Facets of accessibility

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M. Makkai, R. Paré 1989:

Definition. A category $\mathcal{K}$ is called λ-*accessible*, where λ is a regular cardinal, provided that

(1) $\mathcal{K}$ has λ-filtered colimits,

(2) $\mathcal{K}$ has a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-filtered colimit of objects from $\mathcal{A}$.

An object A is λ-presentable if its hom-functor

$$\hom(A, -) : \mathcal{K} \to \mathbf{Set}$$

preserves λ-filtered colimits.

A category is *accessible* if it is λ-accessible for some regular cardinal λ. A cocomplete λ-accessible category is called *locally λ-presentable.* A category is *locally presentable* is it is locally λ-presentable for some regular cardinal λ.

A formula in an infinitary many-sorted logic $L_{\alpha\beta}$ is called

(a) *positive-primitive* if it has the form $(\exists y)\psi(x, y)$ where $\psi(x, y)$ is a conjunction of atomic formulas,

(b) *positive-existential* if it is a disjunction of positive-primitive formulas,

(c) *basic* it it has the form

$$(\forall x)(\varphi(x) \to \psi(x))$$

where $\varphi(x)$ and $\psi(x)$ are positive-existential formulas.

Theorem 1. Accessible categories are precisely the categories equivalent to categories of models of basic theories.

Theorem 2. For each theory T of $L_{\alpha\beta}$, the category of T-models and $L_{\alpha\beta}$-elementary embeddings is accessible.

Given morphisms $f : A \to B$ and $g : C \to D$ in a category $\mathcal{K}$, we write

$$f \Box g$$

if in each commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd}$$

there is a diagonal $d : B \to C$ with $df = u$ and $gd = v$.

For a class $\mathcal{H}$ of morphisms of $\mathcal{K}$ we put

$$\mathcal{H}^{\Box} = \{g | f \Box g \text{ for each } f \in \mathcal{H}\},$$
$$^{\Box}\mathcal{H} = \{f | f \Box g \text{ for each } g \in \mathcal{H}\}.$$

The relation $f \square g$ can be interpreted as

$$f \models g$$

i.e., that the morphism g is a model of a morphism f. Then

$$\mathcal{H} = \mathbf{Mod}\,\mathcal{H}$$

is the class of models of $\mathcal{H}$ while

$$^{\square}\mathcal{H} = \mathbf{Th}\,\mathcal{H}$$

is the theory of $\mathcal{H}$.

The passage to the dual category interchanges models and theories.

An object K is $\mathcal{H}$-*injective* iff $K \to 1$ belongs to $\mathbf{Mod}\,\mathcal{H}$. This means that K is a model of $\mathcal{H}$. Analogously, K is $\mathcal{H}$-*projective* iff $0 \to K$ belongs to $\mathbf{Th}\,\mathcal{H}$.

Given two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms of $\mathcal{K}$, the pair $(\mathcal{L}, \mathcal{R})$ is called a *weak factorization system* if

1. $\mathcal{R} = \mathcal{L}^{\square}$, $\mathcal{L} = {}^{\square}\mathcal{R}$

and

(2) any morphism h of $\mathcal{K}$ has a factorization $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

A *transfinite composition* is the component f_0 of a colimit cocone $f_i : K_i \to K$ of a smooth chain of morphisms $(f_{ij} : K_i \to K_j)_{i<j<\lambda}$ (i.e., λ is a limit ordinal, $f_{jk} \cdot f_{ij} = f_{ik}$ for $i < j < k$ and $f_{ij} : K_i \to K_j$ is a colimit cocone for any limit ordinal $j < \lambda$).

The smallest class of morphisms of $\mathcal{K}$ containing isomorphisms and being closed under transfinite compositions, pushouts of morphisms from $\mathcal{H}$ and retracts (in the category $\mathcal{K}^{\to}$ of morphisms of $\mathcal{K}$) is denoted $\mathrm{cof}(\mathcal{H})$. This class can be interpreted as the deductive closure of $\mathcal{H}$.

Theorem 3. Let $\mathcal{K}$ be a locally presentable category and $\mathcal{C}$ a set of morphisms of $\mathcal{K}$. Then $(\mathrm{cof}(\mathcal{C}), \mathcal{C}^{\square})$ is a weak factorization system in $\mathcal{K}$.

Such weak factorization systems are called *cofibrantly generated.* In homotopy theory, Theorem 4 is called a *small object argument.*

A weak factorization

$$K \to \overline{K} \to 1$$

yields that $\mathcal{K}$ has enough $\mathcal{L}$-injectives. Analogously, it has enough $\mathcal{R}$-projectives. One also gets that $\mathcal{L}$-injectives form a weakly reflective full subcategory of $\mathcal{K}$. Dually, $\mathcal{R}$-projectives form a weakly coreflective full subcategory.

(Flat monomorphisms, cotorsion epimorphisms) is a weak factorization system in the category of R-modules responsible for the existence of flat (pre)covers and cotorsion (pre)envelopes.

Vopěnka's principle says that no accessible category has a large rigid class of objects.

It is a large cardinal axiom – it implies the existence of a proper class of supercompact (even of extendible) cardinals and its consistency follows from the existence of a huge cardinal.

Vopěnka's principle implies (J. Adámek, J. R., 1994)

(R) Every full subcategory $\mathcal{L}$ of a locally presentable category closed under limits is reflective.

(WR) Every full subcategory $\mathcal{L}$ of a locally presentable category closed under products and retracts is weakly reflective.

Both (R) and (WR) are set-theoretical and

$$VP \Rightarrow (WR) \Rightarrow (R)$$

The equivalence of (VP) and (R) is an open problem.

Set-theoretical counterexamples to (R) or (WR) are artificial and it is natural to know what happens for "reasonable" $\mathcal{L}$. For such subcategories one might need less set-theory.

J. Bagaria, C. Casacuberta, A. Matthias, J. R. (2009):

Theorem 4. Assume the existence of a proper class of supercompact cardinals. Then (R) and (WR) hold for each $\mathbf{\Sigma}_2$-definable $\mathcal{L}$.

Let $\mathcal{L}$ be the closure of groups $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$, where κ is a cardinal, under products and retracts in the category of Abelian groups. Then the weak reflectivity of $\mathcal{L}$ lies between the existence of a supercompact cardinal and the existence of a measurable cardinal.

A *model category* is a complete and cocomplete category $\mathcal{K}$ together with three classes of morphisms $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{W}$ called *fibrations*, *cofibrations* and *weak equivalences* such that

(1) $\mathcal{W}$ has the *2-out-of-3 property*, i.e., with any two of f, g, gf belonging to $\mathcal{W}$ also the third morphism belongs to $\mathcal{W}$, and $\mathcal{W}$ is closed under retracts in $\mathcal{K}^{\to}$, and

(2) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems.

Morphisms from $\mathcal{F} \cap \mathcal{W}$ are called *trivial fibrations* while morphisms from $\mathcal{C} \cap \mathcal{W}$ *trivial cofibrations*.

A model category $\mathcal{K}$ is called *cofibrantly generated* provided that the both weak factorization systems above are cofibrantly generated. A cofibrantly generated model category is called *combinatorial* if $\mathcal{K}$ is locally presentable.

$$\operatorname{Ho}\mathcal{K} = \mathcal{K}[\mathcal{W}^{-1}]$$

is called the *homotopy category* of a model category $\mathcal{K}$.

Two distinct model categories $\mathcal{K}_1$ and $\mathcal{K}_2$ can have equivalent homotopy categories. If the equivalence of $\operatorname{Ho}\mathcal{K}_1$ and $\operatorname{Ho}\mathcal{K}_2$ is induced by a suitable adjoint pair on model categories $\mathcal{K}_1$ and $\mathcal{K}_2$, we say that $\mathcal{K}_1$ and $\mathcal{K}_2$ are *Quillen equivalent.*

The model category **Top** of topological spaces is cofibrantly generated but not combinatorial. It is Quillen equivalent to the combinatorial model category **SSet** of simplicial sets.

Theorem 5. (VP) is equivalent to the fact that every cofibrantly generated model category is Quilen equivalent to a combinatorial one.

The implication $\Rightarrow$ was proved by G. Raptis (2008), the converse is due to me (2009).

Almost all important model categories have a combinatorial model. Combinatorial model categories were introduced by J. H. Smith; the following basic result is due to him.

Theorem 6. Let $\mathcal{X}$ be a set of morphisms in a locally presentable category $\mathcal{K}$. Then $\mathcal{C} = \mathrm{cof}(\mathcal{X})$ and $\mathcal{W}$ make $\mathcal{K}$ a combinatorial model category if and only if

(1) $\mathcal{W}$ has the 2-out-of-3 property and is closed under retracts in $\mathcal{K}^{\to}$,

(2) $\mathcal{X}^{\square} \subseteq \mathcal{W}$,

(3) $\mathrm{cof}(\mathcal{X}) \cap \mathcal{W}$ is closed under pushout and transfinite composition, and

(4) $\mathcal{W}$ satisfies the solution set-condition at $\mathcal{X}$.

(4) can be replaced by $\mathcal{W}$ being accessible (as a full subcategory of $\mathcal{K}^{\rightarrow}$). This was also claimed by J. H. Smith; the proofs were independently obtained by J. Lurie and me.

(VP) is equivalent to deleting (4) in Theorem 6.

We have the functor $P : \mathcal{K} \to \operatorname{Ho} \mathcal{K}$.

$\operatorname{Ho} \mathcal{K}$ has products and coproducts (preserved by P) but only weak pullbacks and weak pushouts. Hence $\operatorname{Ho} \mathcal{K}$ has weak limits and weak colimits.

Weak pushouts are constructed in $\mathcal{K}$ as *homotopy pushouts*. Given

$$\begin{tikzcd} B \\ A \arrow[u, "f"] \arrow[r, "g"'] & D \end{tikzcd}$$

a homotopy pushout is a pushout

$$\begin{tikzcd} B_1 \arrow[r, "\overline{g}"] & E \\ A \arrow[u, "f_1"] \arrow[r, "g_1"'] & D_1 \arrow[u, "\overline{f}"'] \end{tikzcd}$$

where $f = f_2 f_1$ and $g = g_2 g_1$ are (cofibration, trivial fibration) factorizations.

Let $\mathcal{K}$ be a pointed model category. Homotopy pushouts

$$\begin{tikzcd}
0 \arrow[r] & \Sigma A \\
A \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

yield the *suspension functor* $\Sigma : \mathcal{K} \to \mathcal{K}$. A pointed model category is called *stable* if Σ is an equivalence in $\mathrm{Ho}\,\mathcal{K}$. $\mathrm{Ho}\,\mathcal{K}$ is then *triangulated* (and thus additive).

A triangulated category has triangles

$$A \to B \to C \to \Sigma A$$

playing the role of short exact sequences.

Examples of stable model categories are spectra or chain complexes over a ring R. As homotopy categories one gets the classical stable homotopy category or derived categories of R.

A full subcategory $\mathcal{L}$ of a triangulated category $\mathcal{T}$ is called *localizing* if it is closed under triangles, coproducts and retracts. Dually, it is called *colocalizing* if it is closed under triangles, products and retracts. These subcategories correspond to torsion and torsion free subcategories in abelian categories.

The following results (partially) solve open problems from M. Hovey, J. H. Palmieri and N. P. Strickland, Axiomatic Stable Homotopy theory (1997). There are due to C. Casacuberta, J. Gutiérrez, J. R.

Theorem 7. Let $\mathcal{K}$ be a locally presentable stable model category. Under (VP), every colocalizing subcategory of $\operatorname{Ho}\mathcal{K}$ is reflective.

Theorem 8. Let $\mathcal{K}$ be a combinatorial stable monoidal model category. Under (VP), every localizing subcategory of $\operatorname{Ho}\mathcal{K}$ is coreflective.

We do not know whether these results are really set-theoretical.

Let $\mathcal{L}$ be a category with coproducts and λ a cardinal. An object A of $\mathcal{L}$ is called λ-*small* if for every morphism $f : A \to \coprod_{i\in I} L_i$ there is a subset J of I of cardinality less than λ such that f factorizes as

$$A \to \coprod_{j\in J} L_j \to \coprod_{i\in I} L_i$$

where the second morphism is the subcoproduct injection.

A. Neeman (2001):

Consider classes $\mathcal{S}$ of λ-small objects of $\mathcal{A}$ such for every morphism $f : S \to \coprod_{i\in I} L_i$ with $S \in \mathcal{S}$ there are morphisms $g_i : S_i \to L_i$ where $S_i \in \mathcal{S}$ for each $i \in I$ such that f factorizes through $\coprod_{i\in I} g_i : \coprod_{i\in I} S_i \to \coprod_{i\in I} L_i$. Since these classes are closed under unions, there is the greatest class $\mathcal{S}$ with this property. Its objects will be called λ-*compact*.

J.R. (2005):

Theorem 9. Let $\mathcal{K}$ be a combinatorial model category. Then there are arbitrarily large regular cardinals λ such that the composition

$$\mathcal{K} \xrightarrow{P} \operatorname{Ho}\mathcal{K} \xrightarrow{E_{P(\mathcal{K}_\lambda)}} \mathbf{Set}^{P(\mathcal{K}_\lambda)^{\mathrm{op}}} .$$

preserves λ-filtered colimits.

Here, $\mathcal{K}_\lambda$ denotes the full subcategory of $\mathcal{K}$ consisting of λ-presentable objects.

Theorem 10. Let $\mathcal{K}$ be a locally λ-presentable model category such that the functor $E_{P(\mathcal{K}_\lambda)}P$ preserves λ-filtered colimits. Then PK is λ-compact for each λ-presentable object K of $\mathcal{K}$.

Let $\mathcal{L}$ be a category with a zero object. We say that a set $\mathcal{G}$ of objects *weakly generates* $\mathcal{L}$ if whenever $\hom(G, L) = \{0\}$ for each $G \in \mathcal{G}$ then $L = 0$.

A. Neeman (2001):

Let $\mathcal{L}$ be a category with coproducts and a zero objects. $\mathcal{L}$ is called *well generated* if it has a weakly generating set of λ-compact objects for some cardinal λ.

Theorem 11. Let $\mathcal{K}$ be a combinatorial pointed model category. Then $\operatorname{Ho}\mathcal{K}$ is well generated.

The existence of a weakly generating set in $\operatorname{Ho}\mathcal{K}$ is due to M. Hovey (1999). The result is important for stable model categories whose homotopy categories are triangulated.

Let λ be a regular cardinal. A λ-*pure monomorphism* is a morphism $f : K \to L$ such that given a commutative square

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ K \arrow[r, "f"'] & L \end{tikzcd}$$

with A and B λ-presentable, then u factorizes through g, i.e., $u = tg$ for some $t : B \to K$.

λ-pure monomorphisms correspond to submodels elementary with respect to positive-primitive formulas of $L_{\lambda\lambda}$.

Let $\mathcal{C}$ be a set of morphisms of a category $\mathcal{K}$ having λ-presentable domains and codomains. The full subcategory $\mathcal{C}^{\triangle}$ consisting of all $\mathcal{C}$-injective objects K of $\mathcal{K}$ is called a λ-*injectivity class*.

J. Adámek, F. Borceux, J.R. (2002):

Theorem 12. Let $\mathcal{K}$ be a locally λ-presentable category. A full subcategory of $\mathcal{K}$ is a λ-injectivity class iff it is closed under products, λ-filtered colimits and λ-pure subobjects.

For $\lambda = \omega$, the result immediately follows from the compactness theorem.

If, in the definition of $f \square g$, we require a unique diagonal, we get the relation

$$f \perp g$$

It leads to factorization systems and orthogonality in the place of weak factorization systems and injectivity. Given a set $\mathcal{C}$ of morphisms of a category $\mathcal{K}$, the full subcategory consisting of all $\mathcal{C}$-orthogonal objects K of $\mathcal{K}$ is called a *small-orthogonality class.*

Theorem 13. Let $\mathcal{K}$ be a locally presentable category and $\mathcal{C}$ a set of morphisms of $\mathcal{K}$. Then $(\mathrm{colim}(\mathcal{C}), \mathcal{C}^{\perp})$ is a factorization system in $\mathcal{K}$.

Here, $\mathrm{colim}\,\mathcal{C}$ denotes the closure of $\mathcal{C}$ under all colimits in $\mathcal{K}^{\rightarrow}$.

Consequently, small-orthogonality classes in locally presentable categories are reflective.

E. R. Fisher (1977):

Theorem 14. (VP) is equivalent to the fact that every limit-closed full subcategory of a locally presentable category is a small-orthogonality class.

An object C of a category $\mathcal{K}$ is orthogonal to a morphism $f : A \to B$ iff

$$\hom(f, C) : \hom(B, C) \to \hom(A, C)$$

is a bijection.

Let $\mathcal{K}$ be a model category. An object C is *homotopy orthogonal* to $f : A \to B$ if

$$\mathrm{map}(f, C) : \mathrm{map}(B, C) \to \mathrm{map}(A, C)$$

is a weak equivalence of simplicial sets.

Here, $\mathrm{map}(A, C)$ is the *homotopy function complex.* It is a simplicial set whose points form $\hom(A, C)$. We denote homotopy orthogonality by $\perp_h$. Homotopy orthogonality implies orthogonality in the homotopy category.

We say that a reflective full subcategory $\mathcal{L}$ of $\mathrm{Ho}\,\mathcal{K}$ is *strict* if its reflector $\mathrm{Ho}\,\mathcal{K} \to \mathcal{L}$ can be strictified, i.e., it equals to $\mathrm{Ho}(L)$ where $L : \mathcal{K} \to \mathcal{K}$.

C. Casacuberta, D. Scevenels, J. H. Smith (2005):

Theorem 15. Under (VP), every strict reflective subcategory of $\mathrm{Ho}\,\mathbf{SSet}$ is a homotopy small-orthogonality class.

The statement is set-theoretical: it implies the existence of a measurable cardinal.

C. Casacuberta, B. Chorny (2006):

Theorem 16. Let $\mathcal{K}$ be a left proper, combinatorial, simplicial model category. Under (VP), every strict reflective subcategory of $\mathrm{Ho}\,\mathcal{K}$ is a homotopy small-orthogonality class.

One can consider homotopy locally presentable categories in the context of simplicial categories (J. R. 2007) or quasicategories (J. Lurie 2003, 2006, A. Joyal 2008).