problems in representation theory, like understanding the characters of tilting modules for a reductive algebraic group, into the problem of determining the

stalks of the parity sheaves. This, in turn, may be rephrased in terms of understanding the failure of the Decomposition Theorem. Indeed, for many proper maps arising in representation theory the Decomposition Theorem is equivalent to the parity sheaves being isomorphic to intersection cohomology complexes.

In recent work de Cataldo and Migliorini [dCM02, dCM05] have given Hodge theoretic proofs of the Decomposition Theorem. In their work a crucial role is played by the case of semi-small resolutions, and certain intersection forms attached to the strata of the target. Indeed, they show that for a semi-small morphism the direct image of the intersection cohomology sheaf splits as a direct sum of intersection cohomology complexes if and only if these forms are non-degenerate.

We extend this observation, showing that the non-degeneracy of the modular reduction of these intersection forms (which are defined over the integers) determine exactly when the decomposition theorem fails in positive characteristic. Moreover, if the direct image of the intersection cohomology sheaf under such a resolution is parity, then the multiplicities of the parity sheaves which occur in the direct image are given in terms of the ranks of these forms.

This observation allows us to reformulate questions in representation theory (for example the determination of the dimension of simple modules for the symmetric group) in terms of such intersection forms.

We now turn to a more detailed summary of our results.

1.2. **The general setting.** Let X be a complex algebraic variety, or a complex G-variety for a connected complex linear algebraic group G. Below we bracket the features that only apply to the equivariant setting. We assume that X is equipped with a Whitney stratification into (G-stable) locally closed smooth subvarieties:

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda.$$

Fix a field k.[1] We assume that the strata of X satisfy a "parity vanishing" condition (see Section 2.2): we require that the (equivariant) cohomology $H^i(X_\lambda, \mathcal{L})$ vanishes for odd i and all (equivariant) local systems $\mathcal{L}$ on X_λ with coefficients in k. For example, if the strata of X are simply connected this amounts to the vanishing of the cohomology of each stratum X_λ over k in odd degree.

Examples of varieties satisfying our assumptions are:

(1) the flag variety G/B stratified by Schubert cells;

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[1] Under the appropriate hypotheses, everything below is valid if k is a complete local ring. See Section 2.2.

(2) the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ viewed as a G-variety;[2]
(3) the affine Grassmannian $G((t))/G[[t]]$ stratified by $G[[t]]$-orbits;
(4) any toric variety, viewed as a T-variety.

Let $D(X)$ denote the full subcategory of the (equivariant) derived category of sheaves of k-vector spaces consisting of complexes whose cohomology is constructible with respect to the stratification.

The central definition is the following:

Definition 1.1. An object $\mathcal{P} \in D(X)$ is an **even (resp. odd) complex** if the stalks and costalks of $\mathcal{P}$ are both concentrated in even (resp. odd) degree. A complex $\mathcal{P}$ is **parity**, if it is either even or odd.

Our first theorem is the following (see Theorem 2.9):

Theorem 1.2. *Given an indecomposable (equivariant) local system $\mathcal{L}$ on some stratum X_λ there is, up to isomorphism, at most one indecomposable parity complex $\mathcal{E}(\lambda, \mathcal{L})$ extending $\mathcal{L}[\dim X_\lambda]$. Moreover, any indecomposable parity complex is isomorphic to $\mathcal{E}(\lambda, \mathcal{L})[i]$ for some $\lambda \in \Lambda$, some (equivariant) local system $\mathcal{L}$ on X_λ, and some integer i.*

Definition 1.3. We call the $\mathcal{E}(\lambda, \mathcal{L})$ **parity sheaves**.

In general the question of existence is more difficult. To this end we introduce the notion of an even morphism (see Section 2.3) and show:

Theorem 1.4. *The direct image under a proper even map of a parity complex is again parity.*

In light of Theorem 1.2, Theorem 1.4 functions as a kind of decomposition theorem and shows the crucial role played by parity sheaves:

Corollary 1.5. *Assume that all intersection cohomology complexes in $D(X)$ are parity. Then, given any proper, even, stratified morphism $\pi : \widetilde{X} \to X$, the Decomposition Theorem holds for $\pi_* \underline{k}_{\widetilde{X}}$. That is, $\pi_* \underline{k}_{\widetilde{X}}$ is a direct sum of shifted intersection cohomology complexes.*

In several important examples in representation theory, one has a proper surjective semi-small map

$$f : \widetilde{X} \to X$$

with $\widetilde{X}$ smooth, and such that the direct image $f_* \underline{k}_{\widetilde{X}}$ of the constant sheaf on $\widetilde{X}$ is parity.

Following de Cataldo and Migliorini [dCM02] we associate to each stratum X_λ a local system $\mathcal{L}_\lambda$ equipped with a bilinear form B_λ. For

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[2]In this case we need the additional assumption that the characteristic of k is not a torsion prime for G (see sections 4.3 for details).

each $x \in X_\lambda$, the stalk of $\mathcal{L}_\lambda$ at x is either zero or canonically isomorphic to the top Borel-Moore homology of $f^{-1}(x)$ in which case the bilinear form is given by an intersection form. We show (see Theorems 3.6 and 3.11):

Theorem 1.6.

(1) *If each B_λ is non-degenerate, then we have an isomorphism*

$$f_* \underline{k}_{\widetilde{X}}[\dim_\mathbb{C} \widetilde{X}] \cong \bigoplus \mathbf{IC}(\mathcal{L}_\lambda).$$

Hence the decomposition theorem holds for $f_* \underline{k}_{\widetilde{X}}[\dim_\mathbb{C} \widetilde{X}]$ if and only if B_λ is non-degenerate and $\mathcal{L}_\lambda$ is semi-simple, for all $\lambda \in \Lambda$.

(2) *In general we have*

$$f_* \underline{k}_{\widetilde{X}}[\dim_\mathbb{C} \widetilde{X}] \cong \bigoplus \mathcal{E}(\lambda, \mathcal{L}_\lambda / \operatorname{rad} B_\lambda)$$

and hence the multiplicity of $\mathcal{E}(\lambda, \mathcal{L})$ in $f_* \underline{k}_{\widetilde{X}}[\dim_\mathbb{C} \widetilde{X}]$ is equal to the multiplicity of $\mathcal{L}$ in $\mathcal{L}_\lambda / \operatorname{rad} B_\lambda$.

1.3. **Applications.** In the second half of the paper, we give a number of examples, hopefully demonstrating the variety of situations where parity sheaves provide interesting objects of study.

We show the existence and uniqueness of parity sheaves on the following varieties:

(1) A (Kac-Moody) flag variety $\mathcal{G}/\mathcal{P}_J$ either viewed as a $\mathcal{P}_I$-variety, or stratified by $\mathcal{P}_I$-orbits, where $\mathcal{P}_I$ and $\mathcal{P}_J$ are standard (finite type) parabolic subgroups (see Section 4.1).
(2) Any toric variety $X(\Delta)$ viewed as a T-variety (see Section 4.2).
(3) The nilpotent cone in $\mathcal{N} \subset \mathfrak{gl}_n$ viewed as a GL_n-variety (see Section 4.3).

The most interesting case is the affine Grassmannian, viewed as a ind-$G(\mathbb{C}[[t]])$-variety (which is a special case of (1) above). Recall that, after fixing a commutative ring k, the geometric Satake theorem ([MV07]) gives an equivalence between the category of $G(\mathbb{C}[[t]])$-equivariant perverse sheaves with coefficients in k under convolution and the tensor category of representations of the split form of the Langlands dual group scheme $G^\vee$ over k.

A particularly interesting class of representations of algebraic groups, the tilting modules, were introduced by Donkin [Don93], following work of Ringel [Rin91] in the context of quasi-hereditary algebras. We show that, under the geometric Satake equivalence, the perverse sheaves corresponding to the tilting modules are parity for all but possibly a finite number of primes.

In the following theorem, let h denote the Coxeter number of G:

Theorem 1.7. *If the characteristic of k is strictly larger than $h+1$ (see Section 5 for better bounds), then for each dominant coweight λ, the parity sheaf $\mathcal{E}(\lambda)$ is perverse and corresponds under geometric Satake to the indecomposable tilting module of $G^\vee$ with highest weight λ.*

We also note that a positive solution to the conjecture of Mirković and Vilonen [MV07, Conj. 13.3], as modified by the first author [Jut08b, Conj. 6.1], would imply that this theorem were true for all good characteristics.

As a consequence of the above theorem one obtains a theory of "q-characters" for tilting modules (see Section 5.3).

We also recall that the dimensions of the irreducible representations of the symmetric groups can be determined from the decomposition of tensor products of the standard representation of GL_n into indecomposable tilting modules and explain how the previous theorems can be used to reinterpret this fact as a geometric statement.

1.4. **Related work.** We comment briefly on ideas related to the current work:

1.4.1. *Soergel's category $\mathcal{K}$.* The idea of considering another class of objects as "replacements" for intersection cohomology complexes when using positive characteristic coefficients is due to Soergel in [Soe00]. He considers the full additive subcategory $\mathcal{K}$ of the derived category of sheaves of k-vector spaces on the flag variety which occur as direct summands of direct images of the constant sheaf on Bott-Samelson resolutions. Furthermore, he shows (using arguments from representation theory) that if the characteristic of k is larger than the Coxeter number, then the indecomposable objects in $\mathcal{K}$ are parametrised by the Schubert cells. In fact the indecomposable objects in Soergel's category $\mathcal{K}$ are parity sheaves, and our arguments provide a geometric way of understanding his result.

1.4.2. *Parity vanishing and intersection cohomology.* The usefulness of some form of parity vanishing in equivariant and intersection cohomology calculations has been pointed out by many authors, as for example in [KL80], [Spr82] (see Remark 4.6 below), [GKM98] and [BJ01].

1.4.3. *The work of Cline, Parshall and Scott.* Cline, Parshall and Scott [CPS93] have considered a collection of objects satisfying a similar parity vanishing condition in the derived category of a highest weight category. They use such objects to reformulate Lusztig's conjecture in terms of the non-vanishing of an Ext-group. In Appendix A, we briefly

comment on the connection between our definition and theirs, as well as a paper of Beilinson, Ginzburg and Soergel [BGS96].

1.4.4. *Tilting perverse sheaves.* In the paper [BBM04], Beilinson, Bezrukavnikov and Mirković define a notion of tilting sheaves. However perverse sheaves satisfying their conditions exist only in the restricted situation where the !- and $*$-extensions from open sets are perverse (as is the case when the strata are affine). The affine Grassmannian stratified by $G[[t]]$-orbits, for example, does not satisfy this condition.

There is, however, a natural generalisation of a definition equivalent to theirs. The more general definition which we will use is: modifying [BBM04, Prop. 1.3] slightly, a perverse sheaf M is tilting with respect to a stratification with strata $i_\nu : X_\nu \to X$, if both M and its dual have a filtration with successive quotients of type $^p i_{\nu!} N_\nu$ for N_ν a perverse sheaf on X_ν. In the case of the affine Grassmannian, under the geometric Satake equivalence, this agrees with the usual notion of tilting module.

On the other hand, this more general definition has the stark disadvantage of being of non-local origin — one of the beautiful properties of tilting sheaves à la [BBM04]. One consequence of Theorem 1.7 is that it gives a local description of the tilting sheaves in the Satake category.

It should be mentioned that in the situation of the finite flag variety, the parity sheaves are not tilting: in almost all characteristics they are intersection cohomology sheaves.

1.4.5. *Intersection cohomology of toric varieties.* Due to work of Bernstein and Lunts [BL94a] and Barthel, Brasselet, Fieseler and Kaup [BBFK99] and the rational equivariant intersection cohomology of toric varieties can be calculated using an algorithm from commutative algebra. The construction is inductive, starting with the open strata and calculating the stalk of the intersection cohomology sheaf at each new stratum as the projective cover of a module of sections.

In general the intersection cohomology of toric varieties with modular coefficients is more difficult to calculate. For example, parity vanishing of stalks need not hold. However it seems likely that the stalks of parity sheaves on toric varieties can be calculated in an similar manner, by mimicking the above algorithm, but where coefficients are taken in a the corresponding field, rather than $\mathbb{Q}$. This will be discussed elsewhere.

1.4.6. *The Braden-MacPherson sheaf and intersection cohomology of Schubert varieties.* Similarly to case of toric varieties, in [BM01] Braden

and MacPherson give an algorithm to inductively calculate the intersection cohomology of Schubert varieties (as well as certain other T-varieties). In a series of papers (see, for example, [Fie07]) Fiebig has shown that the objects one obtains by imitating their algorithm with coefficients of positive characteristic have important representation theoretic applications. Thus it is natural to ask what geometric meaning such a procedure has. In a forthcoming paper [FW] Fiebig and the third author show one obtains in this way the stalks of parity sheaves.

1.4.7. *Quiver and hypertoric varieties.* In addition to the examples considered in this paper, parity sheaves should exist on the quiver varieties introduced by Lusztig and Nakajima. The examples provided by hypertoric varieties should also be interesting and amenable to direct calculations, as here the intersection forms have combinatorial descriptions [HS06].

1.4.8. *Weights and parity sheaves.* If one works instead over a variety X_o defined over a finite field $\mathbb{F}_q$ one can consider the derived category $D_c^b(X_o, \mathbb{Q}_\ell)$ of $\mathbb{Q}_\ell$-sheaves (see [BBD82] for details and notation) and an important role is played by Deligne's theory of weights. It is not clear to what extent this theory can be extended to the categories $D_c^b(X_o, \mathbb{Z}_\ell)$ or $D_c^b(X_o, \mathbb{F}_\ell)$.

However, in all examples considered in this paper one can proceed naively, and say that $\mathcal{F}_o \in D_c^b(X_o, \mathbb{Z}_\ell)$ (resp. $D_c^b(X_o, \mathbb{F}_\ell)$) is pure of weight 0 if $\mathcal{H}^i(\mathcal{F})$ and $\mathcal{H}^i(\mathbb{D}\mathcal{F})$ vanish for odd i and, for all $x \in X_o(\mathbb{F}_{q^n})$ the Frobenius $F_{q^n}^*$ acts on the stalks of $\mathcal{H}^{2i}(\mathcal{F})$ and $\mathcal{H}^{2i}(\mathbb{D}\mathcal{F})$ as multiplication by q^{ni} (the image of q^{ni} in $\mathbb{F}_\ell$ respectively). With this definition one can show that, in all examples considered in this paper, there exist analogues of parity sheaves which are pure of weight 0. Note, however, that the modular analogue of Gabber's theorem is not true: if $\mathcal{F}_o$ in $D_c^b(X_o, \mathbb{Z}_\ell)$ or $D_c^b(X_o, \mathbb{F}_\ell)$ is pure of weight 0, then $\mathcal{F}$ is not necessarily semi-simple.

2. Definition and first properties

2.1. **Notation and assumptions.** In what follows all varieties will be considered over $\mathbb{C}$ and equipped with the classical topology. We fix a complete local principal ideal domain k and all sheaves and cohomology groups are to be understood with coefficients in k. Let $\mathbb{O}$ denote a complete discrete valuation ring (e.g., a finite extension of $\mathbb{Z}_p$), $\mathbb{K}$ its field of fractions (e.g., a finite extension of $\mathbb{Q}_p$), and $\mathbb{F}$ its residue field (e.g., a finite field $\mathbb{F}_q$). Throughout X denotes either a variety, or a G-variety for some connected linear algebraic group G. In sections 2 and 3 we deal with these two situations simultaneously, bracketing the features which only apply in the equivariant situation. In the examples, we will specify the set-up in which we work.

We fix an algebraic stratification (in the sense of [CG97, Definition 3.2.23])

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

of X into smooth connected locally closed (G-stable) subsets. For each $\lambda \in \Lambda$ we denote by $i_\lambda : X_\lambda \to X$ the inclusion and by d_λ the complex dimension of X_λ.

We denote by $D(X)$, or $D(X; k)$ if we wish to emphasise the coefficients, the bounded (equivariant) derived category of k-sheaves on X constructible with respect to the stratification (see [BL94b] for the definition and basic properties of the equivariant derived category). The category $D(X)$ is triangulated with shift functor [1]. We call objects of $D(X)$ complexes. For all $\lambda \in \Lambda$, $\underline{k}_\lambda$ denotes the (equivariant) constant sheaf on X_λ. Given $\mathcal{F}$ and $\mathcal{G}$ in $D(X)$ we set $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_{D(X)}(\mathcal{F}, \mathcal{G})$ and $\mathrm{Hom}^n(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}(\mathcal{F}, \mathcal{G}[n])$. We can form the graded k-module $\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}) := \oplus_{n \in \mathbb{Z}} \mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$.

By our assumptions on k, $D(X)$ is a Krull-Schmidt category (see [LC07]) and hence the endomorphism ring of an indecomposable object in $D(X)$ is local.

For each λ, denote by $\mathrm{Loc}_f(X_\lambda, k)$ or $\mathrm{Loc}_f(X_\lambda)$ the category of (equivariant) local systems of free k-modules on X_λ. We make the following assumptions on our variety X. For each $\lambda \in \Lambda$ and all $\mathcal{L}, \mathcal{L}' \in \mathrm{Loc}_f(X_\lambda)$

we assume:

$$\text{(2.1)} \qquad\qquad \text{Hom}^n(\mathcal{L}, \mathcal{L}') = 0 \text{ for } n \text{ odd}$$

and

$$\text{(2.2)} \qquad\qquad \text{Hom}^n(\mathcal{L}, \mathcal{L}') \text{ is a free } k\text{-module for all } n.$$

Remark 2.1.

(1) When k is a field, all finite dimensional k-modules are free, so the second assumption can be ignored.
(2) Given two local systems $\mathcal{L}, \mathcal{L}' \in \text{Loc}_f(X_\lambda)$ we have isomorphisms:

$$\text{Hom}^\bullet(\mathcal{L}, \mathcal{L}') \cong \text{Hom}^\bullet(\underline{k}_\lambda, \mathcal{L}^\vee \otimes \mathcal{L}') \cong \mathbb{H}^\bullet(\mathcal{L}^\vee \otimes \mathcal{L}').$$

Hence (2.1) and (2.2) are equivalent to requiring that $\mathbb{H}^\bullet(\mathcal{L})$ is a free k-module and vanishes in odd degree, for all $\mathcal{L} \in \text{Loc}_f(X_\lambda)$.

Finally, for $\lambda \in \Lambda$ and $\mathcal{L} \in \text{Loc}_f(X_\lambda)$, we denote by $\mathbf{IC}(\lambda, \mathcal{L})$, or simply $\mathbf{IC}(\lambda)$ if $\mathcal{L} = k$, the intersection cohomology complex on $\overline{X}_\lambda$ with coefficients in $\mathcal{L}$, shifted by d_λ so that it is perverse, and extended by zero on $X \setminus \overline{X}_\lambda$.

2.2. Definition and uniqueness.

Definition 2.2. In the following $? \in \{*, !\}$.

- A complex $\mathcal{F} \in D(X)$ is $?$-**even** (resp. $?$-**odd**) if, for all $\lambda \in \Lambda$, $i_\lambda^? \mathcal{F}$ is an object of $\text{Loc}_f(X_\lambda)$ with cohomology sheaves concentrated in even (resp. odd) degrees.
- A complex $\mathcal{F}$ is $?$-**parity** if it is either $?$-even or $?$-odd.
- A complex $\mathcal{F}$ is **even** (resp. **odd**) if it is both $*$- and $!$-even (resp. odd).
- A complex $\mathcal{F}$ is **parity** if it is either even or odd.

Remark 2.3.

(1) A complex is $?$-even (resp. odd) if and only if the stalks and costalks are free and even (resp. odd).
(2) By (2.1) and a standard devissage argument, $\mathcal{F}$ is $?$-even (resp. odd) if and only if the $i_\lambda^? \mathcal{F}$ are isomorphic to *direct sums* of even (resp. odd) shifts of objects in $\text{Loc}_f(X_\lambda)$
(3) A complex $\mathcal{F}$ is $*$-parity if and only if $\mathbb{D}\mathcal{F}$ is $!$-parity etc.
(4) This definition is a geometric analogue of a notion introduced by Cline-Parshall-Scott in [CPS93]. We discuss the relationship in further detail in Appendix A.

Given a $*$-even $\mathcal{F} \in D(X)$ write X' for the support of $\mathcal{F}$ and choose an open stratum $X_\mu \subset X'$. We denote by i and j the inclusions:

$$X_\mu \xrightarrow{j} X \xleftarrow{i} X' \setminus X_\mu$$

We have a distinguished triangle of $*$-even complexes

(2.3) $$j_! j^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \xrightarrow{[1]}$$

which is the extension by zero of the standard distinguished triangle on X'. Dually, if $\mathcal{G} \in D(X)$ is $!$-even and i, j are as above we have a distinguished triangle of $!$-even complexes

(2.4) $$i_! i^! \mathcal{G} \to \mathcal{G} \to j_* j^* \mathcal{G} \xrightarrow{[1]} .$$

Proposition 2.4. *If $\mathcal{F}$ is $*$-parity and $\mathcal{G}$ is $!$-parity the direct sum of the adjunction morphisms yields an isomorphism of graded k-modules*

$$\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}) \cong \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}^\bullet(i_\lambda^* \mathcal{F}, i_\lambda^! \mathcal{G}).$$

Proof. We may assume without loss of generality, by shifting, that $\mathcal{F}$ (resp. $\mathcal{G}$) is $*$- (resp. $!$-) even. We induct on the number N of $\lambda \in \Lambda$ with $i_\lambda^* \mathcal{F} \neq 0$, with the base case following by adjunction. If $N > 1$ applying $\mathrm{Hom}(-, \mathcal{G})$ to (2.3) yields a long exact sequence

$$\cdots \leftarrow \mathrm{Hom}^n(j_! j^! \mathcal{F}, \mathcal{G}) \leftarrow \mathrm{Hom}^n(\mathcal{F}, \mathcal{G}) \leftarrow \mathrm{Hom}^n(i_* i^* \mathcal{F}, \mathcal{G}) \leftarrow \ldots$$

By induction, $\mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$ vanishes for odd n and for even n, $\mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$ is an extension of $\mathrm{Hom}^n(i_* i^* \mathcal{F}, \mathcal{G})$ by $\mathrm{Hom}^n(j_! j^! \mathcal{F}, \mathcal{G})$. However, by induction both are free k-modules and the result follows. $\square$

Corollary 2.5. *If $\mathcal{F}$ is $*$-even and $\mathcal{G}$ is $!$-odd then*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0.$$

Corollary 2.6. *If $\mathcal{F}$ and $\mathcal{G}$ are indecomposable parity complexes of the same parity and such that $j : X_\mu \to X$ is open in the support of each of them, then we have the adjunction map for X_λ gives a surjection:*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \twoheadrightarrow \mathrm{Hom}(\mathcal{F}, j_* j^* \mathcal{G}) = \mathrm{Hom}(j^* \mathcal{F}, j^* \mathcal{G}).$$

Proof. Apply $\mathrm{Hom}(\mathcal{F}, -)$ to (2.4) and use Corollary 2.5. $\square$

The last corollary says that we can extend morphisms $j^* \mathcal{F} \to j^* \mathcal{G}$ to morphisms $\mathcal{F} \to \mathcal{G}$. Now we want to investigate how parity sheaves behave when restricted to an open union of strata. Before stating the result, let us recall a result from ring theory.

Proposition 2.7. *Let A be a k-algebra, finitely generated as a k-module, and let I be an ideal of A contained in the Jacobson radical of A. Then given any idempotent $e \in A/I$ there exists an idempotent $\tilde{e} \in A$ whose image in A/I is e.*

Proof. This is Theorem 12.3 in [Fei82]. In fact A is complete with respect to the topology defined by I and an induction shows that one may recursively lift e to A/I^m for all m, and hence to A by continuity. $\square$

Proposition 2.8. *Let $U \subset X$ be an open union of strata. Then given an indecomposable parity complex $\mathcal{P}$ on X, its restriction to U is either zero or indecomposable.*

Proof. Suppose that $\mathcal{P}$ has non-zero restriction to U. Proposition 2.4 shows that restriction yields a surjection

$$\mathrm{End}(\mathcal{P}) \twoheadrightarrow \mathrm{End}(\mathcal{P}_{|U})$$

whose kernel is contained in the maximal ideal of $\mathrm{End}(\mathcal{P})$ which is its Jacobson radical because $\mathrm{End}(\mathcal{P})$ is local. By Proposition 2.7 we may lift idempotents from $\mathrm{End}(\mathcal{P}_{|U})$ to $\mathrm{End}(\mathcal{P})$. Hence the identity is the only idempotent in $\mathrm{End}(\mathcal{P}_{|U})$ and $\mathcal{P}_{|U}$ is indecomposable. $\square$

Theorem 2.9. *Let $\mathcal{F}$ be an indecomposable parity complex. Then*
   (1) *the support closure of $\mathcal{F}$ is irreducible, hence of the form $\overline{X}_\lambda$, for some $\lambda \in \Lambda$;*
   (2) *the restriction $i_\lambda^* \mathcal{F}$ is isomorphic to $\mathcal{L}[m]$, for some indecomposable object $\mathcal{L}$ in $\mathrm{Loc}_\mathrm{f}(X_\lambda)$ and some integer m;*
   (3) *any indecomposable parity complex supported on $\overline{X}_\lambda$ and extending $\mathcal{L}[m]$ is isomorphic to $\mathcal{F}$.*

Proof. Suppose for contradiction that X_λ and X_μ are open in the support of $\mathcal{F}$, where λ and μ are two distinct elements of Λ. Let $U = X_\lambda \cup X_\mu$. Then $\mathcal{F}_{|U} \simeq \mathcal{F}_{X_\lambda} \oplus \mathcal{F}_{X_\mu}$, contradicting Proposition 2.8. This proves (1). The assertion (2) also follows from Proposition 2.8.

Now let $\mathcal{G}$ be an indecomposable parity complex supported on $\overline{X}_\lambda$ and such that $i_\lambda^* \mathcal{G} \simeq \mathcal{L}[m]$. By composition, we have inverse isomorphisms $\alpha : i_\lambda^* \mathcal{F} \simeq i_\lambda^* \mathcal{G}$ and $\beta : i_\lambda^* \mathcal{G} \simeq i_\lambda^* \mathcal{F}$. By Corollary 2.6, the restriction $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(i_\lambda^* \mathcal{F}, i_\lambda^* \mathcal{G})$ is surjective. So we can lift α and β to morphisms $\tilde{\alpha} : \mathcal{F} \to \mathcal{G}$ and $\tilde{\beta} : \mathcal{G} \to \mathcal{F}$. By Corollary 2.6 again, the restriction $\mathrm{End}(\mathcal{F}) \to \mathrm{End}(i_\lambda^* \mathcal{F})$ is surjective. Since $\tilde{\beta} \circ \tilde{\alpha}$ restricts to $\beta \circ \alpha = \mathrm{Id}$, the locality of $\mathrm{End}(\mathcal{F})$ implies that $\tilde{\beta} \circ \tilde{\alpha}$ is invertible itself, and similarly for $\tilde{\alpha} \circ \tilde{\beta}$. This proves (3). $\square$

Remark 2.10. If k is a field, one can replace "indecomposable" by "simple" in (2), due to our assumptions on X.

We now introduce the main character of our paper.

Definition 2.11. A **parity sheaf** is an indecomposable parity complex with X_λ open in its support and extending $\mathcal{L}[d_\lambda]$ for some indecomposable $\mathcal{L} \in \mathrm{Loc}_f(X_\lambda)$. When such a complex exists, we will denote it by $\mathcal{E}(\lambda, \mathcal{L})$. If $\mathcal{L}$ is constant we will write $\mathcal{E}(\lambda, k)$, or $\mathcal{E}(\lambda)$ if the coefficient ring k is clear from the context. We call $\mathcal{E}(\lambda, \mathcal{L})$ the parity sheaf associated to the pair $(\lambda, \mathcal{L})$. [3]

Thus any indecomposable parity complex is isomorphic to some shift of a parity sheaf $\mathcal{E}(\lambda, \mathcal{L})$. The reason for the normalisation chosen in the last definition is explained by the following proposition:

Proposition 2.12. *For λ in Λ and $\mathcal{L}$ in $\mathrm{Loc}_f(X_\lambda)$, we have*
$$\mathbb{D}\mathcal{E}(\lambda, \mathcal{L}) \simeq \mathcal{E}(\lambda, \mathcal{L}^\vee).$$

Proof. The definition of parity sheaf is clearly self-dual, so $\mathbb{D}\mathcal{E}(\lambda, \mathcal{L})$ is a parity sheaf. Moreover, it is supported on $\overline{X}_\lambda$ and extends $\mathcal{L}^\vee[d_\lambda]$. By the uniqueness theorem, it is isomorphic to $\mathcal{E}(\lambda, \mathcal{L}^\vee)$.   □

Remark 2.13. When they exist, the parity sheaves are often perverse, and in characteristic zero they usually are the intersection cohomology sheaves. We will see many examples below, and also examples of parity sheaves that are not perverse (see Proposition 4.16).

2.3. **Even resolutions and existence.** In the last subsection, we saw that, if we fix a stratum X_λ and a local system $\mathcal{L}$ on it, then there is at most one parity sheaf $\mathcal{F}$ such that $\overline{\mathrm{supp}\,\mathcal{F}} = \overline{X}_\lambda$ and $i_\lambda^* \mathcal{F} \simeq \mathcal{L}[d_\lambda]$, up to isomorphism.

Now we will give a sufficient condition for the existence of such a parity sheaf. For simplicity, we will only consider the case where $\mathcal{L}$ is trivial (however, $\mathcal{F}$ is allowed to have non-constant local systems on lower strata).

Let us recall the following definition from [GM88, 1.6].

Definition 2.14. Let $X = \sqcup_{\lambda \in \Lambda_X} X_\lambda$ and $Y = \sqcup_{\mu \in \Lambda_Y} Y_\mu$ be stratified varieties. A morphism $\pi : X \to Y$ is **stratified** if

    (1) for all $\mu \in \Lambda_Y$, the inverse image $\pi^{-1}(Y_\mu)$ is a union of strata;
    (2) for each X_λ above Y_μ, the induced morphism $\pi_{\lambda,\mu} : X_\lambda \to Y_\mu$ is a submersion with smooth fibre $F_{\lambda,\mu} = \pi_{\lambda,\mu}^{-1}(y_\mu)$, where y_μ is some chosen base point in Y_μ.

---

[3] More generally, if $\mathcal{L}$ not indecomposable, we will let $\mathcal{E}(\lambda, \mathcal{L})$ denote the direct sum of the parity sheaves associated to the direct summands of $\mathcal{L}$.

Definition 2.15. A stratified morphism π is said to be **even** if for all λ, μ as above, and for any local system $\mathcal{L}$ in $\mathrm{Loc}_{\mathrm{f}}(X_\lambda)$, the cohomology of the fibre $F_{\lambda,\mu}$ with coefficients in $\mathcal{L}_{|F_{\lambda,\mu}}$ is torsion free and concentrated in even degrees.

A class of even morphisms, which are common in geometric representation theory and a motivation for the definition, are those whose stratifications induce "affine pavings" on the fibres — meaning that all of the $F_{\lambda,\mu}$ are affine spaces. Examples of such maps arise in the study of flag manifolds and will be discussed in section 4.

Proposition 2.16. *The direct image of a ?-even (resp. odd) complex under a proper even morphism is again ?-even (resp. odd). The direct image of a parity complex under such a map is parity.*

Proof. First note that if the statement is true for all !-even complexes, then it is true for all !-odd complexes (by shifting). It would then also true for any $*$-parity complex because $\mathcal{F}$ is $*$-even (resp. odd) if and only if $\mathbb{D}\mathcal{F}$ is !-even (resp. odd) by remark 2.3 and

$$\pi_* \mathbb{D}\mathcal{F} \cong \mathbb{D}\pi_! \mathcal{F} = \mathbb{D}\pi_* \mathcal{F}$$

as π is proper.

The second sentence of the theorem is an immediate corollary of the first. Thus it remains to show the first statement for !-even sheaves.

Let $F_\mu = \pi^{-1}(y_\mu)$ denote the fibre and $\Lambda_X(\mu)$ denote the indices of strata in $\pi^{-1}(Y_\mu)$. Thus we have $\pi^{-1}(Y_\mu) = \bigsqcup_{\lambda \in \Lambda_X(\mu)} X_\lambda$. Moreover, $F_\mu = \bigsqcup_{\lambda \in \Lambda_X(\mu)} F_{\lambda,\mu}$ be the fibre of π over y_μ, where, as above, $F_{\lambda,\mu} = F_\mu \cap X_\lambda$. We have the following diagram with Cartesian squares:

$$
\begin{array}{ccccc}
& & i & & \\
F_\mu & \longrightarrow & \pi^{-1}(Y_\mu) & \longrightarrow & X \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\{y_\mu\} & \longrightarrow & Y_\mu & \longrightarrow & Y \\
& & i & &
\end{array}
$$

Note that we abuse notation and denote by i both inclusions $F_\mu \hookrightarrow X$ and $\{y_\mu\} \hookrightarrow Y$, and similarly π denotes any vertical arrow in the above diagram.

Let $\mathcal{P} \in D(X)$ be a !-even complex. We wish to show that $\pi_* \mathcal{P}$ is !-parity. This is equivalent to $i^! \pi_* \mathcal{P}$ being !-parity for all μ. By the proper base change theorem,

$$i^! \pi_* \mathcal{P} \cong \pi_* i^! \mathcal{P} \cong \mathbb{H}^\bullet(F_\mu, i^! \mathcal{P}).$$

We will use the local-global spectral sequence to show that this latter cohomology group is parity.

Choose a filtration $\emptyset = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_m$ of the fibre $\pi^{-1}(y_\mu)$ by closed subsets such that, for all p,

$$F_p \setminus F_{p-1} = F_{\lambda_p, \mu} \qquad \text{for some } \lambda_p \in \Lambda_X(\mu).$$

For all p, let $i_p : F_p \setminus F_{p-1} = F_{\lambda_p, \mu} \hookrightarrow X$ denote the inclusion. The local-global spectral sequence (see for example the proof of Proposition 3.4.4 in [Soe01]) has the form

$$E_1^{p,q} = \mathbb{H}^{p+q}(F_{\lambda_p, \mu}, i_p^! \mathcal{P}) \Rightarrow \mathbb{H}^{p+q}(F_\mu, i^! \mathcal{P})$$

We may express i_p as the composition

$$F_{\lambda_p, \mu} \hookrightarrow X_{\lambda_p} \overset{i_{\lambda_p}}{\hookrightarrow} X$$

where $F_{\lambda_p, \mu}$ is smooth subvariety of X_{λ_p}. It follows that, if d is the (complex) codimension of $F_{\lambda_p, \mu}$ in X_{λ_p}, we have

$$i_p^! \mathcal{P}[2d] \cong (i_{\lambda_p}^! \mathcal{P})_{|F_{\lambda_p, \mu}}.$$

As $\mathcal{P}$ is !-even by assumption, $i_p^! \mathcal{P}$ is isomorphic to a direct sum of local systems in even degree, all obtained by restriction from torsion free local systems on X_{λ_p}.

By assumption, the cohomology of F_μ with values in such local systems is free and concentrated in even degree and so the above spectral sequence degenerates for parity reasons, whence the claim. $\qquad\square$

One practical application of the previous result is that the existence of parity sheaves follows from the existence of even resolutions:

Corollary 2.17. *Suppose that there exists an even stratified proper morphism*

$$\pi : \widetilde{X_\lambda} \to \overline{X}_\lambda \subset X$$

which is an isomorphism over X_λ.

If there exists a parity sheaf on $\widetilde{X_\lambda}$ whose restriction to $\pi^{-1}(X_\lambda) \cong X_\lambda$ is a shifted local system $\mathcal{L}[d_\lambda]$ on X_λ, then there exists a parity sheaf $\mathcal{P}$ on X satisfying

(1) $\overline{\operatorname{supp} \mathcal{P}} = \overline{X}_\lambda$;
(2) $\mathcal{P}_{|X_\lambda} = \mathcal{L}[d_\lambda]$.

In particular, if π is a resolution of singularities, then the above holds for $\mathcal{L} = \underline{k}_\lambda$, since in this case $\underline{k}_{\widetilde{X_\lambda}}[d_\lambda]$ is parity.

Proof. This result follows from the previous proposition. $\qquad\square$

Remark 2.18. In the above situation, if k is a field of characteristic zero then the Decomposition Theorem asserts that the direct image $\pi_* \underline{k}_{\widetilde{X_\lambda}}[d_{\widetilde{X}}]$ is isomorphic to a direct sum of intersection cohomology complexes. We have already seen that $\pi_* \underline{k}_{\widetilde{X_\lambda}}[d_{\widetilde{X}}]$ is parity, and it follows that the parity sheaf $\mathcal{E}(\lambda)$ is isomorphic to the intersection cohomology complex $\mathbf{IC}(\lambda)$. Then it follows that $\mathcal{E}(\lambda, \mathbb{F}_p) \cong \mathbf{IC}(\lambda, \mathbb{F}_p)$ for almost all p, however the question as to whether this holds for a fixed p is very difficult in general.

2.4. **Modular reduction of parity sheaves.** Let $k \to k'$ be a ring homomorphism. In this section, we will consider the behaviour of parity sheaves under the extension of scalars functor, which we denote by

$$k'(-) := k' \overset{L}{\otimes}_k - : D(X; k) \to D(X; k')$$

Lemma 2.19. *Suppose that $\mathcal{F} \in D(X, k)$ is ?-even (resp. odd), then $k'(\mathcal{F})$ is ?-even (resp. odd). In particular, if $\mathcal{F}$ is a parity complex, then so is $k'(\mathcal{F})$.*

Proof. It suffices to prove the ?-even case. It is equivalent to show that the ?-restriction of $k'(\mathcal{F})$ to each point is even. For any complex $\mathcal{F} \in D(X; k)$ we have isomorphisms $i^?(k'(\mathcal{F})) \cong k'(i^?\mathcal{F})$ for i the inclusion of a point (this follows, for example, from Propositions 2.3.5 and 2.5.13 of [KS94]). By definition, if $\mathcal{F}$ is ?-even then the cohomology of $i^?\mathcal{F}$ vanishes in odd degree and is free and therefore flat. Thus $k'(i^?_\lambda\mathcal{F}) = k' \otimes i^?_\lambda\mathcal{F}$ and $k'(\mathcal{F})$ is ?-even. $\square$

We will now restrict our attention to the case when $k = \mathbb{O}$ and $k' = \mathbb{F}$ and $k \to k'$ is the residue map. Recall that $\mathbb{O}$ denotes a complete discrete valuation ring and $\mathbb{F}$ its residue field. We assume that (2.1) and (2.2) holds for $D(X, \mathbb{O})$.

In this case, $k'(-)$ is the modular reduction functor:

$$\mathbb{F}(-) := \mathbb{F} \overset{L}{\otimes}_{\mathbb{O}} - : D(X, \mathbb{O}) \to D(X, \mathbb{F})$$

First we claim that in this situation, the implication of the previous theorem is in fact an equivalence.

Proposition 2.20. *A complex $\mathcal{F} \in D(X; \mathbb{O})$ is ?-even (resp, ?-odd or parity) if and only if $\mathbb{F}\mathcal{F}$ is.*

Remark 2.21. This proposition is analogous to [Ser67, Prop. 42(a)], which states, for a finite group G, a $\mathbb{O}[G]$-module is projective if and only if its reduction is a projective $\mathbb{F}[G]$-module.

Proof. Having proved "only if" it remains to prove "if".

Again, it suffices to check the ?-restrictions to points. As before, we have $i^?(\mathbb{F}\mathcal{F}) \cong \mathbb{F}(i^?\mathcal{F})$. This time, we wish to show that if $i^?(\mathbb{F}\mathcal{F})$ vanishes in odd degree, then $i^?\mathcal{F}$ does too and is a free $\mathbb{O}$-module.

The derived category over a point $D(\mathrm{pt}; \mathbb{O})$ (resp. $D(\mathrm{pt}; \mathbb{F})$) is equivalent to the derived category of $\mathbb{O}$-modules, $D(\mathrm{Mod}_{\mathbb{O}})$ (resp. $\mathbb{F}$-vector spaces, $D(\mathrm{Vect}_{\mathbb{F}})$). The ring $\mathbb{O}$ is hereditary, which implies that any object in $D(\mathrm{Mod}_{\mathbb{O}})$ is isomorphic to its cohomology. Using this one can show that if $i^?\mathcal{F}$ has torsion, then $\mathbb{F}i^?\mathcal{F}$ has cohomology concentrated in two consecutive degrees. Hence $i^?\mathcal{F}$ is a free $\mathbb{O}$-module and is even. $\square$

Proposition 2.22. *If $\mathcal{E} \in D(X, \mathbb{O})$ is a parity sheaf, then $\mathbb{F}\mathcal{E}$ is also a parity sheaf. In other words, for any $\mathcal{L} \in \mathrm{Loc}_{\mathrm{f}}(X_\lambda, \mathbb{O})$, we have*

$$\mathbb{F}\mathcal{E}(\lambda, \mathcal{L}) \cong \mathcal{E}(\lambda, \mathbb{F}\mathcal{L}).$$

Proof. For local systems $\mathcal{L}, \mathcal{L}' \in \mathrm{Loc}_{\mathrm{f}}(X_\lambda, \mathbb{O})$ on X_λ, we have

$$(2.5) \qquad \mathbb{F} \otimes \mathrm{Hom}(\mathcal{L}, \mathcal{L}') \cong \mathrm{Hom}(\mathbb{F}\mathcal{L}, \mathbb{F}\mathcal{L}')$$

Now consider $\mathcal{F}$ (resp. $\mathcal{G}$) in $D(X, \mathbb{O})$ which is $*$- (resp. !-) parity. Then using Proposition 2.4 and (2.5) above for $\mathcal{L} = i_\lambda^*\mathcal{F}, \mathcal{L}' = i_\lambda^!\mathcal{G}$, we see that the natural morphism yields an isomorphism:

$$(2.6) \qquad \mathbb{F} \otimes \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{F}\mathcal{F}, \mathbb{F}\mathcal{G}).$$

Finally, let $\mathcal{F} = \mathcal{G} = \mathcal{E} \in D(X; \mathbb{O})$ be a parity sheaf. It follows that we have a surjection

$$\mathrm{End}(\mathcal{E}) \twoheadrightarrow \mathrm{End}(\mathbb{F}\mathcal{E}).$$

The result then follows from Proposition 2.7. $\square$

Remark 2.23. This is a partial analogue to [Ser67, Prop. 4.2(b)], which states that for each projective $\mathbb{F}[G]$-module F there exists a unique (up to isomorphism) projective $\mathbb{O}[G]$-module whose reduction is isomorphic to F.

2.5. **Torsion primes.**

Our assumptions (2.1) and (2.2) on the space X are quite strict. If we work in the equivariant setting, they might not even be satisfied when X is a single point. However, once we invert a set of prime numbers in k depending on the group G, called the torsion primes, the conditions are satisfied at least for a point. In the examples we consider in this paper, we show that these are the only primes for which our assumptions do not hold. In this subsection, we recall from [Ste75] some facts about torsion primes.

Let Φ be a root system in a rational vector space V. Then we denote by $\Phi^\vee$ the dual root system in V^*, by $Q(\Phi)$ the lattice generated by

the roots, by $P(\Phi)$ the weight lattice (consisting of the elements $\lambda \in V$ which have integral pairings with coroots), and similarly by $Q(\Phi^\vee)$ and $P(\Phi^\vee)$ the coroot lattice and the coweight lattice.

Definition 2.24. A prime p is a **torsion prime** for a root system Φ if $Q(\Phi^\vee)/Q(\Phi_1^\vee)$ has p-torsion for some $\mathbb{Z}$-closed subsystem Φ_1 of Φ.

For a moment, assume that Φ is irreducible. Let us choose some basis Δ of Φ. Let then $\tilde\alpha$ denote the highest root of Φ, and let $\tilde\alpha^\vee = \sum_{\alpha \in \Delta} n_\alpha^\vee \alpha^\vee$ be the decomposition of the corresponding coroot into simple coroots. Finally, let $n^\vee$ denote the maximum of the $n_\alpha^\vee$.

Theorem 2.25. *If Φ is irreducible and p is a prime, then the following conditions are equivalent:*

    (1) *p is a torsion prime for Φ;*
    (2) *$p \le n^\vee$;*
    (3) *p is one of the $n_\alpha^\vee$;*
    (4) *p divides one of the $n_\alpha^\vee$.*

Thus the torsion primes of the irreducible root systems are given by the following table:

A_n, C_n	$B_n (n \ge 3), D_n, G_2$	E_6, E_7, F_4	E_8
none	2	2, 3	2, 3, 5

The torsion primes for a reducible root system Φ are the primes which are torsion primes for some irreducible component of Φ.

Now let G be a reductive group. A reductive subgroup of G is called regular if it contains a maximal torus T of G. Its root system may then be identified with some subsystem of that of G.

Definition 2.26. A prime p is a **torsion prime** for G if $\pi_1(G')$ has p-torsion, for some regular reductive subgroup G' of G whose root system is integrally closed in that of G.

(Actually, the requirement that the subsystem should be integrally closed in the root system of G is irrelevant here, because our base field is $\mathbb{C}$. In [Ste75], this condition is included for some rare cases involving non simply-laced groups over a field of characteristic 2 or 3.)

A reductive group G has the same torsion primes as its derived subgroup. Moreover, the torsion primes of any regular reductive subgroup G' of G are among those of G.

The torsion primes of G are those of its root system Φ together with those of its fundamental group.

Now let us recall a theorem of Borel [Bor61, RS65, Dem73, Kac85].

Theorem 2.27. *The following conditions are equivalent:*

(1) *the prime p is not a torsion prime for G;*
(2) *the cohomology $H^*(G, \mathbb{Z})$ of G has no p-torsion;*
(3) *the cohomology $H^*(B_G, \mathbb{Z})$ of the classifying space of G has no p-torsion.*

Moreover, in that case, $H^*(G, k)$ is an exterior algebra with generators of odd dimensions and is k-free, while $H^*(B_G, k)$ is a polynomial algebra on corresponding generators of one higher dimension.

To conclude, if p is not a torsion prime for G, then $H_G^*(\mathrm{pt}, k)$ is even and torsion-free, and the same is true for any regular reductive subgroup G' of G.

2.6. **Ind-varieties.** In this section we comment on how the results of this section generalise straightforwardly to the slightly more general setting of ind-varieties. Recall that an ind-variety X is a topological space, together with an filtration

$$X_0 \subset X_1 \subset X_2 \subset \dots$$

such that $X = \cup X_n$, each X_n is a complex algebraic variety, and the inclusions $X_n \hookrightarrow X_{n+1}$ are closed embeddings. We will always assume that each X_n carries the classical topology and equip X with the final topology with respect to all inclusions $X_n \hookrightarrow X$. By a stratification of X we mean a stratification of each X_n such that the inclusions preserve the strata. We will also consider the case where X is acted upon by a linear algebraic group G, by which we mean that G acts algebraically on each X_n and the inclusions $X_n \hookrightarrow X_{n+1}$ are G-equivariant. For the basic properties of ind-varieties we refer the reader to [Kum02, Chapter 4].

Now fix a complete local ring k and let

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

be a stratified ind-variety, or an ind-G-variety with G-stable stratification for some linear algebraic group G. We write $D(X)$ for the full subcategory of the bounded (equivariant) derived category of sheaves of k-vector spaces consisting of objects $\mathcal{F}$ such that:

(1) the support of $\mathcal{F}$ is contained in X_n for some n;
(2) the cohomology sheaves of $\mathcal{F}$ are constructible with respect to the stratification.

We assume that 2.1 holds for the strata of X_λ. The notion of parity still makes sense and it is immediate that the analogue of Theorem 2.9 applies. In particular, given any (equivariant) indecomposable local system $\mathcal{L} \in \mathrm{Loc}_{\mathrm{f}}(X_\lambda)$ there is, up to isomorphism and shifts, at most

one indecomposable parity sheaf $\mathcal{E}(\lambda) \in D(X)$ supported on $\overline{X}_\lambda$ and extending $\mathcal{L}[d_\lambda]$.

In what follows we will refer without comment to results which we have proved previously for varieties, but where an obvious analogue holds for ind-varieties.

3. Semi-small maps and intersection forms

In their proof of the decomposition theorem for semi-small maps [dCM02], de Cataldo and Migliorini highlighted the crucial role played by intersection forms associated to the strata of the target: a certain splitting implied by the decomposition theorem is equivalent to these forms being non-degenerate. Then they prove the non-degeneracy using techniques from Hodge theory.

In our situation, where we consider modular coefficients, these forms may be degenerate. In this section we explain how the non-degeneracy of these forms, together with the semi-simplicity of certain local systems, provide necessary and sufficient conditions for the Decomposition Theorem to hold in positive characteristic. For this, we do not have to assume that X satisfies (2.1) or (2.2). If the semi-small map we consider is even, then the direct image will be a direct sum of parity sheaves. Assuming now the parity conditions on the strata (ensuring the uniqueness of parity sheaves), we will see that even if the decomposition fails, one can still use intersection forms to determine the multiplicities of parity sheaves that occur in the direct image of the constant sheaf.

In Section 3.1 we recall the definition and basic properties of intersection forms on Borel-Moore homology. In Section 3.2 we relate these intersection forms to the failure of the Decomposition Theorem for semi-small maps in characteristic p. In Section 3.3 we examine the decomposition of the direct image into parity sheaves in the case where the morphism is even.

We work with a ring of coefficients k as in 2.1. However, to simplify the exposition, in sections 3.2 and 3.3, we will assume that k is a field, and in Section 3.4, we will explain what has to be modified in the case where k is a complete discrete valuation ring $\mathbb{O}$.

All cohomology groups are assumed to have values in k and, as always, dimension always refers to the complex dimension unless otherwise stated.

3.1. **Borel-Moore homology and intersection forms.** In this subsection we recall some basic properties of Borel-Moore homology and intersection forms. For more details the reader is referred to [Ful93] or [CG97].

For any variety X we let $a_X : X \to$ pt denote the projection to a point. The dualising sheaf on X is $\omega_X := a_X^! \underline{k}_{\mathrm{pt}}$. One may define the Borel-Moore homology of X to be

$$H_i^{BM}(X) = H^{-i}(a_{X*}a_X^!\underline{k}_{\mathrm{pt}}) = \mathrm{Hom}(\underline{k}_X, \omega_X[-i]).$$

Let Y be a smooth and connected variety of dimension n. As Y has a canonical orientation (after choosing an orientation on $\mathbb{C}$) we have an isomorphism $\mu_Y : \underline{k}_Y \xrightarrow{\sim} \omega_Y[-2n]$. If we regard μ_Y as an element of $H_{2n}^{BM}(Y)$ it is called the **fundamental class**. Even if Y is singular of dimension n, $H_{2n}^{BM}(Y)$ is still freely generated by the fundamental classes of the irreducible components of Y of maximal dimension.

Now suppose that $F \xhookrightarrow{i} Y$ is a closed embedding of a variety F into a smooth variety of dimension n. For all m we have a canonical isomorphism

$$H_m^{BM}(F) \cong H^{2n-m}(Y, Y - F).$$

Recall that their exists a cup product on relative cohomology. We may use this to define an intersection form of F inside Y:

$$
\begin{array}{ccccc}
H_p^{BM}(F) & \otimes & H_q^{BM}(F) & \to & H_{p+q-2n}^{BM}(F) \\
\downarrow{\sim} & & \downarrow{\sim} & & \downarrow{\sim} \\
H^{2n-p}(Y, Y-F) & \otimes & H^{2n-q}(Y, Y-F) & \xrightarrow{\cup} & H^{4n-p-q}(Y, Y-F)
\end{array}
$$

Note that this product depends on the inclusion $F \hookrightarrow Y$. It is particularly interesting when F is connected, proper and half-dimensional inside Y. In this case we obtain an intersection form

$$H_{\mathrm{top}}^{BM}(F) \otimes H_{\mathrm{top}}^{BM}(F) \to H_0^{BM}(F) = k$$

where top denotes the real dimension of F. From the above comments, $H_{\mathrm{top}}^{BM}(F)$ has a basis given by the irreducible components of maximal dimension of F. It also follows that this intersection form over any ring is obtained by extension of scalars from the corresponding form over $\mathbb{Z}$.

The effect of forming the Cartesian product with a smooth and contractible space on Borel-Moore homology is easy to describe (and will be needed below). If U is an open contractible subset of $\mathbb{C}^m$, then for any $i \in \mathbb{Z}$, we have canonical isomorphisms

$$H_i^{BM}(X \times U) \cong H_{i-2m}^{BM}(X).$$

These isomorphisms are compatible with the intersection forms of $F \hookrightarrow Y$ and $F \times U \hookrightarrow Y \times U$.

3.2. **The Decomposition Theorem for semi-small maps.** In this Section, we do not assume that the stratified variety X satisfies the conditions (2.1) or (2.2). Moreover, in this Section and the next, we assume that k is a field, leaving the case where k is $\mathbb{O}$ to Section 3.4.

In 3.2.1 we explain how multiplicities of indecomposable objects in a k-linear Krull-Schmidt category are encoded in certain bilinear forms. In 3.2.2 we examine the splitting at the "most singular point" and relate it to an intersection form on the fibre. This is used in 3.2.3 to relate the Decomposition Theorem to the non-degeneracy of the intersection forms attached to each stratum.

3.2.1. *Bilinear forms and multiplicities in Krull-Schmidt categories.* We assume that k is a field. Let H and H' be finite dimensional k-vector spaces and consider a bilinear map

$$B : H \times H' \to k.$$

We define:

$$^{\perp}B := \{\alpha \in H \mid B(\alpha, \beta) = 0 \text{ for all } \beta \in H'\},$$
$$B^{\perp} := \{\beta \in H' \mid B(\alpha, \beta) = 0 \text{ for all } \alpha \in H\}.$$

Then B induces a perfect pairing

$$H/^{\perp}B \times H'/B^{\perp} \to k.$$

and we have equalities

$$\dim(H/^{\perp}B) = \operatorname{rank} B = \dim(H'/B^{\perp}).$$

If $H = H'$ and B is a symmetric bilinear form then we write $\operatorname{rad} B$ instead of $^{\perp}B = B^{\perp}$.

Now let $\mathcal{C}$ be a Krull-Schmidt k-linear category with finite dimensional Hom-spaces. Let $a \in \mathcal{C}$ denote an indecomposable object. Given any object $x \in \mathcal{C}$ we can write $x \simeq a^{\oplus m} \oplus y$ such that a is not a direct summand in y. The integer m is called the **multiplicity** of a in x. This multiplicity is well-defined because $\mathcal{C}$ is Krull-Schmidt.

Now assume that $\operatorname{End}(a) = k$. Composition gives us a pairing:

$$(3.1) \qquad \begin{array}{rccc} B &:& \operatorname{Hom}(a, x) \times \operatorname{Hom}(x, a) & \longrightarrow & \operatorname{End}(a) = k \\ & & (\alpha, \beta) & \longmapsto & \beta \circ \alpha. \end{array}$$

Lemma 3.1. *The multiplicity of a in x is equal to the rank of B.*

Proof. Choose an isomorphism $\phi : x \xrightarrow{\sim} a^{\oplus m} \oplus y$. Inclusion and projection define subspaces $\operatorname{Hom}(a, y) \subset \operatorname{Hom}(a, x)$ and $\operatorname{Hom}(y, a) \subset \operatorname{Hom}(x, a)$. We will show that

$$(3.2) \qquad \operatorname{Hom}(a, y) = {}^{\perp}B \qquad \text{and} \qquad \operatorname{Hom}(y, a) = B^{\perp}.$$

Thus these subspaces do not depend on the choice of ϕ, and this will show that the multiplicity of a in x is equal to the rank of B.

The isomorphism ϕ induces isomorphisms

$$\mathrm{Hom}(a, x) \simeq \mathrm{Hom}(a, a^{\oplus m}) \oplus \mathrm{Hom}(a, y),$$
$$\mathrm{Hom}(x, a) \simeq \mathrm{Hom}(a^{\oplus m}, a) \oplus \mathrm{Hom}(y, a).$$

For α in $\mathrm{Hom}(a, x)$ and β in $\mathrm{Hom}(x, a)$, we write $\alpha = \alpha_1 \oplus \alpha_2$ and $\beta = \beta_1 \oplus \beta_2$ for the corresponding decompositions. Thus we have $B(\alpha, \beta) = \beta_1 \alpha_1 + \beta_2 \alpha_2$.

We actually have

$$(3.3) \qquad\qquad\qquad \beta_2 \alpha_2 = 0.$$

Otherwise, since $\mathrm{End}(a) = k$, we could assume that $\beta_2 \alpha_2 = \mathrm{Id}$, in which case $e := \alpha_2 \beta_2$ would be an idempotent in $\mathrm{End}(y)$, and a would be a direct summand of y, contrary to our assumption.

Thus we have $B(\alpha, \beta) = \beta_1 \alpha_1$ for all α and β. Hence we have inclusions

$$(3.4) \qquad \mathrm{Hom}(a, y) \subset {}^{\perp}B \qquad \text{and} \qquad \mathrm{Hom}(y, a) \subset B^{\perp},$$

and B induces a bilinear form

$$\widetilde{B} : \mathrm{Hom}(a, a^{\oplus m}) \times \mathrm{Hom}(a^{\oplus m}, a) \longrightarrow \mathrm{End}(a) = k.$$

Now $\mathrm{Hom}(a, a^{\oplus m}) \simeq \mathrm{End}(a)^{\oplus m} \simeq k^{\oplus m}$ and similarly $\mathrm{Hom}(a^{\oplus m}, a) \simeq k^{\oplus m}$. With these identifications $\widetilde{B}$ is just the standard bilinear form on $k^{\oplus m}$, hence it is non-degenerate, and we have equalities in (3.4). $\square$

Let us assume further that $\mathcal{C}$ is equipped with a duality

$$D : \mathcal{C} \xrightarrow{\sim} \mathcal{C}^{op}$$

and we have isomorphisms $a \xrightarrow{\sim} Da$ and $x \xrightarrow{\sim} Dx$. Then, using these isomorphisms, we may identify $\mathrm{Hom}(a, x)$ and $\mathrm{Hom}(x, a)$. In which case the composition (3.1) is given by a symmetric bilinear form on $H = \mathrm{Hom}(a, x) = \mathrm{Hom}(x, a)$ and the multiplicity of a in x is equal to the rank of this form.

3.2.2. *Splitting at the most singular point.* Consider a proper surjective semi-small morphism

$$\pi : \widetilde{X} \to X$$

with $\widetilde{X}$ smooth and assume that $\widetilde{X}$ (and hence X) is of dimension $2n$. Because π is semi-small the dimension of each fibre is bounded by n.

Assume that there exists a point $s \in X$ such that $F := \pi^{-1}(s)$ has dimension n and form the Cartesian diagram:

$$
\begin{array}{ccc}
F & \xrightarrow{\;\tilde{i}\;} & \widetilde{X} \\
\downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\
\{s\} & \xrightarrow{\;i\;} & X
\end{array}
$$

Let B_s denote the intersection form on $H^{BM}_{\mathrm{top}}(F)$ associated to the inclusion $F \hookrightarrow \widetilde{X}$.

Proposition 3.2. *The multiplicity of $i_* \underline{k}_s$ as a direct summand of $\pi_* \underline{k}_{\widetilde{X}}[2n]$ is equal to the rank of B_s.*

Proof. By the discussion of the previous section if B denotes the pairing given by composition:

$$
B : \mathrm{Hom}(i_* \underline{k}_s, \pi_* \underline{k}_{\widetilde{X}}[2n]) \times \mathrm{Hom}(\pi_* \underline{k}_{\widetilde{X}}[2n], i_* \underline{k}_s) \to k
$$

Then the multiplicity of $i_* \underline{k}_s$ in $\pi_* \underline{k}_{\widetilde{X}}[2n]$ is given by the rank of B. A string of adjunctions gives canonical identifications:

$$
\mathrm{Hom}(i_* \underline{k}_s, \pi_* \underline{k}_{\widetilde{X}}[2n]) = H^{BM}_{\mathrm{top}}(F),
$$
$$
\mathrm{Hom}(\pi_* \underline{k}_{\widetilde{X}}[2n], i_* \underline{k}_s) = H^{BM}_{\mathrm{top}}(F).
$$

Hence we are interested in a pairing

$$
(3.5) \qquad\qquad H^{BM}_{\mathrm{top}}(F) \times H^{BM}_{\mathrm{top}}(F) \to k.
$$

By the lemma below, this is the intersection form. The proposition then follows. $\qquad\square$

Lemma 3.3. *The pairing in (3.5) is the intersection form.*

Proof. By definition we may identify the intersection form with the cup product on relative cohomology. Let ω_1 and ω_2 be classes in $H^{2n}(\widetilde{X}, \widetilde{X} - F)$. We may represent them by morphisms

$$
\omega_1 : \underline{k}_{\widetilde{X}} \to \tilde{i}_! \tilde{i}^! \underline{k}_{\widetilde{X}}[2n] \qquad \omega_2 : \underline{k}_{\widetilde{X}} \to \tilde{i}_! \tilde{i}^! \underline{k}_{\widetilde{X}}[2n].
$$

Their cup product is the morphism

$$
\omega_1 \cup \omega_2 : \underline{k}_{\widetilde{X}} \to \tilde{i}_! \tilde{i}^! \underline{k}_{\widetilde{X}}[2n] \to \underline{k}_{\widetilde{X}}[2n] \to \tilde{i}_! \tilde{i}^! \underline{k}_{\widetilde{X}}[4n]
$$

where the middle morphism is the adjunction. By chasing the various identifications and adjunctions it is then routine to verify that this agrees with the above pairing given by composition. $\qquad\square$

The proof of Proposition 3.2 has the following corollary:

Corollary 3.4. *The natural morphism*

$$H^0(i^! \pi_* \underline{k}_{\widetilde{X}}[2n]) \to H^0(i^* \pi_* \underline{k}_{\widetilde{X}}[2n])$$

may be canonically identified with the morphism

$$H^{BM}_{\mathrm{top}}(F) \to H^{BM}_{\mathrm{top}}(F)^*$$

induced by the intersection form.

3.2.3. *A criterion for the Decomposition Theorem.* In this subsection we assume that X is a connected, equidimensional variety equipped with an algebraic stratification into connected strata

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda.$$

We write d_X for the dimension of X and, as usual, write d_λ and i_λ for the dimension and inclusion of X_λ respectively. We fix a smooth variety $\widetilde{X}$ and a stratified proper surjective semi-small morphism

$$f : \widetilde{X} \to X.$$

We want to understand when the perverse sheaf $f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ decomposes as a direct sum of intersection cohomology sheaves. This may be thought of as a global version of the previous section.

As we have assumed that the stratification of X is algebraic [CG97, 3.2.23], at each point $x \in X_\lambda$, we can choose a stratified slice N_λ to X. We obtain a Cartesian diagram with $\widetilde{N_\lambda}$ smooth:

(3.6)
$$\begin{array}{ccc} F_\lambda & \xrightarrow{\tilde{i}} & \widetilde{N_\lambda} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \{x\} & \xrightarrow{i} & N_\lambda \end{array}$$

Note that, as f is semi-small, the dimension of F is less than or equal to $\frac{1}{2}(d_X - d_\lambda)$. If equality holds we say that X_λ is **relevant** (see [BM83]).

Definition 3.5. The **intersection form associated to** X_λ is the intersection form on $H^{BM}_{d_X - d_\lambda}(F_\lambda)$ given by the inclusion $F_\lambda \hookrightarrow \widetilde{N_\lambda}$.

Note that $H^{BM}_{d_X - d_\lambda}(F)$ is non-zero if and only if X_λ is relevant. Using the discussion at the end of Section 3.1 it is straightforward to see that the above intersection form does not depend on the choice of x. For each λ, we define

(3.7)
$$\mathcal{L}_\lambda := \mathcal{H}^{-d_\lambda}(i_\lambda^! f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]).$$

Note that $\mathcal{L}_\lambda$ is a local system on X_λ which is non-zero if and only if X_λ is relevant. The aim of this subsection is to show:

Theorem 3.6. *Suppose that the intersection forms associated to all strata are non-degenerate. Then one has an isomorphism:*

$$f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus_{\lambda \in \Lambda} \mathbf{IC}(\mathcal{L}_\lambda).$$

In that case, the full Decomposition Theorem holds if and only if each local system $\mathcal{L}_\lambda$ is semi-simple.

Remark 3.7. It is a deep result of de Cataldo and Migliorini [dCM02] that in fact, these intersection forms are definite (and hence non-degenerate) over $\mathbb{Q}$. The semi-simplicity of each $\mathcal{L}_\lambda$ over a field of characteristic zero is much more straightforward (and is also pointed out in [dCM02]): For relevant strata X_λ the stalks of $\mathcal{L}_\lambda$ have a basis at any point $x \in X_\lambda$ consisting of the irreducible components of maximal dimension in the fibre $f^{-1}(s)$. The monodromy action permutes these components. Hence each local system factors through a representation of a finite group, and hence is semi-simple.

Remember that here, we do not assume condition (2.1) . Otherwise, the semi-simplicity of the local systems would be automatic.

Assume that X_λ is a closed stratum and set $\mathcal{F} := f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$. For any $\mathcal{L} \in \mathrm{Loc}_{\mathrm{f}}(X_\lambda)$ we are interested in the pairing

$$\mathrm{Hom}(i_{\lambda *}\mathcal{L}[d_\lambda], \mathcal{F}) \times \mathrm{Hom}(\mathcal{F}, i_{\lambda *}\mathcal{L}[d_\lambda]) \to \mathrm{End}(i_{\lambda *}\mathcal{L}[d_\lambda]).$$

Applying two adjunctions on each side, this is equivalent to determining the pairing

$$\mathrm{Hom}(\mathcal{L}, \mathcal{L}_\lambda) \times \mathrm{Hom}(\mathcal{L}_\lambda^\vee, \mathcal{L}) \to \mathrm{End}(\mathcal{L})$$

where, given morphisms

$$\mathcal{L} \xrightarrow{f} \mathcal{L}_\lambda \text{ and } \mathcal{L}_\lambda^\vee \xrightarrow{g} \mathcal{L}$$

their pairing is given by the composition

$$\mathcal{L} \xrightarrow{f} \mathcal{L}_\lambda \to i_\lambda^! \mathcal{F}[-d_\lambda] \to i_\lambda^* \mathcal{F}[-d_\lambda] \to \mathcal{L}_\lambda^\vee \xrightarrow{g} \mathcal{L}$$

where all morphisms except f and g are canonical. Hence it is important to understand the morphism

$$(3.8) \qquad\qquad D_\lambda : \mathcal{L}_\lambda \to \mathcal{L}_\lambda^\vee.$$

In the following lemma (and its proof), we use the notations in the diagram (3.6).

Lemma 3.8. *Given $x \in X_\lambda$ as above, the stalk of D_λ at x may be canonically identified with the morphism*

$$H_{d_X - d_\lambda}^{BM}(F_\lambda) \to H_{d_X - d_\lambda}^{BM}(F_\lambda)^*$$

induced by the intersection form associated to X_λ.

Proof. Without loss of generality we may assume that $X_\lambda = U$, $X = N_\lambda \times U$ and $\widetilde{X} = \widetilde{N_\lambda} \times U$, for some contractible open subset $U \subset \mathbb{C}^{d_\lambda}$. It follows that the stalk of D_λ may be identified with the morphism

$$\mathcal{H}^0(i^! \pi_* \underline{k}_{\widetilde{N_\lambda}}[d_{\widetilde{X}} - d_\lambda]) \to \mathcal{H}^0(i^* \pi_* \underline{k}_{\widetilde{N_\lambda}}[d_{\widetilde{X}} - d_\lambda])$$

in which case the result follows from Corollary 3.4. $\square$

Applying the adjunction $(- \otimes \mathcal{L}_\lambda, - \otimes \mathcal{L}_\lambda^\vee)$ to D_λ we obtain a morphism

$$B_\lambda : \mathcal{L}_\lambda \otimes \mathcal{L}_\lambda \to \underline{k}_\lambda$$

and it follows from the above lemma that the stalk of this morphism at each point $x \in X_\lambda$ is given by the intersection form on $H^{BM}_{d_X - d_\lambda}(\pi^{-1}(x))$.

Let j denote the open inclusion of the complement of X_λ. We are now in a position to prove:

Proposition 3.9. *We have that $i_{\lambda*}\mathcal{L}_\lambda$ is a direct summand of $f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ if and only if the intersection form associated to X_λ is non-degenerate. If this is the case we have an isomorphism*

$$f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \simeq i_{\lambda*}\mathcal{L}_\lambda[d_\lambda] \oplus j_{!*}j^* f_* \underline{k}_{\widetilde{X}}[d_X].$$

Note that Theorem 3.6 follows by a simple induction over the stratification.

Proof. The above discussion shows that $i_{\lambda*}\mathcal{L}_\lambda[d_\lambda]$ is a direct summand of $f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \iff D_\lambda$ is an isomorphism $\iff B_\lambda$ is non-degenerate $\iff$ the intersection form associated to X_λ is non-degenerate.

Now assume that $i_{\lambda*}\mathcal{L}_\lambda[d_\lambda]$ is a direct summand of $f_* \underline{k}_{\widetilde{X}}[d_X]$ and write $f_* \underline{k}_{\widetilde{X}}[d_X] \simeq i_{\lambda*}\mathcal{L}_\lambda[d_\lambda] \oplus \mathcal{F}$ for some perverse sheaf $\mathcal{F}$. Then $\mathcal{F}$ is necessarily self-dual because $f_* \underline{k}_{\widetilde{X}}[d_X]$ and $i_{\lambda*}\mathcal{L}_\lambda[d_\lambda]$ are. Also $\mathcal{H}^m(i_\lambda^* \mathcal{F}) = 0$ for $m \geq -d_\lambda$. Hence $\mathcal{F} \cong j_{!*}j^* f_* \underline{k}_{\widetilde{X}}[d_X]$ by the characterisation of $j_{!*}$ given in [BBD82, Proposition 2.1.9]. $\square$

3.3. **Decomposing parity sheaves.** In this section we keep the notation from the previous section and assume additionally that our stratified variety X satisfies (2.1) and so that the parity sheaf $\mathcal{E}(\lambda, \mathcal{L})$ corresponding to an irreducible local system $\mathcal{L} \in \mathrm{Loc}_\mathrm{f}(X_\lambda)$ is well-defined up to isomorphism if it exists.

We also assume that the semi-small morphism

$$f : \widetilde{X} \to X$$

is even (see Section 2.3). It follows that $f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ may be decomposed into a direct sum of indecomposable parity sheaves

$$f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus \mathcal{E}(\lambda, \mathcal{L})^{\oplus m(\lambda, \mathcal{L})}$$

and we would like to determine the multiplicities $m(\lambda, \mathcal{L})$.

Remark 3.10. We do not assume that $\mathcal{E}(\lambda, \mathcal{L})$ exist for all pairs $(\lambda, \mathcal{L})$. However, all the indecomposable summands of the direct image are indecomposable parity sheaves, so existence will follow for any pair for which the multiplicity is > 0. Moreover, by semi-smallness, these $\mathcal{E}(\lambda, \mathcal{L})$ will be perverse.

Recall the pairing

$$B_\lambda : \mathcal{L}_\lambda \otimes \mathcal{L}_\lambda \to \underline{k}_\lambda$$

introduced in the last section. We define the radical of B_λ fibrewise as in Section 3.2.1 and obtain in this way a sub-local system

$$\operatorname{rad} B_\lambda \subset \mathcal{L}_\lambda.$$

Theorem 3.11. *We have an isomorphism*

$$f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus \mathcal{E}(\lambda, \mathcal{L}_\lambda / \operatorname{rad} B_\lambda).$$

In particular, the multiplicity $m(\lambda, \mathcal{L})$ of an indecomposable parity sheaf $\mathcal{E}(\lambda, \mathcal{L})$ in $f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ is equal to the multiplicity of $\mathcal{L}$ in $\mathcal{L}_\lambda / \operatorname{rad} B_\lambda$.

Proof. By Proposition 2.8 it is enough to prove the theorem for a closed stratum $X_\lambda \subset X$. Note that we have an exact sequence of local systems

$$\operatorname{rad} B_\lambda \hookrightarrow \mathcal{L}_\lambda \twoheadrightarrow \mathcal{L}_\lambda / \operatorname{rad} B_\lambda.$$

Our assumption (2.1) guarantees that this sequence splits and hence

$$\mathcal{L}_\lambda \cong \operatorname{rad} B_\lambda \oplus \mathcal{L}_\lambda / \operatorname{rad} B_\lambda$$

and B_λ restricts to a non-degenerate form on $\mathcal{L}_\lambda / \operatorname{rad} B_\lambda$. The results of the previous section then show that

$$f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \mathcal{L}_\lambda / \operatorname{rad} B_\lambda \oplus \mathcal{E}$$

for some parity complex $\mathcal{E}$ having no direct summand supported on X_λ. The theorem follows. $\square$

3.4. **Working over a ring.** In this section we assume that $k = \mathbb{O}$ is a complete discrete valuation ring. In this case some constructions performed in Sections 3.2 and 3.3 are not canonical; here we outline the necessary changes.

Throughout this section, "modular reduction" means extension of scalars from $\mathbb{O}$ to $\mathbb{F} = \mathbb{O}/\mathfrak{m}$. We will denote the modular reduction of a $\mathbb{O}$-module H by $\overline{H}$.

Given a bilinear form

$$B : H \times H' \to \mathbb{O}$$

between finitely generated free $\mathbb{O}$-modules we can consider the modular reduction $\overline{B}$ and obtain subspaces ${}^{\perp}\overline{B} \subset \overline{H}$ and $\overline{B}^{\perp} \subset \overline{H'}$.

We can choose direct summands $R \subset H$ and $R' \subset H'$ such that $\overline{R} = {}^{\perp}\overline{B}$ and $\overline{R'} = \overline{B}^{\perp}$. Note, however, that unless ${}^{\perp}\overline{B} = 0$ or ${}^{\perp}\overline{B} = H$, the choice of R is not canonical (and similarly for R').

However, as in the case of a field, B induces a perfect pairing

$$H/R \times H'/R' \to \mathbb{O}$$

and we have

$$\operatorname{rank} H/R = \dim \overline{H}/\overline{R} = \operatorname{rank} \overline{B} = \dim \overline{H'}/\overline{R'} = \operatorname{rank} H'/R'.$$

Now let $\mathcal{C}$ be a Krull-Schmidt category over $\mathbb{O}$ with finitely generated Hom-spaces. Consider objects a and x in $\mathcal{C}$ with a indecomposable. Assume further that $\operatorname{End}(a) = \mathbb{O}$ and that $\operatorname{Hom}(a, x)$ and $\operatorname{Hom}(x, a)$ are free $\mathbb{O}$-modules. Composition yields a bilinear form

$$B : \operatorname{Hom}(a, x) \times \operatorname{Hom}(x, a) \to \operatorname{End}(a) = \mathbb{O}.$$

The analogue of Lemma 3.1 is the following:

Lemma 3.12. *The multiplicity of a in x is equal to the rank of the modular reduction $\overline{B}$ of B.*

Proof. The proof is similar to that of Lemma 3.1. As before, choose an isomorphism $x \cong a^{\oplus m} \oplus y$ such that $y \in \mathcal{C}$ does not contain a as a direct summand. Instead of (3.2), we want to show the following: $\operatorname{Hom}(a, y)$ is a direct summand of $\operatorname{Hom}(a, x)$ whose modular reduction is ${}^{\perp}\overline{B}$, and $\operatorname{Hom}(y, a)$ is a direct summand of $\operatorname{Hom}(x, a)$ whose modular reduction is $\overline{B}^{\perp}$.

We replace the statement of (3.3) and the argument following it by the following: $\beta_2 \alpha_2$ lies in the maximal ideal of $\operatorname{End}(a) = \mathbb{O}$. Otherwise, the same argument as in the proof of Lemma 3.1 would show that a is a direct summand of y. It follows that $\overline{\operatorname{Hom}(a, y)} \subset {}^{\perp}\overline{B}$ and $\overline{\operatorname{Hom}(y, a)} \subset \overline{B}^{\perp}$. Now, as in Lemma 3.1, $\overline{B}$ induces a bilinear form

$$\overline{\operatorname{Hom}(a, a^{\oplus m})} \times \overline{\operatorname{Hom}(a^{\oplus m}, a)} \to \mathbb{F}$$

which may be identified with the canonical form on $\mathbb{F}^m$, and hence is non-degenerate. The lemma then follows. $\qquad\square$

It follows from Section 3.1 that the intersection forms associated over $\mathbb{O}$ or $\mathbb{F}$ are obtained via extension of scalars from intersection forms defined over $\mathbb{Z}$. One may then use the above lemma to show that all the results in Section 3.2.2 remain is true with coefficients in $\mathbb{O}$ if one instead considers the modular reduction of the intersection forms.

In Section 3.2.3 the definition of the local systems $\mathcal{L}_\lambda$ in (3.7) still makes sense and yields local systems whose stalks are non-zero if and only if X_λ is relevant, in which case they are canonically isomorphic to the top Borel-Moore homology of the fibre. In particular, each local system $\mathcal{L}_\lambda$ has torsion free stalks.

If the modular reduction of the intersection form associated to X_λ is non-degenerate then the morphism D_λ is an isomorphism (as follows from Nakayama's lemma) which implies Proposition 3.9. It follows that we have an isomorphism (with notation as in Section 3.2.3):

$$f_*\underline{\mathbb{O}}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus_{\lambda \in \Lambda} \mathbf{IC}(X, \mathcal{L}_\lambda).$$

Note, however that the local systems $\mathcal{L}_\lambda$ are never semi-simple if they are non-zero.

We now turn to Section 3.3. Recall that, for each stratum X_λ, we have a split exact sequence

$$0 \longrightarrow \operatorname{rad}\overline{B}_\lambda \longrightarrow \overline{\mathcal{L}}_\lambda \longrightarrow \overline{\mathcal{L}}_\lambda/\operatorname{rad}\overline{B}_\lambda \longrightarrow 0.$$

Choosing a splitting, we get an idempotent $e \in \operatorname{End}(\overline{\mathcal{L}}_\lambda)$. Now $\operatorname{End}(\overline{\mathcal{L}}_\lambda) = \operatorname{End}(\mathcal{L}_\lambda)/\mathfrak{m}\operatorname{End}(\mathcal{L}_\lambda)$ and $\mathfrak{m}\operatorname{End}(\mathcal{L}_\lambda)$ is included in the Jacobson radical of $\operatorname{End}(\mathcal{L}_\lambda)$, hence by Proposition 2.7, we may lift e to an idempotent E of $\operatorname{End}(\mathcal{L}_\lambda)$, and hence lift the preceding exact sequence to a split exact sequence of torsion free local systems

$$0 \longrightarrow \mathcal{R}_\lambda \longrightarrow \mathcal{L}_\lambda \longrightarrow \mathcal{Q}_\lambda \longrightarrow 0$$

such that $\overline{\mathcal{R}}_\lambda = \operatorname{rad}\overline{B}_\lambda$ and $\overline{\mathcal{Q}}_\lambda = \overline{\mathcal{L}}_\lambda/\operatorname{rad}\overline{B}_\lambda$.

It then follows as in the proof of Theorem 3.11 that we have

$$f_*\underline{\mathbb{O}}_{\widetilde{X}}[d_{\widetilde{X}}] \simeq \bigoplus_{\lambda \in \Lambda} \mathcal{E}(\lambda, \mathcal{Q}_\lambda).$$

Alternatively, given (2.6), we could deduce results over $\mathbb{O}$ from results over $\mathbb{F}$ by lifting idempotents.

4. Applications

4.1. (Kac-Moody) Flag varieties. In this section we show the existence and uniqueness of parity sheaves on (Kac-Moody) flag varieties. The reader unfamiliar with Kac-Moody flag varieties may keep the important case of a (finite) flag variety in mind. The standard reference for Kac-Moody Schubert varieties is [Kum02].

We fix some notation. Let A be a generalised Cartan matrix of size l and let $\mathfrak{g}(A)$ denote the corresponding Kac-Moody Lie algebra with Weyl group W, Bruhat order $\leq$, length function ℓ and simple reflections $S = \{s_i\}_{i=1,\dots l}$. To A one may also associate a Kac-Moody

group $\mathcal{G}$ and subgroups N, $\mathcal{B}$ and $\mathcal{T}$ with $\mathcal{B} \supset \mathcal{T} \subset N$. Given any subset $I \subset \{1, \ldots, l\}$ one has a standard parabolic subgroup $\mathcal{P}_I$ containing $\mathcal{B}$. The group $\mathcal{T}$ is a connected algebraic torus, $\mathcal{B}$, N, $\mathcal{P}_I$ and $\mathcal{G}$ all have the structure of pro-algebraic groups and $(\mathcal{G}, \mathcal{B}, N, S)$ is a Tits system with Weyl group canonically isomorphic to W. Finally, the set $\mathcal{G}/\mathcal{P}_I$ may be given the structure of an ind-variety and is called a Kac-Moody flag variety.

Example 4.1. If A is a Cartan matrix then $\mathfrak{g}(A)$ is a semi-simple finite-dimensional complex Lie algebra and $\mathcal{G}$ is the semi-simple and simply connected complex linear algebraic group with Lie algebra $\mathfrak{g}$, $\mathcal{B}$ is a Borel subgroup, $\mathcal{T} \subset \mathcal{B}$ is a maximal torus, N is the normaliser of $\mathcal{T}$ in $\mathcal{G}$, $\mathcal{P}_I$ is a standard parabolic and $\mathcal{G}/\mathcal{P}_I$ is a partial flag variety.

Example 4.2. If A is now a Cartan matrix of size $l-1$ and $\mathfrak{g}(A)$ is the corresponding Lie algebra with semisimple simply-connected group G, one can obtain a generalised Cartan matrix $\tilde{A}$ by adding an l-th row and column with the values:

$$a_{l,l} = 2, a_{l,j} = -\alpha_j(\theta^\vee), a_{j,l} = -\theta(\alpha_j^\vee),$$

where $1 \leq j \leq l - 1$, α_i are the simple roots of $\mathfrak{g}(A)$ and θ is the highest root. The corresponding Kac-Moody Lie algebra $\mathfrak{g}(\tilde{A})$ (resp. group $\mathcal{G}$) is the so-called (untwisted) affine Kac-Moody Lie algebra (resp. group) defined in [Kum02, Chapter 13]. It turns out that the associated Kac-Moody flag varieties have an alternative description. Let $\mathcal{K} = \mathbb{C}((t))$ denote the field of Laurent series and $\mathcal{O} = \mathbb{C}[[t]]$ the ring of Taylor series. In this case, the Kac-Moody flag variety $\mathcal{G}/\mathcal{P}_I$, where $I = \{1, \ldots, l-1\}$, can be identified with the quotient $G(\mathcal{K})/G(\mathcal{O})$ also know as the affine Grassmannian. Here $G(\mathcal{K})$ (resp. $G(\mathcal{O})$) denotes the group of $\mathcal{K}$(resp. $\mathcal{O}$)-points of G.

Given a subset $I \subset \{1, \ldots, l\}$ we denote by A_I the submatrix of A consisting of those rows and columns indexed by I. For any such I, A_I is a generalised Cartan matrix Recall that a subset $I \subset \{1, \ldots, l\}$ is of **finite type** if A_I is a Cartan matrix. Equivalently, the subgroup $W_I \subset W$ generated by the simple reflections s_i for $i \in I$ is finite. Below we will only consider subsets $I \subset \{1, \ldots, l\}$ of finite type.[4]
For any two subsets $I, J \subset \{1, \ldots, l\}$ of finite type we define

$${}^I W^J := \{w \in W \mid s_i w > w \text{ and } w s_j > w \text{ for all } i \in I, j \in J\}.$$

---

[4] Much of the theory that we develop below is also valid $\mathcal{G}/\mathcal{P}_J$ even when J is not of finite type, but we will not make this explicit.

The orbits of $\mathcal{P}_I$ on $\mathcal{G}/\mathcal{P}_J$ give rise to a Bruhat decomposition:

$$\mathcal{G}/\mathcal{P}_J = \bigsqcup_{w \in {}^I W^J} \mathcal{P}_I w \mathcal{P}_J / \mathcal{P}_J = \bigsqcup_{w \in {}^I W^J} {}^I X_w^J.$$

The Bruhat decomposition gives an algebraic stratification of $\mathcal{G}/\mathcal{P}_J$.

If $I = \emptyset$ each ${}^I X_w^J$ is isomorphic to an affine space of dimension $\ell(w)$. In general the decomposition of ${}^I X_w^J$ into orbits under $\mathcal{B}$ gives a cell decomposition

$$(4.1) \qquad\qquad {}^I X_w^J = \bigsqcup_{x \in W_I w W_J \cap {}^\emptyset W^J} \mathbb{C}^{\ell(x)}.$$

In the following proposition we analyse the strata ${}^I X_w^J$.

Proposition 4.3. *Let k be a ring.*

(1) *The graded k-module $H^\bullet({}^I X_w^J, k)$ is torsion free and concentrated in even degree.*

(2) *The same is true of $H^\bullet_{\mathcal{P}_I}({}^I X_w^J, k)$ if all the torsion primes for A_I are invertible in k.*

Moreover, any local system or $\mathcal{P}_I$-equivariant local system on ${}^I X_w^J$ is constant.

Proof. The first statement follows from the fact that (4.1) provides an affine paving of ${}^I X_w^J$. This also shows that ${}^I X_w^J$ is simply connected, by the long exact sequence for relative homotopy groups. and hence any local system on ${}^I X_w^J$ is constant. Note if H is the reductive part of the stabiliser of a point in ${}^I X_w^J$ then H is isomorphic to a regular reductive subgroup of a semi-simple connected and simply connected algebraic group with Lie algebra $\mathfrak{g}(A_I)$. It follows that any $\mathcal{P}_I$-equivariant local system on ${}^I X_w^J$ is constant. We also have

$$H^\bullet_{\mathcal{P}_I}({}^I X_w^J, \mathbb{Z}) \cong H^\bullet_H(\mathrm{pt}, \mathbb{Z}).$$

By Theorem 2.27 this has no p-torsion for p not a torsion prime for A_I and the result follows. $\qquad\qquad\square$

For the rest of this section we fix a complete local principal ideal domain k.

Fix $I, J \subset \{1, \dots, l\}$ of finite type. We consider the following situations:

$$(4.2) \qquad X = \mathcal{G}/\mathcal{P}_J, \text{ an ind-variety stratified by the } \mathcal{P}_I\text{-orbits;}$$

$$(4.3) \qquad\qquad X = \mathcal{G}/\mathcal{P}_J, \text{ an ind-}\mathcal{P}_I\text{-variety.}$$

If we are in situation (4.3), we assume additionally that whenever we choose $I \subset \{1, \dots, l\}$ of finite type, the torsion primes of A_I are invertible in k.

In either case we let $D_I(\mathcal{G}/\mathcal{P}_Y) := D(X, k) = D(X)$ be as in Section 2.1 (see also Section 2.6). Proposition 4.3 shows that the stratified ind-$(\mathcal{P}_I)$-variety $\mathcal{G}/\mathcal{P}_Y$ satisfies (2.1) and (2.2). By Theorem 2.9, it follows that there exists up to isomophism at most one parity sheaf with support $\overline{^I X_w^J}$ for each $w \in {}^I W^J$.

The first aim of this section is to show:

Theorem 4.4. *Suppose that we are in situation* (4.2) *or* (4.3)*. For each $w \in {}^I W^J$, there exists, up to isomorphism, one parity sheaf $\mathcal{E}(w) \in D_I(\mathcal{G}/\mathcal{P}_Y)$ such that the closure of its support is $\overline{^I X_w^J}$.*

Recall that, if we are in the situation (4.3) then, given any three subsets $I, J, K \subset \{1, \ldots, l\}$ of finite type there exists a bifunctor

$$D_I(\mathcal{G}/\mathcal{P}_J) \times D_J(\mathcal{G}/\mathcal{P}_K) \to D_I(\mathcal{G}/\mathcal{P}_K)$$
$$(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} * \mathcal{G}$$

called convolution (see [Spr82, MV07]). It is defined using the convolution diagram (of topological spaces):

$$\mathcal{G}/\mathcal{P}_J \times \mathcal{G}/\mathcal{P}_K \xleftarrow{p} \mathcal{G} \times \mathcal{G}/\mathcal{P}_K \xrightarrow{q} \mathcal{G} \times_{\mathcal{P}_J} \mathcal{G}/\mathcal{P}_K \xrightarrow{m} \mathcal{G}/\mathcal{P}_K$$

where p is the natural projection, q is the quotient map and m is the map induced by multiplication. One sets

$$\mathcal{F} * \mathcal{G} := m_* \mathcal{K} \quad \text{where} \quad q^* \mathcal{K} \cong p^*(\mathcal{F} \boxtimes \mathcal{G}).$$

For the existence of $\mathcal{K}$ and how to make sense of $\mathcal{G} \times \mathcal{G}/\mathcal{P}_K$ algebraically, we refer the reader to [Nad05, Sections 2.2 and 3.3].

The second goal of this Section is to show:

Theorem 4.5. *Suppose that we are in situation* (4.3)*. Then convolution preserves parity: if $\mathcal{F} \in D_I(\mathcal{G}/\mathcal{P}_J)$ and $\mathcal{G} \in D_J(\mathcal{G}/\mathcal{P}_K)$ are parity complexes, then so is $\mathcal{F} * \mathcal{G} \in D_I(\mathcal{G}/\mathcal{P}_K)$.*

Remark 4.6. The case of finite flag varieties was considered in [Spr82]. There Springer gives a new proof, due to MacPherson and communicated to him by Brylinski, of the fact that the characters of intersection cohomology complexes on the flag variety give the Kazhdan-Lusztig basis of the Hecke algebra. This uses parity considerations in an essential way. See also [Soe00].

Before turning to the proofs we prove some properties about the canonical quotient maps between Kac-Moody flag varieties and recall the construction of (generalised) Bott-Samelson varieties. Unless we state otherwise, in all statements below we assume that we are in either situation (4.2) or (4.3).

If $J \subset K$ are subsets of $\{1, \ldots, l\}$ the canonical quotient map

$$\pi : \mathcal{G}/\mathcal{P}_J \to \mathcal{G}/\mathcal{P}_K.$$

is a morphism of ind-varieties.

Proposition 4.7. *If K is of finite type then both π_* and π^* preserve parity.*

Proof. Because a complex is parity if and only if it is parity after applying the forgetful functor, it is clearly enough to deal with the non-equivariant case (i.e. that we are in situation (4.2)). Moreover, as the stratification of $\mathcal{G}/\mathcal{P}_K$ by $\mathcal{B}$-orbits refines the stratification by $\mathcal{P}_I$-orbits we may assume without loss of generality that $I = \emptyset$. By [Kum02, Proposition 7.1.5], π is a stratified proper morphism between the stratified ind-varieties $\mathcal{G}/\mathcal{P}_J$ and $\mathcal{G}/\mathcal{P}_K$. Moreover, the same proposition shows that the restriction of π to a stratum in $\mathcal{G}/\mathcal{P}_K$ is simply a projection between affine spaces. If follows that π is even and hence π_* preserves parity complexes by Proposition 2.16.

For π^* note that π^* certainly preserves $*$-even complexes. However our assumptions on K guarantee that π is a smooth morphism with fibres of some (complex) dimension d. Hence $\pi^! \cong \pi^*[2d]$ and so π^* also preserves !-even complexes. $\qquad\square$

From now on we write π_K^J for the quotient morphism $\mathcal{G}/\mathcal{P}_J \to \mathcal{G}/\mathcal{P}_K$.

Now, let $I_0 \subset J_0 \supset I_1 \subset J_1 \supset \ldots J_{n-1} \supset I_n$ be a sequence of finite type subsets of $\{1, \ldots, l\}$. Consider the space

$$\mathrm{BS}(0, \ldots, n) := \mathcal{P}_{J_0} \times^{\mathcal{P}_{I_1}} \mathcal{P}_{J_1} \times^{\mathcal{P}_{I_2}} \ldots \mathcal{P}_{J_{n-2}} \times^{\mathcal{P}_{I_{n-1}}} \mathcal{P}_{J_{n-1}} / \mathcal{P}_{I_n}$$

defined as the quotient of $\mathcal{P}_{J_0} \times \mathcal{P}_{J_1} \times \cdots \times \mathcal{P}_{J_{n-1}}$ by $\mathcal{P}_{I_1} \times \mathcal{P}_{I_2} \times \cdots \times \mathcal{P}_{I_n}$ where $(q_1, q_2, \ldots, q_n)$ acts on $(p_0, p_1, \ldots, p_{n-1})$ by

$$(p_0 q_1^{-1}, q_1 p_1 q_2^{-1}, \ldots, q_{n-1} p_{n-1} q_n^{-1}).$$

This space is a projective algebraic variety with $\mathcal{P}_{I_0}$-action. It is called a (generalised) Bott-Samelson variety. (In the case where $I_i = \emptyset$ and $|J_i| = 1$ for all i, this space is constructed in [Kum02, 7.1.3]. The general case is discussed in [GL05]). We will denote points in this variety by $[p_0, p_1 \ldots, p_{n-1}]$. For $j = 1, \ldots, n$ we have a morphism of ind-varieties

$$f_j : \mathrm{BS}(0, 1, \ldots, n) \to \mathcal{G}/\mathcal{P}_{I_j}$$
$$[p_0, p_1, \ldots, p_n] \mapsto p_0 \ldots p_{j-1} \mathcal{P}_{I_j}.$$

Below, the map f_n will play a special role and we will denote it simply by f if the context is clear. The map

$$BS(i_1, \ldots, i_n) \to \mathcal{G}/\mathcal{P}_{I_1} \times \mathcal{G}/\mathcal{P}_{I_2} \times \cdots \times \mathcal{G}/\mathcal{P}_{I_n}$$
$$p \mapsto (f_1(p), f_2(p), \ldots, f_n(p))$$

given by their product is a closed embedding with image (see [GL05, Section 7])
(4.4)

$$\left\{ \begin{array}{c} (x_1, x_2 \ldots, x_n) \in \\ \mathcal{G}/\mathcal{P}_{I_1} \times \mathcal{G}/\mathcal{P}_{I_2} \times \cdots \times \mathcal{G}/\mathcal{P}_{I_n} \end{array} \middle| \begin{array}{c} \pi_{J_0}^{I_1}(x_1) = \mathcal{P}_{J_0} \text{ and} \\ \pi_{J_j}^{I_j}(x_j) = \pi_{J_j}^{I_{j+1}}(x_{j+1}) \\ \text{for } j = 1, 2, \ldots, n-1. \end{array} \right\}.$$

It follows that all the squares in the following diagram are Cartesian:

$$\begin{array}{ccccccc}
\mathrm{BS}(0, 1, \ldots, n) & \longrightarrow & \mathrm{BS}(1, \ldots, n) & \longrightarrow & \cdots \longrightarrow & \mathrm{BS}(n-1, n) & \longrightarrow \mathrm{BS}(n) \\
\downarrow & & \downarrow & & & \downarrow & \downarrow {\scriptstyle \pi_{J_{n-1}}^{I_n}} \\
\mathrm{BS}(0, \ldots, n-1) & \to & \mathrm{BS}(1, \ldots, n-1) & \longrightarrow & \cdots \longrightarrow & \mathrm{BS}(n-1) \xrightarrow{\pi_{J_{n-1}}^{I_{n-1}}} & \mathcal{G}/\mathcal{P}_{J_{n-1}} \\
\downarrow & & \downarrow & & & \downarrow & \\
\vdots & & \vdots & & \ddots \longrightarrow & \mathcal{G}/\mathcal{P}_{J_{n-2}} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{BS}(0, 1) & \longrightarrow & \mathrm{BS}(1) & \longrightarrow & \mathcal{G}/\mathcal{P}_{J_2} & & \\
\downarrow & & \downarrow & & & & \\
\mathrm{BS}(0) & \xrightarrow{\pi_{J_1}^{I_1}} & \mathcal{G}/\mathcal{P}_{J_1} & & & & \\
\downarrow {\scriptstyle \pi_{J_0}^{I_1}} & & & & & & \\
\mathcal{G}/\mathcal{P}_{J_0} & & & & & &
\end{array}$$

Proposition 4.8. *The sheaf* $f_* \underline{k}_{\mathrm{BS}(0,1,\ldots,n)} \in D_{I_0}(\mathcal{G}/\mathcal{P}_{I_n})$ *is parity.*

Proof. (See [Soe00].) Repeated use of proper base change applied to the above diagram gives us an isomorphism

$$f_* \underline{k}_{\mathrm{BS}(1,\ldots,n)} \cong \pi_{J_{n-1}}^{I_n *} \pi_{J_{n-1}*}^{I_{n-1}} \cdots \pi_{J_1}^{I_2 *} \pi_{J_1 *}^{I_1} \pi_{J_0}^{I_1 *} \underline{k}_{\mathcal{P}_{J_0}}$$

where $\underline{k}_{\mathcal{P}_{J_0}}$ denotes the skyscraper sheaf on the point $\mathcal{P}_{J_0} \in \mathcal{G}/\mathcal{P}_{J_0}$. However $\underline{k}_{\mathcal{P}_{J_0}}$ is certainly parity and the result follows from Proposition 4.7. □

We can now prove Theorems 4.4 and 4.5:

Proof. Fix subsets $I, J \subset \{1, \ldots, l\}$ of finite type and choose $w \in {}^I W^J$. By Theorem 2.9 it is enough to show that there exists at least one parity sheaf $\mathcal{E}$ such that the closure of the support of $\mathcal{E}$ is $\overline{{}^I X_w^J}$.

In fact one may show (see [Wil08, Proposition 1.3.4]) that there exists a sequence $I = I_0 \subset J_0 \supset I_1 \subset J_1 \supset \ldots J_{n-1} \supset I_n = J$ such that, if BS denotes the corresponding generalised Bott-Samelson variety, the morphism

$$f : \mathrm{BS} \to \mathcal{G}/\mathcal{P}_J$$

has image $\overline{{}^I X_w^J}$ and is an isomorphism over ${}^I X_w^J$. We have just seen that $f_* \underline{k}_{\mathrm{BS}}$ is parity, and Theorem 4.4 follows.

We now turn to Theorem 4.5 and assume we are in the situation (4.3). By the uniqueness of parity sheaves, and the above remarks, it is enough to show that if

$$I = I_0 \subset J_0 \supset I_1 \subset \cdots \subset J_{n-1} \supset I_n = J$$
$$J = I_n \subset J_{n+1} \supset I_{n+1} \subset \cdots \subset J_{m-1} \supset I_m = K$$

are two sequences of finite type subsets of $\{1, \ldots, l\}$, BS_1 and BS_2 are the corresponding generalised Bott-Samelson varieties and $f_1 : \mathrm{BS}_1 \to \mathcal{G}/\mathcal{P}_J$ and $f_2 : \mathrm{BS}_2 \to \mathcal{G}/\mathcal{P}_K$ then

$$f_{1*}\underline{k}_{\mathrm{BS}_1} * f_{2*}\underline{k}_{\mathrm{BS}_2} \in D_I(\mathcal{G}/\mathcal{P}_K)$$

is parity.

However, if BS denotes the Bott-Samelson variety associated to the concatenation $I = I_0 \subset J_0 \supset \cdots \supset I_n \subset \cdots \subset J_{m-1} \supset I_m = K$ and $f : \mathrm{BS} \to \mathcal{G}/\mathcal{P}_K$ is the multiplication morphism then

$$f_{1*}\underline{k}_{\mathrm{BS}_1} * f_{2*}\underline{k}_{\mathrm{BS}_2} \cong f_*\underline{k}_{\mathrm{BS}}$$

and the result follows from the proposition above. $\square$

Remark 4.9.

(1) Such theorems have been established for the finite flag varieties if k is a field of characteristic larger than the Coxeter number by Soergel in [Soe00].

(2) An important special case of the above is the affine Grassmannian. In this case, parity sheaves are closely related to tilting modules (see Section 5).

4.2. **Toric varieties.** For notation, terminology, and basic properties of toric varieties we refer the reader to [Ful93] and [Mus05]. In this section T denotes a connected algebraic torus and $M = X^*(T)$ and $N = X_*(T)$ denote the character and cocharacter lattices respectively. If L is a lattice we set $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$.

Recall that a fan in N is a collection Δ of polyhedral, convex cones in $N_{\mathbb{Q}}$ closed under taking faces and intersections. To a fan Δ in N one may associate a toric variety $X(\Delta)$ which is a connected normal

T-variety. There are finitely many orbits of T on $X(\Delta)$ and the decomposition into orbits gives a stratification

$$X(\Delta) = \bigsqcup_{\tau \in \Delta} X_\tau$$

indexed by the cones of Δ. For example the zero cone $\{0\}$ always belongs to Δ and $X_{\{0\}}$ is an open dense orbit, canonically identified with T.

In this section we fix a complete local principle ideal domain k, take

(4.5) $X = X(\Delta)$ as a T-variety

and let $D_T(X(\Delta)) = D(X)$ be as in Section 2.1. We use the notation of Section 2 without further comment.

Theorem 4.10. *For each orbit O_τ, there exists up to isomorphism one parity sheaf $\mathcal{E}(\tau) \in D_T(X(\Delta))$ such that the closure of its support is $\overline{O}_\tau$. Moreover, given any proper toric morphism*

$$f : X(\Delta) \to X(\tilde{\Delta})$$

the direct image $f_*\mathcal{E}(\tau)$ of a parity sheaf on $X(\Delta)$ is a parity complex on $X(\tilde{\Delta})$.

Let $\tau \in \Delta$ and let $N(\tau)$ denote the intersection of N with the linear span of τ. Then $N(\tau)$ determines a connected subtorus $T_\tau \subset T$.

Lemma 4.11. *The stabiliser of a point $x \in X_\tau$ is T_τ and is therefore connected.*

Proof. This follows from the last exercise of Section 3.1 in [Ful93]. □

We now turn to the proof of the theorem.

Proof. By the quotient equivalence, the categories of T-equivariant local systems on X_τ and T_τ-equivariant local systems on a point are equivalent. Hence any torsion free equivariant local system on X_τ is isomorphic to a direct sum of copies of the trivial local system $\underline{k}_\tau$. We have

$$\mathrm{Hom}^\bullet(\underline{k}_\tau, \underline{k}_\tau) = H_T^\bullet(X_\tau) = H_{T_\tau}^\bullet(\mathrm{pt})$$

which is torsion free and vanishes in odd degree. It follows that the T-variety $X(\Delta)$ satisfies (2.1) and (2.2). By Theorem 2.9, we conclude that for each $\tau \in \Delta$ there exists at most one parity sheaf $\mathcal{E}(\tau)$ supported on $\overline{O}_\tau$ and satisfying $i_\tau^*\mathcal{E}(\tau) \cong \underline{k}_\tau[d_\tau]$.

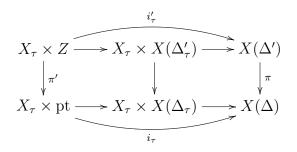It remains to show existence. Recall the following properties of toric varieties:

(1) For $\tau \in \Delta$, $\overline{O}_\tau$ is a toric variety for T/T_τ ([Ful93, Section 3.1]).

(2) For any fan Δ there exists a refinement Δ' of Δ such that $X(\Delta')$ is quasi-projective and the induced T-equivariant morphism

$$\pi : X(\Delta') \to X(\Delta)$$

is a resolution of singularities ([Ful93, Section 2.6]).

(3) For all τ in Δ we have a Cartesian diagram (all morphisms are T-equivariant):

Here $X(\Delta_\tau)$ and $X(\Delta'_\tau)$ are toric varieties for T_τ corresponding to $\tau \subset N(\tau)_\mathbb{Q}$ (resp. those cones in Δ' contained in τ). The first two vertical maps are the projections.

By (1) it suffices to show the existence of $\mathcal{E}(\tau)$ when τ is the zero cone (corresponding to the open T-orbit). For this it suffices to show that $\pi_* \underline{k}_{X(\Delta')}$ is even. In fact, as $\underline{k}_{X(\Delta')}[d_\tau]$ is self-dual and π is proper, we need only show that $\pi_* \underline{k}_{X(\Delta')}$ is $*$-even.

By proper base change we have $i_\tau^* \pi_* \underline{k}_{X(\Delta')} \cong \pi'_* \underline{k}_{X_\tau \times Z}$. Under the quotient equivalence $D_T(X_\tau) \xrightarrow{\sim} D_{T_\tau}(\mathrm{pt})$, the sheaf $\pi'_* \underline{k}_{X_\tau \times Z}$ corresponds to $\tilde{\pi}_* \underline{k}_Z \in D_{T_\tau}(\mathrm{pt})$, where $\tilde{\pi} : Z \to \mathrm{pt}$ is the projection (of T_τ-varieties). We will see in the proposition below that $\tilde{\pi}_* \underline{k}_Z$ is always $*$-even. This proves the first part of the theorem.
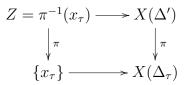
For the second part, note that given any $\tau \in \Delta$ we can find a toric morphism

$$f' : X(\Delta') \to X(\Delta)$$

such that $X(\Delta')$ is smooth and quasi-projective, and $f_* \underline{k}_{X(\Delta')}$ contains $\mathcal{E}(\tau)$ as a direct summand. Then $(f' \circ f)_* \underline{k}_{X(\Delta')}$ is parity and contains $f_* \mathcal{E}(\tau)$ as a direct summand. Hence $f_* \mathcal{E}(\tau)$ is parity as claimed. □

Proposition 4.12. *Let $\tau \subset N_\mathbb{Q}$ be a polyhedral, convex cone, such that $\langle \tau \rangle = N_\mathbb{Q}$, Δ_τ the corresponding fan, and Δ' a refinement of τ such that the corresponding toric variety $X(\Delta')$ is smooth and quasi-projective. Let x_τ denote the unique T-fixed point of $X(\Delta_\tau)$. Consider*

the Cartesian diagram:

$$Z = \pi^{-1}(x_\tau) \longrightarrow X(\Delta')$$
$$\downarrow \pi \qquad\qquad \downarrow \pi$$
$$\{x_\tau\} \longrightarrow X(\Delta_\tau)$$

Then $\pi_* \underline{k}_Z \in D_T(\mathrm{pt})$ is a direct sum of equivariant constant sheaves concentrated in even degree.

Proof. It is enough to show that the T-equivariant cohomology of Z with integral coefficients is free over $H_T^\bullet(\mathrm{pt}, \mathbb{Z})$ and concentrated in even-degree. We will show that the integral cohomology of Z is free, and generated by the classes of T-stable closed subvarieties. The result then follows by the Leray-Hirsch lemma (see [Bri, Proof of Theorem 4]).

We claim in fact that Z has a T-stable affine paving, which implies the result by the long exact sequence of compactly supported cohomology. The argument is a straightforward adaption of [Dan78, 10.3 – 10.7] (which the reader may wish to consult for further details).

As $X(\Delta')$ is assumed to be quasi-projective we can find a piecewise linear function $g : N_\mathbb{Q} \to \mathbb{Q}$ which is strictly convex with respect to Δ'. In other words, g is continuous, convex and for each maximal cone $\sigma \in \Delta'$, g is given on σ by $m_\sigma \in M$. The function g allows us to order the maximal cones of σ as follows: We fix a generic point $x_0 \in N_\mathbb{Q}$ lying in a cone of Δ' and declare that $\sigma' > \sigma$ if $m_{\sigma'}(x_0) > m_\sigma(x_0)$. If σ' and σ satisfy $\sigma' > \sigma$ and intersect in codimension 1, then their intersection is said to be a positive wall of σ. Given a maximal cone σ we define $\gamma(\sigma)$ to be the intersection of σ with all its positive walls.

It is then easy to check (remembering that $X(\Delta')$ is assumed smooth) that if we set

$$C(\sigma) = \bigsqcup_{\gamma(\sigma) \subset \omega \subset \sigma} X_\omega$$

then $C(\sigma)$ is a locally closed subset of $X(\Delta')$ isomorphic to an affine space of dimension equal to the codimension of $\gamma(\sigma)$ in $N_\mathbb{Q}$. Lastly note that

$$Z = \bigsqcup X_\sigma$$

were the union takes place over those cones in Δ' which are not contained in any wall of τ. Hence the order on maximal cones yields a filtration of Z by T-stable closed subspaces $\cdots \subset F_{\sigma_{i+1}} \subset F_{\sigma_i} \subset \ldots$ such that $F_{\sigma_{i+1}} \setminus F_{\sigma_i}$ is isomorphic to an affine space for all i. The result then follows. $\qquad\square$

Remark 4.13. With notation as above, $X(\Delta')$ retracts equivariantly onto Z. With this in mind, the above arguments (together with the reduction to the quasi-projective case in [Dan78]) can be used to establish the equivariant formality (over $\mathbb{Z}$) of convex smooth toric varieties. The elegant Mayer-Vietoris spectral sequence argument of [BZ03] may then be used to identify the equivariant cohomology ring with piecewise integral polynomials on the fan. We have been unable to find such a result in the literature.

4.3. **Nilpotent cones.**

4.3.1. *Case of the general linear group.* Let $\mathcal{N} \subset \mathfrak{gl}_n$ be the nilpotent cone. The orbits of $G = GL_n$ on $\mathcal{N}$ are naturally parametrised by partitions λ of n, according to Jordan type. Let us denote each such orbit by $\mathcal{O}_\lambda$, and let x_λ be an element of $\mathcal{O}_\lambda$. We use exponents to indicate multiple entries in λ. So we can write a partition as $\lambda = (i_1^{r_1}, \ldots, i_m^{r_m})$, with $i_1 > i_2 > \cdots > i_m > 0$.

There are several features that are particular to the case of $G = GL_n$. First, the centralisers $C_G(x_\lambda)$ are connected, so in the G-equivariant setting we only have to deal with constant local systems. Secondly, there is a semi-small resolution for each nilpotent orbit closure, whose fibres admit affine pavings. This is why we deal with this case first. We will consider the case of a general reductive group in the next subsection.

In general (2.1) is not satisfied by the $\mathcal{O}_\lambda$ if we use ordinary cohomology. However, if we consider instead the equivariant derived category $D^b_{GL_n}(\mathcal{N})$ then we have

$$(4.6) \qquad H^\bullet_G(\mathcal{O}_\lambda) = H^\bullet_{G_{x_\lambda}}(\mathrm{pt}) = H^\bullet_{G^{\mathrm{red}}_{x_\lambda}}(\mathrm{pt}).$$

It is known that $G^{\mathrm{red}}_{x_\lambda} \simeq GL_{r_1} \times GL_{r_2} \times \cdots \times GL_{r_m}$, if $\lambda = (i_1^{r_1}, \ldots, i_m^{r_m})$ as above. Now the classifying space of GL_{r_i} is a direct limit of Grassmannians, all of which admit cell decompositions, hence $H^\bullet_{GL_r}(\mathrm{pt})$ is concentrated in even degrees, and is torsion-free if we use integer coefficients. Thus

$$H^\bullet_{GL_{r_1} \times \cdots \times GL_{r_m}}(\mathrm{pt}) = H^\bullet_{GL_{r_1}}(\mathrm{pt}) \otimes \cdots \otimes H^\bullet_{GL_{r_m}}(\mathrm{pt})$$

has the same property. Hence Theorem 2.9 is applicable. For each λ, there is at most one parity sheaf $\mathcal{E}(\lambda)$ with support $\overline{\mathcal{O}}_\lambda$ (up to isomorphism).

For existence, we will use a property which is specific to type A. Namely, for each partition λ of n we have a G-equivariant semi-small resolution of singularities

$$T^*(G/P_{\lambda'}) \to \overline{\mathcal{O}}_\lambda$$

whose fibres admit affine pavings [HS79] (here λ' denotes the partition conjugate to λ). It follows that there does exist a parity (even) sheaf $\mathcal{E}(\lambda)$ with support $\overline{\mathcal{O}}_\lambda$. Moreover in this case it is perverse (by semi-smallness).

4.3.2. *Case of a general reductive group.* Now we consider the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ in the Lie algebra of any connected reductive group G. For groups other than GL_n, several difficulties arise. First, the centralisers G_x are not necessarily connected, already for SL_n. So we must allow non-trivial G-equivariant local systems. They correspond to representations of the finite group $A_G(x) = G_x/G_x^0$.

If $\mathcal{O}_x$ denotes the orbit of $x \in \mathcal{N}$, we still have

$$H_G(\mathcal{O}_x) = H_{G_x}(\mathrm{pt}) = H_{G_x^0}(\mathrm{pt})$$

if the order of $A_G(x)$ is invertible in k.

But the reductive part of the centraliser $C_x := (G_x^0)^{\mathrm{red}}$ is a reductive group for which $H_{C_x}^\bullet(\mathrm{pt}, k)$ is more complicated than for a product of GL_{r_i}. In particular, there may be torsion over $\mathbb{Z}_p$, and for $\mathbb{F}_p$ there may be cohomology in odd degrees. To avoid these problems, we will stay away from torsion primes.

Now assume that p is not a torsion prime for G, and that it does not divide the orders of the $A_G(x)$ for x nilpotent. Then for each pair $(\mathcal{O}, \mathcal{L})$ consisting of a nilpotent orbit together with an irreducible G-equivariant local system, there is at most one parity sheaf $\mathcal{E}(\mathcal{O}, \mathcal{L})$ with support $\overline{\mathcal{O}}$ extending $\mathcal{L}[d_\mathcal{O}]$.

It is known [Lus86, V, Theorem 24.8] that the intersection cohomology complexes of nilpotent orbit closures, with coefficients in any irreducible G-equivariant local system in characteristic zero, are even. Thus a similar result holds for almost all characteristics. However, work still needs to be done to determine precise bounds on p for parity sheaves to exist, resp. to be perverse, resp. to be intersection cohomology sheaves.

Let us give some partial results on existence. Most importantly, let us consider the case of the subregular orbit $\mathcal{O}_{\mathrm{reg}}$ in $\mathcal{N}$. Then Springer's resolution $\pi : G \times^B \mathfrak{u} \to \overline{\mathcal{O}}_{\mathrm{reg}} = \mathcal{N}$ is semi-small. Moreover, the cohomology of its fibres is free over $\mathbb{Z}$, and concentrated in even degrees, by [DCLP88]. Thus π is even and semi-small, and $\mathcal{E}(\overline{\mathcal{O}}_{\mathrm{reg}})$ exists and is perverse. By Remark 3.10, we also have existence of $\mathcal{E}(\overline{\mathcal{O}}, \mathcal{L})$ for all pairs appearing with non-zero multiplicity in the direct image $\mathcal{P} := \pi_* \underline{k}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]$, that is, those pairs for which $\mathcal{L}$ appears with non-zero multiplicity in $\mathcal{H}^{-d_\mathcal{O}}(\mathcal{P})/\operatorname{rad} \mathcal{H}^{-d_\mathcal{O}}(\mathcal{P})$. By semi-smallness, all of these are perverse.

More generally, one can consider the case of a Richardson orbit $\mathcal{O}$. Then there is a parabolic subgroup P of G, with unipotent radical U_P, such that $\mathcal{O} \cap \mathfrak{u}_P$ is dense in $\mathfrak{u}_P$. Then we have a semi-small resolution $G \times^P \mathfrak{u}_P \to \overline{\mathcal{O}}$. However, we do not know in general if it is even.

For a general nilpotent orbit $\mathcal{O}$, let us recall how to construct a resolution of $\overline{\mathcal{O}}$ [Pan91]. Let x be an element of $\mathcal{O} \cap \mathfrak{u}$. By the Jacobson-Morozov theorem, there is an $\mathfrak{sl}_2$-triplet (x, h, y) in $\mathfrak{g}$. The semi-simple element h induces a grading on $\mathfrak{g}$, and we can choose the triplet so that all the simple roots have degree 0, 1 or 2. Let $P_{\mathcal{O}}$ be the parabolic subgroup of G corresponding to the set of simple roots with degree zero. Then there is a resolution of the form $\pi_{\mathcal{O}} : \tilde{\mathcal{N}}_{\mathcal{O}} \to \overline{\mathcal{O}}$, where $\tilde{\mathcal{N}}_{\mathcal{O}} = G \times^{P_{\mathcal{O}}} \mathfrak{g}_{\geq 2}$ is a G-equivariant sub-bundle of $T^*(G/P_{\mathcal{O}}) = G \times^{P_{\mathcal{O}}} \mathfrak{u}_P$, and $\pi_{\mathcal{O}}$ is the restriction of the moment map. This resolution is semi-small only in the case of a Richardson orbit. To settle the question of existence for $\mathcal{E}(\overline{\mathcal{O}}, k)$ in general, one should solve the following problem.

Question 4.14. Is the resolution $\pi_{\mathcal{O}} : \tilde{\mathcal{N}}_{\mathcal{O}} \to \overline{\mathcal{O}}$ even for any coefficients ? Otherwise, for which primes p is the resolution even with respect to $\mathbb{Z}_p$ or $\mathbb{F}_p$ coefficients ?

Finally, for non-trivial local systems, one should also consider coverings of nilpotent orbit closures.

4.3.3. *Minimal singularities.* Suppose that G is simple. Then there is a unique minimal (non-trivial) nilpotent orbit in $\mathfrak{g}$. We denote it by $\mathcal{O}_{\min}$. It is of dimension $d := 2h^\vee - 2$, where $h^\vee$ is the dual Coxeter number [Wan99]. We will describe the parity sheaf $\mathcal{E}(\overline{\mathcal{O}}_{\min})$, which always exists in this case. Indeed, we have a resolution of singularities

$$\pi : E := G \times^P \mathbb{C}x_{\min} \longrightarrow \overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \cup \{0\}$$

where $x_{\min}$ is a highest weight vector of the adjoint representation and P is the parabolic subgroup of G stabilising the line $\mathbb{C}x_{\min}$. It is an isomorphism over $\mathcal{O}_{\min}$, and the fibre above 0 is the null section, isomorphic to G/P, which has even cohomology. Hence π is an even resolution, and $\pi_* \underline{k}_E[d]$ is even.

Let us first mention two general facts that happen when we have a stratification into two orbits.

Lemma 4.15. *Suppose* $X = U \sqcup \{0\}$ *is a stratified variety (thus 0 is the only singular point). We denote by* $j : U \to X$ *and* $i : \{0\} \to X$ *the inclusions.*

(1) *Let* $\mathcal{P}$ *be a* $*$-*even complex on* X. *Then we have a short exact sequence*

$$0 \longrightarrow {}^p j_! j^* \mathcal{P} \longrightarrow {}^p H^0 \mathcal{P} \longrightarrow i_* {}^p i_* \mathcal{P} \longrightarrow 0.$$

(2) *If $\mathcal{F}$ is any perverse sheaf on X whose composition factors are one time $\mathbf{IC}(X, \mathbb{F})$ and a certain number a of times $\mathbf{IC}(0, \mathbb{F})$, then $\mathcal{H}^m(\mathcal{F})_0 \simeq \mathcal{H}^m(\mathbf{IC}(X, \mathbb{F}))$ for all $m \leq -2$.*

Proof. We have a distinguished triangle

$$ j_! j^* \mathcal{P} \longrightarrow \mathcal{P} \longrightarrow i_* i^* \mathcal{P} \xrightarrow{[1]} $$

which gives rise to a long exact sequence of perverse cohomology sheaves, which ends with:

$$ i_*{}^p H^{-1} i^* \mathcal{P} \longrightarrow {}^p j_! j^* \mathcal{P} \longrightarrow {}^p H^0 \mathcal{P} \longrightarrow i_*{}^p i^* \mathcal{P} \longrightarrow 0. $$

Now, ${}^p H^{-1} i^* \mathcal{P}$ is identified with $(\mathcal{H}^{-1}\mathcal{P})_0$ which is zero since $\mathcal{P}$ is $*$-even. This proves (1).

For (2), we proceed by induction on a. The result is trivial for $a = 0$. Now suppose $a > 1$. There is a perverse sheaf $\mathcal{G}$ such that we have a short exact sequence of one of the two following forms:

$$ (4.7) \qquad\qquad 0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathbf{IC}(0, \mathbb{F}) \longrightarrow 0 $$

$$ (4.8) \qquad\qquad 0 \longrightarrow \mathbf{IC}(0, \mathbb{F}) \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 $$

and we can consider the corresponding long exact sequence for the cohomology of the stalk at zero. From $\mathcal{H}^m(\mathbf{IC}(0, \mathbb{F}))_0 = 0$ for $m \leq -1$, we deduce in both cases that $\mathcal{H}^m(\mathcal{F})_0$ is isomorphic to $\mathcal{H}^m(\mathcal{G})_0$ for $m \leq -2$ (at least). The result follows by induction. $\qquad\square$

Proposition 4.16. *The following conditions are equivalent:*

(1) *the parity sheaf $\mathcal{E}(\overline{\mathcal{O}}_{\min}, \mathbb{F})$ is perverse;*
(2) *the standard sheaf ${}^p j_!(\mathbb{F}_{\mathcal{O}_{\min}}[d])$ is $*$-even.*
(3) *the standard sheaf ${}^p j_!(\underline{\mathbb{O}}_{\mathcal{O}_{\min}}[d])$ has torsion free stalks;*
(4) *for all $m < d$, the cohomology group $H^m(\mathcal{O}_{\min}, \mathbb{Z})$ has no p-torsion;*
(5) *the prime p is not in the list corresponding to the type of G in the following table:*

A_n	B_n, C_n, D_n, F_4	G_2	E_6, E_7	E_8
$-$	2	3	$2, 3$	$2, 3, 5$

Proof. First suppose that $\mathcal{E}(\overline{\mathcal{O}}_{\min}, \mathbb{F})$ is perverse. Then both $\mathcal{E}(\overline{\mathcal{O}}_{\min}, \mathbb{F})$ and ${}^p j_!(\mathbb{F}_{\mathcal{O}_{\min}}[d])$ are perverse sheaves with composition factors one time $\mathbf{IC}(\overline{\mathcal{O}}_{\min}, \mathbb{F})$, and a certain number of times $\mathbf{IC}(0, \mathbb{F})$. By Lemma 4.15 (2), we have

$$ \mathcal{H}^m({}^p j_!(\mathbb{F}_{\mathcal{O}_{\min}}[d]))_0 \simeq \mathcal{H}^m(\mathbf{IC}(\overline{\mathcal{O}}_{\min}, \mathbb{F}))_0 \simeq \mathcal{H}^m(\mathcal{E}(\overline{\mathcal{O}}_{\min}, \mathbb{F}))_0 $$

for $m \leq -2$. Since $({}^p j_!(\mathbb{F}_{\mathcal{O}_{\min}}[d]))_0$ is concentrated in degrees ≤ -2, this proves that ${}^p j_!(\mathbb{F}_{\mathcal{O}_{\min}}[d])$ is $*$-even. Thus (1) $\implies$ (2).

Now assume that ${}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d])$ is $*$-even. Let $\mathcal{P} := \pi_* \underline{k}_E[d]$. It is parity. By Lemma 4.15 (1), we have a short exact sequence

$$0 \longrightarrow {}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d]) \longrightarrow {}^p H^0 \mathcal{P} \longrightarrow i_* {}^p i^* \mathcal{P} \longrightarrow 0.$$

Since the extreme terms are $*$-even, we deduce that ${}^p H^0 \mathcal{P}$ is $*$-even as well. But ${}^p H^0 \mathcal{P}$ is self-dual, because $\mathcal{P}$ is. Thus ${}^p H^0 \mathcal{P}$ is parity, and $\mathcal{E}(\mathcal{O}_{\min}, \mathbb{F})$ must be a direct summand of ${}^p H^0 \mathcal{P}$. It follows that $\mathcal{E}(\mathcal{O}_{\min}, \mathbb{F})$ is perverse. Thus (2) $\Longrightarrow$ (1).

That (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) is proved in [Jut08a, Jut08b]. Briefly, the stalk ${}^p \mathcal{J}_!(\overline{\mathcal{O}}_{\min}, \mathbb{Z}_p)_0$ is given by a shift of $H^*(\mathcal{O}_{\min}, \mathbb{Z}_p)$ truncated in degrees $\leq d - 2$, and $H^{d-1}(\mathcal{O}_{\min}, \mathbb{Z}) = 0$, so (3) $\Longleftrightarrow$ (4). Now, by a case-by-case calculation [Jut08a], one finds that (4) $\Longleftrightarrow$ (5).

The fact that $H^{d-1}(\mathcal{O}_{\min}, \mathbb{O}) = 0$ implies that ${}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d]) = \mathbb{F} \otimes_{\mathbb{O}}^L {}^p j_!(\underline{\mathbb{O}}_{\mathcal{O}_{\min}}[d])$ by [Jut09]. Thus (2) $\Longleftrightarrow$ (3) by Proposition 2.20. $\qquad\square$

Finally, let us recall from [Jut09] when the standard sheaf is equal to the intersection cohomology sheaf for a minimal singularity.

Proposition 4.17. *Let Φ denote the root system of G, with some choice of positive roots. Let Φ' denote the root subsystem of Φ generated by the long simple roots. Let H denote the fundamental group of Φ', that is, the quotient of its weight lattice by its root lattice. We have a short exact sequence*

$$0 \longrightarrow i_*(\mathbb{F} \otimes_{\mathbb{Z}} H) \longrightarrow {}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d]) \longrightarrow \mathbf{IC}(\overline{\mathcal{O}}_{\min}, \mathbb{F}) \longrightarrow 0$$

Thus ${}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d]) \simeq \mathbf{IC}(\overline{\mathcal{O}}_{\min}, \mathbb{F})$ when p does not divide H.

5. Parity sheaves on the affine Grassmannian

5.1. **Parity sheaves and tilting modules.** In this section, let G be the adjoint form of a split simple group scheme over a field k of characteristic p, with maximal split torus T and Borel subgroup $B \supset T$. Let $G^\vee$ denote its Langlands dual group over $\mathbb{C}$, with dual torus $T^\vee$. As is in the example 4.2, let $\mathcal{K} = \mathbb{C}((t))$ denote the field of Laurent series and $\mathcal{O} = \mathbb{C}[[t]]$ the ring of Taylor series. Let $\mathcal{G}r = G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$ denote the affine Grassmannian for the complex simple and simply-connected group $G^\vee$.

The geometric Satake theorem [MV07] states that representations of the Langlands dual group G over k is tensor equivalent to the category of $G^\vee(\mathcal{O})$-equivariant perverse sheaves with coefficients in k on $\mathcal{G}r$ equipped with the convolution product described in Section 4.1.

The $G^\vee(\mathcal{O})$-orbits are labelled by the set Λ of dominant weights of G. We will denote the orbit corresponding to a weight λ by $\mathcal{G}r_\lambda$. Recall [MV07, Prop. 13.1] that, for λ a dominant weight, the standard

(resp. costandard) representations $\Delta(\lambda)$ (resp. $\nabla(\lambda)$) go under this equivalence to the standard sheaves ${}^p\mathcal{J}_!(\lambda) := {}^p j_{\lambda!}\underline{k}_\lambda[d_\lambda]$ where $j_\lambda :$ $\mathcal{G}r_\lambda \to \mathcal{G}r$ is the inclusion of the orbit and $\underline{k}_\lambda$ the constant sheaf on $\mathcal{G}r_\lambda$ (resp. ${}^p\mathcal{J}_*(\lambda) := {}^p j_{\lambda*}\underline{k}_\lambda[d_\lambda]$). A representation of G is said to be a tilting module if it has both a filtration by standard modules and by costandard modules (see [Jan03, Chapter E] for a survey of results concerning tilting modules). Under geometric Satake, tilting modules clearly correspond to perverse sheaves with the same property. There exists a unique indecomposable tilting module for each highest weight. We will denote it by $T(\lambda)$ and the corresponding tilting sheaf[5] by $\mathcal{T}(\lambda)$.

In this section we consider $\mathcal{G}r$ as a ind-$G^\vee(\mathcal{O})$-variety. In Section 4.1 we have seen that if p is not a torsion prime for $G^\vee$, then for all $\lambda \in \Lambda$ there exists a parity sheaf $\mathcal{E}(\lambda)$ on $\mathcal{G}r$ with support equal to $\overline{\mathcal{G}r}_\lambda$.

Theorem 5.1. *For p satisfying the conditions of the following proposition, $\mathcal{E}(\lambda) = \mathcal{T}(\lambda)$.*

Proposition 5.2. *All indecomposable tilting modules appear as direct summands of tensor products of the indecomposable tilting modules with minuscule or highest short root highest weights of G, under the following conditions for p depending on the root datum for G:*

$$
\begin{array}{cc}
A_n & any\ p \\
B_n & p > n - 1 \\
D_n & p > n - 2 \\
C_n & p > n \\
E_6, F_4, G_2 & p > 3 \\
E_7 & p > 19 \\
E_8 & p > 31
\end{array}
$$

Remark 5.3. The bounds for types B_n and D_n can be improved to $p > 2$, as was explained to us by Jantzen using his sum formula. He also suggested a better way to prove it, which we hope to include in a forthcoming version of this article.

We postpone the proof of the proposition until the next section. The remainder of this section will be dedicated to showing that the proposition implies the theorem.

First, we proceed by checking the equivalence of the indecomposable tilting and parity sheaves in the two special cases mentioned in the statement of the proposition.

---

[5]Warning: the notion of tilting sheaf used here is not the same as that of [BBM04]. See Remark 1.4.4 for an explanation.

Lemma 5.4. *Let μ be a minuscule highest weight and α_0 denote the highest short root of G. Then*

1. *$\mathcal{E}(\mu) = \mathcal{T}(\mu)$;*
2. *$^pH^0\mathcal{E}(\alpha_0)$ is tilting;*
3. *if p is a good prime for G, then $\mathcal{E}(\alpha_0)$ is perverse and $\mathcal{E}(\alpha_0) = \mathcal{T}(\alpha_0)$;*
4. *if moreover $p \nmid n + 1$ in type A_n, resp. $p \nmid n$ in type B_n, then actually $\mathcal{E}(\alpha_0) = \mathcal{T}(\alpha_0) = \mathbf{IC}(\alpha_0)$.*

Proof. (1) The $G^\vee(\mathcal{O})$-orbit in $\mathcal{G}r$ corresponding to the minuscule highest weight μ is closed, thus $\mathbf{IC}(\mu) = {}^p\mathcal{J}_!(\mu) = {}^p\mathcal{J}_*(\mu) = \underline{k}_\mu[d_\mu]$, which implies $\mathcal{E}(\mu) = \mathcal{T}(\mu) = \underline{k}_\mu[d_\mu]$.

(2) Recall [MOV05, 2.3.3] that the orbit closure $\overline{\mathcal{G}r}_{\alpha_0}$ consists of two strata, a point $\mathcal{G}r_0$ and its complement $\mathcal{G}r_{\alpha_0}$, and that the singularity is equivalent to that of the orbit closure of the minimal orbit of the corresponding nilpotent cone of $\mathfrak{g}^\vee = \mathrm{Lie}(G^\vee)$. So we can apply the results of Subsection 4.3.3, for $G^\vee$ instead of G. We still denote by $\mathcal{P}$ the direct image of the shifted constant sheaf under the even resolution described there. By Lemma 4.15 (1), we have a short exact sequence:

$$(5.1) \qquad 0 \to {}^p\mathcal{J}_!(\alpha_0) \to {}^pH^0\mathcal{P} \to {}^pH^0 j_{0*}j_0^*\mathcal{P} \to 0$$

We have constructed a filtration by standard sheaves and, as $^pH^0\mathcal{P}$ is self-dual ($^pH^0$ is preserved by duality), duality gives a filtration by costandard sheaves. Thus $^pH^0\mathcal{P}$ corresponds to a tilting module under the geometric Satake equivalence.

(3) By Proposition 4.16, the standard perverse sheaf $^p\mathcal{J}_!(\alpha_0)$ is $*$-even. Recall the short exact sequence 5.1 from the proof of (2). The left and right hand terms are $*$-even, so the middle is as well. The sheaf $^pH^0\mathcal{P}$ is self-dual and thus also $!$-even. We conclude that $^pH^0\mathcal{P}$ is parity. By part (2), it is also tilting. Splitting it into parity sheaves and into tilting sheaves, as there is a unique indecomposable summand with full support, we get $\mathcal{E}(\alpha_0) = \mathcal{T}(\alpha_0)$.

(4) Recall from Proposition 4.17 that we have a short exact sequence

$$0 \longrightarrow j_{0,*}k \otimes_{\mathbb{Z}} H \longrightarrow {}^p\mathcal{J}_!(\alpha_0) \longrightarrow \mathbf{IC}(\alpha_0) \longrightarrow 0$$

(See the proposition for the definition of H, replacing G by $G^\vee$.) So we have $^p\mathcal{J}_!(\alpha_0) \simeq \mathbf{IC}(\alpha_0) \simeq {}^p\mathcal{J}_*(\alpha_0) \simeq \mathcal{T}(\alpha_0)$ as soon as p does not divide H. Assuming that p is good, we only need to add the conditions stated for A_n and B_n. □

Mirković and Vilonen have conjectured that standard sheaves with $\mathbb{Z}$ coefficients on the affine Grassmannian are torsion free [MV07]. This

conjecture is equivalent to the standard sheaves being $*$-parity for all fields. Actually the minimal nilpotent orbit singularities provide counterexamples to this conjecture, in all types but in type A_n: see [Jut08b], where the conjecture is modified to exclude bad primes. We get the following reformulation:

Conjecture 5.5. *If p is a good prime for G, then the standard sheaves with coefficients in a field of characteristic p are $*$-parity.*

Remark 5.6. If this conjecture is true, it would imply our theorem 5.1 for all p which are good primes. To see this, assume the conjecture were true, i.e. the standard sheaves were $*$-parity. We claim it follows that all tilting sheaves would then be $*$-parity. This is because any perverse sheaf which has a filtration with successive quotients that are $*$-parity is $*$-parity. On the other hand, tilting sheaves are self-dual and thus would also be !-parity. Having shown that the tilting sheaves would be parity, it would follow that $\mathcal{T}(\lambda) = \mathcal{E}(\lambda)$.

It is therefore tantalising to ask if the conjecture is equivalent to the tilting sheaves being parity. Slightly stronger, one could ask if the parity sheaves being perverse implies the conjecture. By Proposition 4.16, this stronger claim is true for the highest short root.

Proof of Theorem 5.1. The previous lemma shows that for good primes, minuscule highest weights and the highest short root, the indecomposable parity sheaves and tilting sheaves agree.

Now suppose that the indecomposable tilting sheaves $\mathcal{T}(\lambda), \mathcal{T}(\mu)$ are parity for two highest weights λ, μ. We claim that if an indecomposable tilting module $T(\nu)$ occurs as a direct summand of the tensor product $T(\lambda) \otimes T(\mu)$, then the tilting sheaf $\mathcal{T}(\nu)$ is parity. To see this, note that the tensor product corresponds to the convolution of sheaves $\mathcal{T}(\lambda) * \mathcal{T}(\mu)$, which is parity by Theorem 4.5 and perverse. Thus each indecomposable summand is parity and $\mathcal{T}(\nu) = \mathcal{E}(\nu)$ as was to be shown.

Proposition 5.2, which will be proved in the next section, says that for p greater than the bounds, every tilting module occurs as a direct summand of a tensor product of $\mathcal{T}(\lambda)$'s for λ either minuscule or the highest short root. Lemma 5.4 shows that, for good p (and thus for all p satisfying our bound), $\mathcal{T}(\lambda) = \mathcal{E}(\lambda)$ for minuscule highest weights and the highest short root. We can then apply the previous paragraph to the situation of Proposition 5.2 and the theorem is proved. $\square$

5.2. **Generating the tilting modules.** We give a case-by-case proof of the proposition 5.2. Because every indecomposable tilting module

appears as a direct summand of a tensor product of fundamental tilting modules, it suffices to show that, for p greater than the bounds given above, all the fundamental tilting modules appear as direct summands of tensor products of those tilting modules corresponding to minuscule weights or the highest short root.

In several occasions we needed to know for which primes the Weyl modules remain simple, and found the answer in [Jan03], [Jan91] or [Lüb01].

Remark 5.7. We do not know in all cases the exact bound for p for the statement in the proposition to hold. However, Stephen Donkin pointed out to us that it fails in some cases for $p \leq n$ in type C_n.

In what follows, We use Bourbaki's notation [Bou68, Planches] for roots, simple roots, fundamental weights, etc.

5.2.1. *Type A_n.* All fundamental weights are minuscule, so there is nothing to prove.

5.2.2. *Type B_n.* The weight ϖ_1 is the dominant short root and ϖ_n is minuscule. We have $V(\varpi_n) = L(\varpi_n) = T(\varpi_n)$ for all p, and $V(\varpi_1) = L(\varpi_1) = T(\varpi_1)$ for $p > 2$.

For $1 \leq i \leq n - 1$, we have $\Lambda^i V(\varpi_1) \simeq V(\varpi_i)$ [Bou75, Chap. VIII, §13]. Moreover, this module is simple, hence tilting, as soon as $p > 2$. If $p > i$, then the ith exterior power splits as a direct summand of the ith tensor power. Thus the claim is true for $p > n - 1$.

5.2.3. *Type C_n.* The highest short root is ϖ_2, and the weight ϖ_1 is minuscule. We have $V(\varpi_1) = L(\varpi_1) = T(\varpi_1)$ for all p, and $V(\varpi_2) = L(\varpi_2) = T(\varpi_2)$ for $p \nmid n$.

For $p > n$, we have $V(\lambda_i) = L(\lambda_i) = T(\lambda_i)$ for all i, and

$$\Lambda^i V(\varpi_1) \simeq V(\varpi_i) \oplus V(\varpi_{i-2}) \oplus \cdots$$

as this is true at the level of characters, and there are no extension between tilting modules. Thus the claim is true for $p > n$.

5.2.4. *Type D_n.* There are three minuscule weights: $\varpi_1, \varpi_{n-1}, \varpi_n$. The highest (short) root is ϖ_2. The Weyl modules for minuscule weights are simple, hence tilting, for all p. The Weyl modules for the other fundamental weights are simple, hence tilting, as soon as $p > 2$. Moreover, we have $\Lambda^i V(\varpi_1) \simeq V(\varpi_i)$ for $1 \leq i \leq n - 2$. Thus the claim is true for $p > n - 2$.

5.2.5. *Type E_6.* The minuscule weights are ϖ_1 and ϖ_6, and the highest (short) root is ϖ_2. Thus $V(\varpi_1)$ and $V(\varpi_6)$ are simple, hence tilting, for any p, and $V(\varpi_2)$ is simple, hence tilting, for $p \neq 3$.

We have $\Lambda^2 V(\varpi_1) \simeq V(\varpi_3)$ and $\Lambda^2 V(\varpi_6) \simeq V(\varpi_5)$, so $T(\varpi_3)$, resp. $T(\varpi_5)$, appears as a direct summand of $T(\varpi_1)^{\otimes 2}$, resp. $T(\varpi_6)^{\otimes 2}$, as soon as $p > 2$ (and they are simple Weyl modules).

We have

$$\Lambda^2 V(\varpi_2) \simeq V(\varpi_4) \oplus V(\varpi_2)$$

in characteristic zero, and it is still true in characteristic $p > 3$, as $V(\varpi_4)$ is simple for $p > 3$ and $V(\varpi_2)$ is simple for $p \neq 3$. Thus $T(\varpi_4)$ appears as a direct summand of $T(\varpi_2)^{\otimes 2}$ as soon as $p > 3$.

5.2.6. *Type E_7.* The weight ϖ_7 is minuscule, and the highest (short) root is ϖ_1.

For $p > 19$, we have

$$V(\varpi_1)^{\otimes 2} \quad \simeq \quad V(2\varpi_1) \oplus V(\varpi_1) \oplus V(\varpi_3) \oplus V(\varpi_6) \oplus V(0)$$

$$V(\varpi_6) \otimes V(\varpi_7) \quad \simeq \quad V(\varpi_6 + \varpi_7) \oplus V(\varpi_1 + \varpi_7) \oplus V(\varpi_2) \oplus \\ V(\varpi_5) \oplus V(\varpi_7)$$

$$V(\varpi_5) \otimes V(\varpi_7) \quad \simeq \quad V(\varpi_5 + \varpi_7) \oplus V(\varpi_2 + \varpi_7) \oplus V(\varpi_1 + \varpi_6) \oplus \\ V(\varpi_3) \oplus V(\varpi_6) \oplus V(\varpi_4)$$

and all these Weyl modules are tilting modules.

5.2.7. *Type E_8.* There is no minuscule weight. The highest (short) root is ϖ_8. For $p > 31$, we have

$$V(\varpi_8)^{\otimes 2} \quad \simeq \quad V(2\varpi_8) \oplus V(\varpi_7) \oplus V(\varpi_1) \oplus V(\varpi_8) \oplus V(0)$$

$$V(\varpi_7) \otimes V(\varpi_8) \quad \simeq \quad V(\varpi_7 + \varpi_8) \oplus V(\varpi_1 + \varpi_8) \oplus V(2\varpi_8) \oplus \\ V(\varpi_8) \oplus V(\varpi_7) \oplus V(\varpi_6) \oplus V(\varpi_2) \oplus V(\varpi_1)$$

$$V(\varpi_6) \otimes V(\varpi_8) \quad \simeq \quad V(\varpi_6 + \varpi_8) \oplus V(\varpi_7 + \varpi_8) \oplus V(\varpi_2 + \varpi_8) \oplus \\ V(\varpi_1 + \varpi_8) \oplus V(\varpi_1 + \varpi_7) \oplus V(\varpi_7) \oplus \\ V(\varpi_6) \oplus V(\varpi_5) \oplus V(\varpi_3) \oplus V(\varpi_2)$$

$$V(\varpi_5) \otimes V(\varpi_8) \quad \simeq \quad V(\varpi_5 + \varpi_8) \oplus V(\varpi_1 + \varpi_6) \oplus V(\varpi_2 + \varpi_8) \oplus \\ V(\varpi_6 + \varpi_8) \oplus V(\varpi_1 + \varpi_2) \oplus V(\varpi_3 + \varpi_8) \oplus \\ V(\varpi_2 + \varpi_7) \oplus V(\varpi_7) \oplus V(\varpi_6) \oplus \\ V(\varpi_5) \oplus V(\varpi_4) \oplus V(\varpi_3)$$

and all these Weyl modules are tilting modules.

5.2.8. *Type F_4.* The short dominant root is ϖ_4. For $p \neq 3$, we have $V(\varpi_4) = L(\varpi_4) = T(\varpi_4)$.

The Weyl module $V(\varpi_1)$, resp. $V(\varpi_3)$, is simple (hence tilting) for $p > 2$, resp. $p > 3$ [Jan91]. For $p > 3$, we have

$$\Lambda^2 V(\varpi_4) \simeq V(\varpi_1) \oplus V(\varpi_3)$$

as this is true in characteristic zero, and there are no non-trivial extensions between these two simple (hence tilting) Weyl modules.

Similarly, we have

$$\Lambda^3 V(\varpi_4) \simeq V(\varpi_2) \oplus V(\varpi_1 + \varpi_4) \oplus V(\varpi_3)$$

in characteristic zero, and it is still true in characteristic $p > 3$, as the three Weyl modules on the right hand side are simple for $p > 3$ (actually the middle one is also simple for $p = 3$), and there are no non-trivial extensions between them, as above.

So, for $p > 3$, we can get $T(\varpi_1)$ and $T(\varpi_3)$ as direct summands of $T(\varpi_4)^{\otimes 2}$, and $T(\varpi_2)$ as a direct summand of $T(\varpi_4)^{\otimes 3}$.

5.2.9. *Type G_2.* The short dominant root is ϖ_1. The orbit closure $\mathcal{G}r_{\varpi_1}$ has a minimal singularity of type g_2 at 0. The Weyl module $V(\varpi_1)$ is simple, hence tilting, as soon as $p > 2$. However, the sheaf $\mathcal{J}_!(\varpi_1, \mathbb{F}_p)$ is parity only for $p > 3$.

For $p > 3$, we have $\Lambda^2 V(\varpi_1) \simeq V(\varpi_1) \oplus V(\varpi_2)$, as this is true in characteristic zero, and these Weyl modules are simple for $p > 3$.

5.3. **q-Characters for tilting modules.** Having proved that the tilting sheaves are parity for p sufficiently large, we are able to deduce a number of corollaries. The first of which is that there are naturally graded characters, or q-characters for tilting modules. More precisely,

Corollary 5.8. *The stalk of the tilting sheaf $\mathcal{T}(\lambda)$ at a point in $\mathcal{G}r_\nu$ for p larger than the bound in proposition 5.2 has the same dimension as the weight space $T(\lambda)^\nu$ and thus the dimension of the weight space has natural graded refinement.*

Proof. We need a little bit of notation in order to recall the weight functors of Mirković-Vilonen. Let t^λ denote the point of $\mathcal{G}r$ obtained from the character $\lambda \in \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathrm{Hom}(\mathbb{C}^\times, T^\vee)$ by composing with the natural inclusions $\mathrm{Spec}(\mathcal{K}) \to \mathbb{C}^\times$ and $T^\vee \to G^\vee$. Let 2ρ denote the sum of the positive roots of G. It give an action of $\mathbb{C}^\times$ on $\mathcal{G}r$ by composing it with the action of $T^\vee$ on $\mathcal{G}r$.

Recall that by Mirković-Vilonen [MV07], the weight space functor F_ν corresponds under geometric Satake to the cohomology with compact

support of the restriction to the subvariety

$$S_\nu = \{x \in \mathcal{G}r | \lim_{s \to 0} 2\rho(s)x = t^\nu, \}$$

i.e. the attracting set of t^ν for the $\mathbb{C}^\times$-action defined by 2ρ.

In the terminology of Braden [Bra03], this is an example of a hyperbolic localisation. Namely, t^ν is an isolated fixed point of the $\mathbb{C}^\times$-action 2ρ, and if we denote the inclusions $f^+ : \{t^\nu\} \to S_\nu$, $g^+ : S_\nu \to \mathcal{G}r$, then the hyperbolic localisation $\mathcal{F}^{!*} = (f^+)^!(g^+)^*\mathcal{F}$ of a sheaf $\mathcal{F} \in D(\mathcal{G}r)$ is equal to $H_c^*(\mathcal{F}|_{S_\nu})$, the Mirković-Vilonen functor applied to $\mathcal{F}$.

As explained in [Bra03, Prop. 3], the local Euler characteristic of a sheaf $\mathcal{F}$ at a point x is equal to the Euler characteristic of any hyperbolic localisation of $\mathcal{F}$. Therefore, the stalk of a perverse sheaf in the Satake category at the point t^ν has an Euler characteristic of absolute value equal to the dimension of the ν-th weight space of the corresponding representation of $G^\vee$. On the other hand, the parity sheaf $\mathcal{T}(\lambda)$ has stalk concentrated in even or odd degree, thus the dimension of the stalk and weight space are equal. The stalk has a natural grading and thus the dimension of the weight space inherits a natural grading. □

5.4. **Simple S_d-modules.** One source of interest in tilting modules for GL_n is that they can be used to compute the dimensions of irreducible representations of the symmetric group S_d. In this section, we briefly remind the reader how this is done (a more complete survey can be found in [Jan03, E.14-17]) and then illustrate how the above theory may be used to express these dimensions in terms of the ranks of certain intersection forms associated to the affine Grassmannian.

We first recall Schur-Weyl duality. Let V be a vector space over a field k of dimension n, then V is a simple $GL(V)$-module and one has obvious commuting $GL(V)$ and S_d actions on the space $V^{\otimes d}$. If $n \geq d$ then the group algebra injects into the commutant of the $GL(V)$-action (consider $e_1 \otimes e_2 \otimes \cdots \otimes e_d$). In formulas

$$kS_d \hookrightarrow \operatorname{End}_{GL(V)}(V^{\otimes d}).$$

It is a fact known as Schur-Weyl duality that this is actually an isomorphism. It is called a duality because S_d and $GL(V)$ are in fact commutants of each other. This fact was first proved by Carter and Lusztig [CL74] and a geometric proof is to appear in [Mau].

Now the standard representation V of $GL(V)$ is a minuscule representation and thus a tilting module. Recall that the tensor product of tilting modules is tilting and any tilting module is a direct sum of indecomposable tilting modules $T(\lambda)$. There is therefore a decomposition

of $V^{\otimes d}$ into indecomposable tilting modules

$$V^{\otimes d} \cong \bigoplus_{\lambda \in \Lambda} T(\lambda)^{\oplus m_\lambda}.$$

We then have an isomorphism

$$kS_d \cong \mathrm{End}_{GL(V)} \left(\bigoplus_{\lambda \in \Lambda} T(\lambda)^{\oplus m_\lambda} \right).$$

This isomorphism allows us to write $\mathrm{Id} = \sum \mathrm{pr}_\lambda$ where pr_λ denotes the projection onto $T(\lambda)^{\oplus m_\lambda}$. Each pr_λ is central, and hence exactly one pr_λ can act non-trivially on a simple kS_d module. Now $\mathrm{End}_{GL(V)}(T(\lambda)^{\oplus m_\lambda})$ and $\mathrm{End}_{GL(V)}(T(\lambda))$ are Morita equivalent and $T(\lambda)$ is indecomposable so $\mathrm{End}_{GL(V)}(T(\lambda))$ is a local ring, hence there is only one simple $\mathrm{End}_{GL(V)}(T(\lambda)^{\oplus m_\lambda})$-module, and its dimension is m_λ. One then concludes that the dimensions of the simple kS_d-modules are $\{m_\lambda \mid \lambda \in \Lambda\}$.

We now translate this onto the affine Grassmannian. By the geometric Satake theorem, the tensor product $V^{\otimes d}$ corresponds to the perverse sheaf

$$\mathbf{IC}(\mathcal{G}r_{\varpi_1})^{*d},$$

whose decomposition into indecomposable perverse sheaves we would like to understand. Recall that because V is minuscule, $\mathbf{IC}(\varpi_1) = \mathbf{IC}(\mathcal{G}r_{\varpi_1})$ is the constant sheaf $\underline{k}_{\varpi_1}[n]$ in all characteristics.

Let P_{ϖ_1} denote the inverse image of $\mathcal{G}r_{\varpi_1}$ under the quotient map $G(\mathcal{K}) \to \mathcal{G}r$. Let

$$\mathrm{Res}(n\varpi_1) = P_{\varpi_1} \times_{G(\mathcal{O})} P_{\varpi_1} \times_{G(\mathcal{O})} \ldots P_{\varpi_1}/G(\mathcal{O}).$$

and

$$\pi : \mathrm{Res}(n\varpi_1) \to \mathcal{G}r$$

denote the multiplication. It is a semi-small resolution of singularities of $\overline{\mathcal{G}r}_{d\varpi_1}$, which is smooth of dimension nd and thus one has an isomorphism

$$[\mathbf{IC}(\varpi_1)]^{*d} \cong \pi_* \underline{k}_{\mathrm{Res}(d\varpi_1)}[nd]$$

This implies that the perverse sheaf $[\mathbf{IC}(\varpi_1)]^{*d}$ is parity by the Theorem 4.5 that convolution preserves parity (or alternatively by our Theorem 5.1 that the tilting sheaves are parity).

Hence our earlier machinery applies and we are obtain by the conclusion of section 3.3:

$$\pi_* \underline{k}_{\mathrm{Res}(d\varpi_1)}[nd] \cong \bigoplus_{\lambda \in \Lambda} \mathcal{E}(\lambda)^{r_\lambda}$$

where r_λ denotes the rank (over k) of the intersection form attached to $\mathcal{G}r_\lambda$. It then follows that the dimensions of the simple kS_d-modules are also given by these ranks.

Appendix A. Connections with work of Cline-Parshall-Scott

In the paper of Cline-Parshall-Scott [CPS93], the authors study the implications of certain parity vanishing of Ext groups in the derived category of a highest weight category $\mathcal{C}$. Let Λ denote the weight poset and for each $\lambda \in \Lambda$, let $\Delta(\lambda)$ be the standard object (or 'Weyl object') and $\nabla(\lambda)$ the costandard object (or 'induced object'). Cline-Parshall-Scott consider objects $M \in D^b(\mathcal{C})$ such that

$$\mathrm{Ext}^k_{D^b(\mathcal{C})}(M, \nabla(\lambda)) = 0$$

for either all odd or even k (they say such an object is in $\mathcal{E}^L$ or $\mathcal{E}^L[1]$). Our notion of $*$-parity could be understood as a geometric analogue of this definition in the category of sheaves $D(X)$. To see this, recall that the costandard objects in the category of perverse sheaves are the perverse $*$-extensions. By the adjunction of the pair $(j^*, {}^p j_*)$,

$$\mathrm{Ext}_{D(X)}(\mathcal{M}, {}^p(j_\lambda)_* \mathcal{L}) = \mathrm{Ext}_{D(X_\lambda)}(\mathcal{M}|_{X_\lambda}, \mathcal{L}).$$

If the complex $\mathcal{M}$ is $*$-parity, then $\mathcal{M}|_{X_\lambda}$ has parity-vanishing cohomology and by our assumption of vanishing $\mathrm{Hom}^n(\mathcal{L}', \mathcal{L})$ in odd degree, we get that the Ext-groups satisfy parity-vanishing. Similarly for !-parity complexes and Cline-Parshall-Scott's objects in $\mathcal{E}^R$.

Using their parity objects, the main theorem of [CPS93] reduces the Lusztig conjecture to a statement about the non-vanishing of Ext^1 between simple modules with p-regular highest weights which are mirror images of each other across adjacent p-alcoves.

If one considers a space X which satisfies our conditions and such that the category of perverse sheaves on it is a highest weight category, then it makes sense to compare, at least for perverse sheaves, our geometric notion of parity with the algebraic one of [CPS93].

Beilinson-Ginzburg-Soergel [BGS96] consider the case when the stratification is made up of affine cells and the coefficients are $\mathbb{C}$. In this case, they note that there is a natural functor from the algebraic to the geometric and that it is an equivalence. In such a situation, for example on the flag variety G/B stratified by B-orbits (or the affine Grassmannian by its Iwahori orbits) with respect to the constructible derived category, the vanishing studied here should coincide with that of Cline-Parshall-Scott.

More generally, if the strata of X are not contractible (for example the Satake category), or if we work in the equivariant derived category, the Ext groups are, in some sense, much bigger in the geometric category, $D(X)$, than in the derived category of perverse sheaves.

References

[BB81]   A. Beĭlinson and J. Bernstein. Localisation de $\mathfrak{g}$-modules. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(1):15–18, 1981. 2

[BBD82]   A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analyse et topologie sur les espaces singuliers, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982. 2, 9, 27

[BBFK99] G. Barthel, J.-P. Brasselet, K.-H. Fieseler, and L. Kaup. Equivariant intersection cohomology of toric varieties. In *Algebraic geometry: Hirzebruch 70 (Warsaw, 1998)*, volume 241 of *Contemp. Math.*, pages 45–68. Amer. Math. Soc., Providence, RI, 1999. 8

[BBM04]   A. Beilinson, R. Bezrukavnikov, and I. Mirković. Tilting exercises. *Mosc. Math. J.*, 4(3):547–557, 782, 2004. 7, 8, 42

[BGS96]   A. Beilinson, V. Ginzburg, and W. Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.*, 9(2):473–527, 1996. 7, 50

[BJ01]   M. Brion and R. Joshua. Vanishing of odd-dimensional intersection cohomology. II. *Math. Ann.*, 321(2):399–437, 2001.

[BK81]   J.-L. Brylinski and M. Kashiwara. Kazhdan-Lusztig conjecture and holonomic systems. *Invent. Math.*, 64(3):387–410, 1981. 2

[BL94a]   A. Beauville and Y. Laszlo. Conformal blocks and generalized theta functions. *Comm. Math. Phys.*, 164(2):385–419, 1994. 8

[BL94b]   J. Bernstein and V. Lunts. *Equivariant sheaves and functors*, volume 1578 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994. 10

[BM83]   W. Borho and R. MacPherson. Partial resolutions of nilpotent varieties. In *Analyse et topologie sur les espaces singuliers, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 23–74. Soc. Math. France, Paris, 1983. 25

[BM01]   T. Braden and R. MacPherson. From moment graphs to intersection cohomology. *Math. Ann.*, 321(3):533–551, 2001. 8

[Bor61]   A. Borel. Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes. *Tôhoku Math. J. (2)*, 13:216–240, 1961. 19

[Bou68]   N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines.* Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968. 45

[Bou75]   N. Bourbaki. *Éléments de mathématique.* Hermann, Paris, 1975. Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de

Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées, Actualités Scientifiques et Industrielles, No. 1364. 45

[Bra03]   T. Braden. Hyperbolic localization of intersection cohomology. *Transform. Groups*, 8(3):209–216, 2003. 47, 48

[Bri]   M. Brion. http://www-fourier.ujf-grenoble.fr/ bonavero/articles/ecoledete/brion.ps. notes from a summer school on the geometry of toric varieties, `http://www-fourier.ujf-grenoble.fr/~bonavero/articles/ecoledete/brion.ps`. 36

[BZ03]   J.-L. Brylinski and B. Zhang. Equivariant todd classes for toric varieties. Preprint arXiv/0311318, 2003. 37

[CG97]   N. Chriss and V. Ginzburg. *Representation theory and complex geometry*. Birkhäuser Boston Inc., Boston, MA, 1997. 10, 21, 25

[CL74]   R. W. Carter and G. Lusztig. On the modular representations of the general linear and symmetric groups. *Math. Z.*, 136:193–242, 1974. 48

[CPS93]   E. Cline, B. Parshall, and L. Scott. Abstract Kazhdan-Lusztig theories. *Tohoku Math. J. (2)*, 45(4):511–534, 1993. 7, 11, 49, 50

[Dan78]   V. I. Danilov. The geometry of toric varieties. *Uspekhi Mat. Nauk*, 33(2(200)):85–134, 247, 1978. 36, 37

[DCLP88]   C. De Concini, G. Lusztig, and C. Procesi. Homology of the zero-set of a nilpotent vector field on a flag manifold. *J. Amer. Math. Soc.*, 1(1):15–34, 1988.

[dCM02]   M. A. A. de Cataldo and L. Migliorini. The hard Lefschetz theorem and the topology of semismall maps. *Ann. Sci. École Norm. Sup. (4)*, 35(5):759–772, 2002. 4, 5, 20, 26

[dCM05]   M. A. A. de Cataldo and L. Migliorini. The Hodge theory of algebraic maps. *Ann. Sci. École Norm. Sup. (4)*, 38(5):693–750, 2005. 4

[Dem73]   M. Demazure. Invariants symétriques entiers des groupes de Weyl et torsion. *Invent. Math.*, 21:287–301, 1973. 19

[Don93]   S. Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993. 6

[Fei82]   W. Feit. *The representation theory of finite groups*, volume 25 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1982. 12

[Fie07]   P. Fiebig. Sheaves on affine Schubert varieties, modular representations and Lusztig's conjecture. Preprint arXiv:0711.0871, 2007. 3, 8

[Ful93]   W. Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry. 21, 34, 35

[FW]   P. Fiebig and G. Williamson. The *p*-smooth locus of Schubert varieties. in preparation. 3, 8

[GKM98]   M. Goresky, R. Kottwitz, and R. MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998.

[GL05]   S. Gaussent and P. Littelmann. LS galleries, the path model, and MV cycles. *Duke Math. J.*, 127(1):35–88, 2005. 32

[GM88]   M. Goresky and R. MacPherson. *Stratified Morse theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. 14

[HS79]   R. Hotta and N. Shimomura. The fixed-point subvarieties of unipo-
           tent transformations on generalized flag varieties and the Green func-
           tions. Combinatorial and cohomological treatments cent. *Math. Ann.*,
           241(3):193–208, 1979. 38

[HS06]   T. Hausel and E. Swartz. Intersection forms of toric hyperkähler vari-
           eties. *Proc. Amer. Math. Soc.*, 134(8):2403–2409 (electronic), 2006. 9

[Jan91]   J. C. Jantzen. First cohomology groups for classical Lie algebras. In
           *Representation theory of finite groups and finite-dimensional algebras
           (Bielefeld, 1991)*, volume 95 of *Progr. Math.*, pages 289–315. Birkhäuser,
           Basel, 1991. 46

[Jan03]   J. C. Jantzen. *Representations of algebraic groups*, volume 107 of *Math-
           ematical Surveys and Monographs*. American Mathematical Society,
           Providence, RI, second edition, 2003. 42, 46, 48

[JMW09]   D. Juteau, C. Mautner, and G. Williamson. Perverse sheaves and mod-
           ular representation theory. preprint arXiv:0901.3322, 2009. 2

[Jut07]   D. Juteau. *Modular Springer correspondence and decomposition matri-
           ces*. PhD thesis, Université Paris 7 Denis Diderot, 2007. Available at
           www.math.jussieu.fr/~juteau/manuscrit/these-juteau.pdf. 2

[Jut08a]   D. Juteau. Cohomology of the minimal nilpotent orbit. *Transform.
           Groups*, 13(2):355–387, 2008. 41

[Jut08b]   D. Juteau. Modular representations of reductive groups and geometry
           of affine grassmannians. preprint arXiv:0804.2041, 2008. 7, 41, 44

[Jut09]   D. Juteau. Decomposition numbers for perverse sheaves. *Ann. Inst.
           Fourier*, 59, 2009. To appear. 41

[Kac85]   V. G. Kac. Torsion in cohomology of compact Lie groups and Chow
           rings of reductive algebraic groups. *Invent. Math.*, 80(1):69–79, 1985. 19

[KL80]   D. Kazhdan and G. Lusztig. Schubert varieties and Poincaré duality.
           In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ.
           Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI,
           pages 185–203. Amer. Math. Soc., Providence, R.I., 1980.

[KS94]   M. Kashiwara and P. Schapira. *Sheaves on manifolds*, volume 292 of
           *Grundlehren der Mathematischen Wissenschaften [Fundamental Prin-
           ciples of Mathematical Sciences]*. Springer-Verlag, Berlin, 1994. With a
           chapter in French by Christian Houzel, Corrected reprint of the 1990
           original. 17

[Kum02]   S. Kumar. *Kac-Moody groups, their flag varieties and representation
           theory*, volume 204 of *Progress in Mathematics*. Birkhäuser Boston Inc.,
           Boston, MA, 2002. 20, 28, 29, 31, 32

[LC07]   J. Le and X.-W. Chen. Karoubianness of a triangulated category. *J.
           Algebra*, 310(1):452–457, 2007. 10

[Lüb01]   F. Lübeck. Small degree representations of finite Chevalley groups in
           defining characteristic. *LMS J. Comput. Math.*, 4:135–169 (electronic),
           2001.

[Lus86]   G. Lusztig. Character sheaves. *Adv. Math.*, 1985-1986. I 56:193–237,
           1985 ; II 57:226–265, 1985 ; III 57:266–315, 1985 ; IV 59:1–63, 1986 ; V
           61:103–155, 1986. 39

[Mau]   C. Mautner. Geometric Schur functors. in preparation. 3, 48

[MOV05]   A. Malkin, V. Ostrik, and M. Vybornov. The minimal degeneration singularities in the affine Grassmannians. *Duke Math. J.*, 126(2):233–249, 2005. 43

[Mus05]   M. Mustata. Lecture notes on toric varieties. Lecture notes, `http://www.math.lsa.umich.edu/~mmustata/toric_var.html`, 2005. 34

[MV07]   I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007. 2, 6, 7, 31, 41, 43, 47

[Nad05]   D. Nadler. Perverse sheaves on real loop Grassmannians. *Invent. Math.*, 159(1):1–73, 2005. 31

[Pan91]   D. I. Panyushev. Rationality of singularities and the Gorenstein property of nilpotent orbits. *Funktsional. Anal. i Prilozhen.*, 25(3):76–78, 1991.

[Rin91]   C. M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.*, 208(2):209–223, 1991. 6

[RS65]   M. Rothenberg and N. E. Steenrod. The cohomology of classifying spaces of H-spaces. *Bull. Amer. Math. Soc.*, 71:872–875, 1965. 19

[Ser67]   J.-P. Serre. *Représentations linéaires des groupes finis*. Hermann, Paris, 1967. 17, 18

[Soe00]   W. Soergel. On the relation between intersection cohomology and representation theory in positive characteristic. *J. Pure Appl. Algebra*, 152(1-3):311–335, 2000. 2, 7, 33, 34

[Soe01]   W. Soergel. Langlands' philosophy and Koszul duality. In *Algebra—representation theory (Constanta, 2000)*, volume 28 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 379–414. Kluwer Acad. Publ., Dordrecht, 2001. 15

[Spr82]   T. A. Springer. Quelques applications de la cohomologie d'intersection. In *Bourbaki Seminar, Vol. 1981/1982*, volume 92 of *Astérisque*, pages 249–273. Soc. Math. France, Paris, 1982. 31

[Ste75]   R. Steinberg. Torsion in reductive groups. *Advances in Math.*, 15:63–92, 1975. 18, 19

[Wan99]   W. Wang. Dimension of a minimal nilpotent orbit. *Proc. Amer. Math. Soc.*, 127(3):935–936, 1999. 39

[Wil08]   G. Williamson. *Singular Soergel bimodules*. PhD thesis, Albert-Ludwigs-Universität Freiburg, 2008. Available at `http://people.maths.ox.ac.uk/williamsong/`. 33

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