The solution of time harmonic wave equations using complete families of elementary solutions

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Outline

- Introduction
  - Least squares methods for Helmholtz:
  - Ultra Weak Variational Formulation
- Discretization
  - Using plane waves
  - Using finite elements
- Maxwell's equations
- Comments and conclusions

Model Problem

Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, find u such that

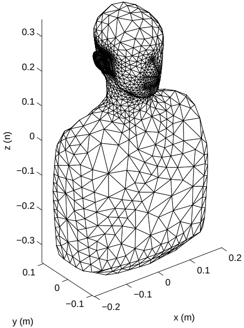
$$\begin{aligned} \Delta u + \kappa^2 u &= 0 \text{ in } \Omega \\ \frac{\partial u}{\partial n} - \mathrm{i}\kappa u &= g \text{ on } \Gamma := \partial\Omega. \end{aligned}$$

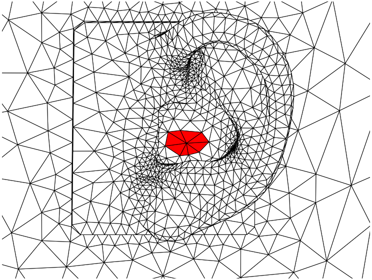
Assume κ is real.

The presentation focuses on “volume” methods for the Helmholtz equation.

Human hearing [Huttunen]

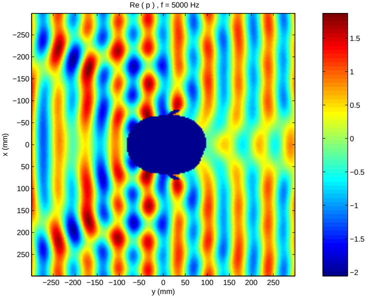
Investigate coupling of sound into the human ear (head related transfer function) across the audible spectrum.

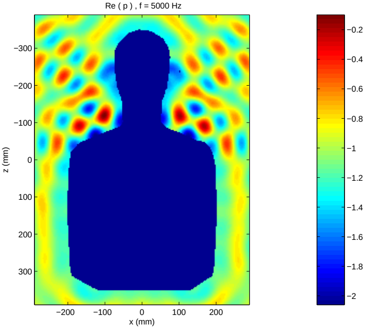




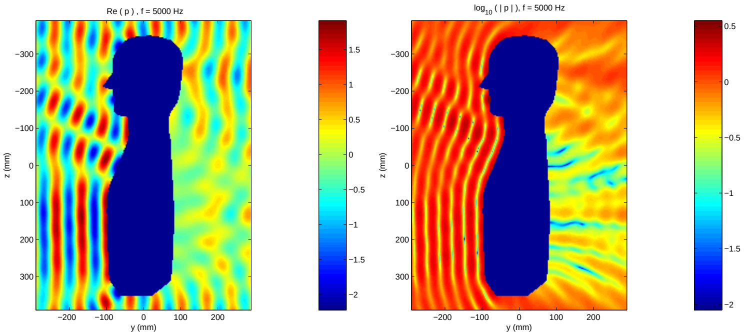
Human hearing (continued)

Real part of the pressure field at 5kHz (using the two meshes we can compute the audible range to 20kHz)





Human hearing (continued)



The Least Squares Method

Use the mesh $\mathcal{T}_h$ consisting of tetrahedra or triangles T_j, $1 \leq j \leq J_h$, of maximum diameter h.
At first we shall consider the "exact" least squares method so that

- Define u_j to be any $H^1(T_j)$ solution of the Helmholtz equation on T_j.
- To compute the exact solution we need to enforce continuity of u and $\partial u/\partial n$ on interior faces/edges and enforce the boundary condition.

Later we can discretize u_j for each j.

Some notation:
$\Gamma_{j,k} = \partial T_j \cap \partial T_k$ and $\Gamma_j = \partial T_j \cap \Gamma$. We denote by n_j the outward normal to T_j.

Required continuity between elements

For the piecewise defined function $(u_1, u_2, \cdots, u_{J_h})$ to be a global solution of the Helmholtz equation we need

$$\frac{\partial u_k}{\partial n_k} + \mathrm{i}\kappa u_k = -\frac{\partial u_j}{\partial n_j} + \mathrm{i}\kappa u_j$$
$$\frac{\partial u_k}{\partial n_k} - \mathrm{i}\kappa u_k = -\frac{\partial u_j}{\partial n_j} - \mathrm{i}\kappa u_j$$

on $\Gamma_{j,k}$.
The boundary condition on Γ_j is then

$$\frac{\partial u_j}{\partial n} - \mathrm{i}\kappa u_j = g$$

Least squares functional

Let $\vec{u} = (u_1, u_2, \cdots, u_{J_h})$ where u_j satisfies the Helmholtz equation in $H^1(T_j)$. To assure a global solution fitting the boundary condition we could minimize the functional $\mathcal{J}(\vec{u})$ over all such functions:

$$\begin{aligned}\mathcal{J}(\vec{u}) &= \sum_j \sum_{k \neq j} \left\| \left(-\frac{\partial u_j}{\partial \boldsymbol{n}_j} + \mathrm{i}\kappa u_j \right) - \left(\frac{\partial u_k}{\partial \boldsymbol{n}_k} + \mathrm{i}\kappa u_k \right) \right\|^2_{L^2(\partial T_j)} \\ &\quad + \sum_j \left\| \left(-\frac{\partial u_j}{\partial \boldsymbol{n}_j} + \mathrm{i}\kappa u_j \right) + g \right\|^2_{L^2(\Gamma_j)}\end{aligned}$$

There is a unique minimizer (existence and uniqueness of the standard boundary value problem) at least if the domain is smooth.

Rewriting the Least Squares Method

Let $X = \Pi_{j=1}^{J_h} L^2(\partial T_j)$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_X$. Let $\mathcal{X} \in X$ and write $\mathcal{X} = (\mathcal{X}_1, \cdots, \mathcal{X}_{J_h})$. Define

- $\Pi : X \to X$ such that

$$\begin{aligned} \Pi\mathcal{X}_j|_{\Gamma_{j,k}} &= \mathcal{X}_k|_{\Gamma_{j,k}} \text{ when } \Gamma_{j,k} \neq \phi \\ \Pi\mathcal{X}_j|_{\Gamma_j} &= 0 \text{ when } \Gamma_j \neq \phi. \end{aligned}$$

- $F : X \to X$ so that if $F(\mathcal{X}) = (F_1(\mathcal{X}_1), \cdots, F_{J_h}(\mathcal{Y}\mathcal{X}_{J_h}))$ and if $w_j \in H^1(T_j)$ satisfies

$$\begin{aligned} \Delta w_j + \kappa^2 w_j &= 0 \text{ in } T_j \\ \frac{\partial w_j}{\partial n_j} + i\kappa w_j &= \mathcal{X}_j \text{ on } \partial T_j \end{aligned}$$

  then $F_j(\mathcal{X}_j) = -\partial w_j/\partial n_j + \mathrm{i}\kappa w_j$ on ∂T_j

The Least Squares Method obscured

Let

$$\mathcal{X}_k = \left(\frac{\partial u_k}{\partial n_k} + \mathrm{i}\kappa u_k\right)\bigg|_{\partial T_k}, \quad \mathcal{X} = (\mathcal{X}_1, \cdots, \mathcal{X}_{J_h}),$$

the we may write the least squares problem as the problem of finding $\mathcal{X} \in X$ that minimizes

$$\mathcal{J}(\mathcal{X}) = \|F\mathcal{X} - \Pi\mathcal{X} + \tilde{g}\|_X^2.$$

where $\tilde{g} \in X$ is such that $\tilde{g}|_\Gamma = g$ and vanishes on other faces in the mesh (we have just parametrized the solution by it's boundary impedance trace on each element).

Lemma (Cessenat and Després)

$\|\Pi\|_{X \to X} \leq 1$ *and* F *is an isometry* ($F^*F = I$).

A New Method

We know that there is a unique minimizer $\mathcal{X} \in X$ such that $\mathcal{J}(\mathcal{X}) = 0$ so

$$F\mathcal{X} - \Pi\mathcal{X} + \tilde{g} = 0.$$

Operating by F^* we get

$$\mathcal{X} - F^*\Pi\mathcal{X} = -F^*\tilde{g}$$

This is the Ultra Weak Variational Formulation (UWVF) of the Helmholtz equation [Cessenat & Després]. Compare to Least Squares normal equations

$$(I - F^*\Pi)^*(I - F^*\Pi)\mathcal{X} = -(I - F^*\Pi)F^*\tilde{g}$$

We expect the UWVF to be better conditioned. The UWVF can also be seen as

- An upwind discontinuous Galerkin method (see also work of Gabard and Hiptmair, Perugia et al) using special degrees of freedom and test functions.

The UWVF equations

It is convenient to write a Galerkin formulation: Find $\mathcal{X} \in X$ such that

$$\langle \mathcal{X}, \mathcal{Y} \rangle - \langle \Pi \mathcal{X}, F(\mathcal{Y}) \rangle = \langle \tilde{g}, F(\mathcal{Y}) \rangle.$$

for all $\mathcal{Y} \in X$.

To clarify: suppose T_j is an interior tetrahedron surrounded by four other tetrahedra

$$\int_{\partial T_j} \mathcal{X}_j \overline{\mathcal{Y}_j} \, ds - \sum_{k \neq j} \int_{\Gamma_{k,j}} \mathcal{X}_k \overline{F_j(\mathcal{Y}_k)} \, ds = 0.$$

The Discrete UWVF [Cessenat & Després]

For each element T_k we choose p_k directions $\boldsymbol{d}_j$ on the unit circle (or unit sphere [Sloan]) and define the solution on that element to be a sum of traces of plane waves

$$\mathcal{X}_k^h = \sum_{j=1}^{p_k} x_j^k \left(\frac{\partial \exp(\mathrm{i}\kappa \boldsymbol{d}_j \cdot \boldsymbol{x})}{\partial n_k} + \mathrm{i}\kappa \exp(\mathrm{i}\kappa \boldsymbol{d}_j \cdot \boldsymbol{x}) \right) \bigg|_{\partial T_k}$$

The test function is, for $1 \leq r \leq p_k$,

$$\mathcal{Y}_k^h = \left(\frac{\partial \exp(\mathrm{i}\kappa \boldsymbol{d}_r \cdot \boldsymbol{x})}{\partial n_k} + \mathrm{i}\kappa \exp(\mathrm{i}\kappa \boldsymbol{d}_r \cdot \boldsymbol{x}) \right) \bigg|_{\partial T_k}$$

In this case $F_k(\mathcal{Y}_k^h)$ is easy to compute:

$$F_k(\mathcal{Y}_k^h) = \left(- \frac{\partial \exp(\mathrm{i}\kappa \boldsymbol{d}_r \cdot \boldsymbol{x})}{\partial n_k} + \mathrm{i}\kappa \exp(\mathrm{i}\kappa \boldsymbol{d}_r \cdot \boldsymbol{x}) \right) \bigg|_{\partial T_k}$$

Properties of the acoustic UWVF

Uniform mesh: Number of DoF $= O(h^{-d}p^{d-1})$.

- [Cessenat/Després, 2D] Let $p = 2\mu + 1$, $\mu > 0$,

$$\|\mathcal{X} - \mathcal{X}^h\|_{L^2(\Gamma)} \leq C(\kappa)h^{\mu-1/2}\|u\|_{\mathcal{C}^{\mu+1}(\Omega)}$$

- [Monk/Buffa] Using DG techniques, for a convex 2D domain with u^h reconstructed from $\mathcal{X}^h$,

$$\|u - u^h\|_{L^2(\Omega)} \leq C(\kappa)h^{\mu-1}\|u\|_{\mathcal{C}^{\mu+1}(\Omega)}$$

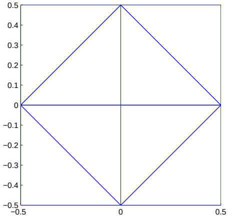
- [Hiptmair, Moiola, Perugia, 2009] General DG, 2D, 3D, explicit constants and p-version error estimate (using k-dependent norms):

$$\kappa\|u - u_h\|_{L^2(\Omega)} \leq Ch^{r-1}\left(\frac{\log(p)}{p}\right)^{r-1/2}\|u\|_{r+1,\kappa,\Omega}$$

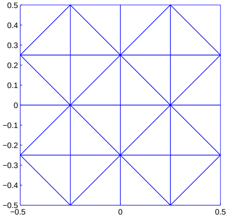
- The discrete problem has the form $(B - C)\boldsymbol{x} = \boldsymbol{b}$ where B is Hermitian positive definite and the eigenvalues of $B^{-1}C$ lie in the closure of the unit disk excluding 1

Numerical results: 2D mesh refinement

We take $u(\boldsymbol{x}) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|\boldsymbol{x} - \boldsymbol{x}_0|)$ with $\boldsymbol{x}_0$ is outside the computational domain.



$h = 0.5$



$h = 0.25$

Results for $k = 20$, $M = 15$ ($p = 7$)

Mesh size h	$L^2(\Omega)$ Error (%)	Order	cond(D)	Order
0.50	4.38	-	0.64×10^2	-
0.25	0.01873	7.9	0.20×10^6	-11.6
0.10	1.51×10^{-5}	7.8	0.22×10^{11}	-12.7

Measured global convergence $O(h^{7.8})$.

Cessenat and Després predict that the condition number of D will increase $O(h^{-12})$

Results for $k = 40$, $M = 21$ ($p = 10$).

Mesh size h	$L^2(\Omega)$ Error (%)	Order	cond(D)	Order
0.50	25.2	-	8.7	-
0.25	0.0337	9.55	0.94×10^5	-13.4
0.1	2.32×10^{-6}	10.5	0.14×10^{13}	-18

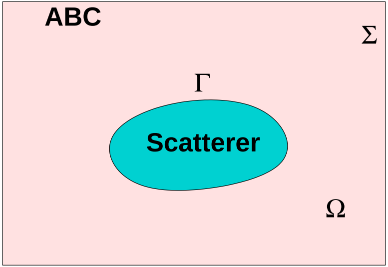
Measured global convergence $O(h^{10})$.

Cessenat and Després predict that the condition number of D increases $O(h^{-18})$.

A Model Scattering Problem [Huttunen & Monk]

Let $\Omega \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) with disjoint boundaries Γ and Σ. Approximate u which satisfies

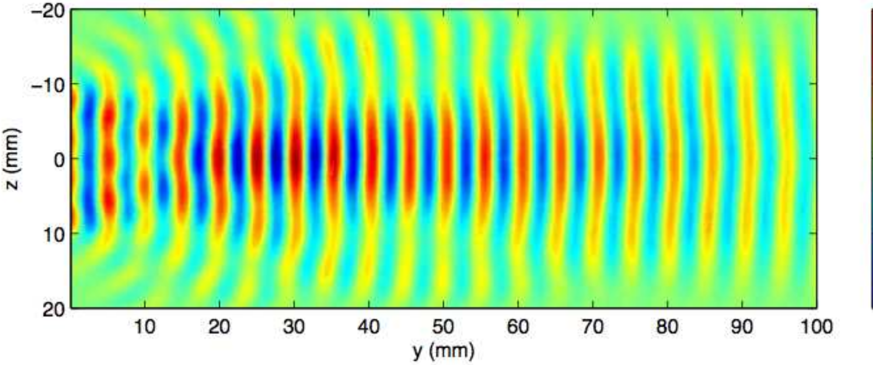
$$\begin{aligned} \Delta u + \kappa^2 u &= 0 \text{ in } \Omega \\ u &= g \text{ on } \Gamma \\ \frac{\partial u}{\partial \nu} - i\kappa u &= 0 \text{ on } \Sigma \end{aligned}$$



where g describes the incoming plane wave. The region Ω is meshed with simplicial elements and the UWVF applied there.
ABC = Absorbing Boundary Condition

Comparison to FEMLAB in 3D acoustics [Huttunen]

FEMLAB P_2 FEM with low order ABC.



Comparison continued

FEMLAB (two meshes):

f (kHz)	h (mm)	Elem.	CPU (s)	Error (%)	Mem (GB)
100	3	101 978	448	30.88	1.4
150	1.8	478 471	4699	25.39	2.5
200	1.8	478 471	5321	20.64	2.5
300	1.8	478 471	5391	30.13	2.5

UWVF (one mesh, variable # directions):

f (kHz)	h (mm)	Elem.	CPU (s)	Error (%)	Mem (GB)
100	15	16 926	275	28.56	0.2
150	15	16 926	353	23.22	0.3
200	15	16 926	449	20.07	0.4
300	15	16 926	854	18.96	1.1

UWVF near a singularity and at low κ

- Near a singularity the plane wave UWVF requires very small elements
- If κ is small the plane wave UWVF becomes poorly conditioned.

We now examine a way of using standard piecewise polynomial basis functions on each element within UWVF framework.

Recall the (slightly modified) definition of F

$F_j : L^2(\partial T_j) \to L^2(\partial T_j)$ is defined using an auxiliary function $w_j \in H^1(T_j)$ that satisfies

$$\begin{aligned} \Delta w_j + \kappa^2 w_j &= 0 \text{ in } T_j, \\ \frac{1}{\mathrm{i}\kappa}\frac{\partial w_j}{\partial n_j} + w_j &= \mathcal{X}_j \text{ on } \partial T_j, \end{aligned}$$

then

$$F_j(\mathcal{X}_j) = -\frac{1}{\mathrm{i}\kappa}\frac{\partial w_j}{\partial n_j} + w_j \text{ on } \partial T_j$$

Basic idea: Approximate F_j by a finite element method inside each element

Joint work with J. Schoeberl and A. Sinwel

Computing F_j by a mixed method

Let $\mathcal{X}_j \in L^2(\partial T_j)$ and define $(w_j, \boldsymbol{v}_j) \in L^2(T_j) \times H(\mathrm{div}; T_j)$ such that

$$
\begin{aligned}
-\mathrm{i}\kappa w_j &= \nabla \cdot \boldsymbol{v}_j \text{ in } T_j \\
-\mathrm{i}\kappa \boldsymbol{v}_j &= \nabla w_j \text{ in } T_j \\
-\boldsymbol{v}_j \cdot \boldsymbol{n}_j + w_j &= \mathcal{X}_j \text{ on } \partial T_j
\end{aligned}
$$

where $\boldsymbol{n}_j$ is the unit outward normal to T_j then F_j is given by

$$F_j(\mathcal{X}_j) = \boldsymbol{v}_j \cdot \boldsymbol{n}_j + w_j = \mathcal{X}_j + 2\boldsymbol{n}_j \cdot \boldsymbol{v}_j \text{ on } \partial T_j.$$

Raviart-Thomas elements

We use Raviart-Thomas subspaces $U \subset L^2(T_j)$ and $V \subset H(\mathrm{div}; T_j)$:

$$
\begin{aligned}
U_h &:= \text{degree } p \text{ polynomials on } T_j \\
V_h &:= \text{vector-valued polynomials of degree } p+1 \text{ on } T_j \\
&\quad\ \text{for which the normal component on } \partial T_j \text{ is of degree } p
\end{aligned}
$$

A discrete approximation to F_j using RT_p

We can compute $(u_h, \boldsymbol{v}_h) \in U_h \times V_h$ such that

$$\begin{aligned}\int_{T_j} (\nabla \cdot \boldsymbol{v}_{j,h} + \mathrm{i}\kappa w_{j,h})\,\xi\, dV &= 0 \text{ for all } \xi \in U_h \\ \int_{T_j} w_{j,h} \nabla \cdot \boldsymbol{\tau} - \mathrm{i}\kappa \boldsymbol{v}_{j,h} \cdot \boldsymbol{\tau}\, dV &= \int_{\partial T_j} (\mathcal{X}_j + \boldsymbol{v}_{j,h} \cdot \boldsymbol{n}_j)\,\boldsymbol{\tau} \cdot \boldsymbol{n}_j\, dA \\ & \qquad \text{for all } \boldsymbol{\tau} \in V_h.\end{aligned}$$

Then we define $F_{T_j,h} : L^2(\partial T_j) \to L^2(\partial T_j)$ by

$$F_{T_j,h}(\mathcal{X}_j) = \mathcal{X}_j + 2\boldsymbol{v}_{j,h} \cdot \boldsymbol{n}_j \text{ on } \partial K.$$

Lemma

If h is small enough, $F_{K,h} : L^2(\partial K) \to L^2(\partial K)$ is an isometry.

FE basis on the edges

The basis functions on each element used to construct U_h and V_h are the standard Raviart-Thomas RT_p elements. The fully discrete finite element UWVF is obtained by letting

$$X_{h,j} = \left\{ \boldsymbol{n}_j \cdot \boldsymbol{v}_j|_{\partial K} \mid \forall \boldsymbol{v}_j \in V_h \right\}$$

and using $F_{j,h}$ on each element.

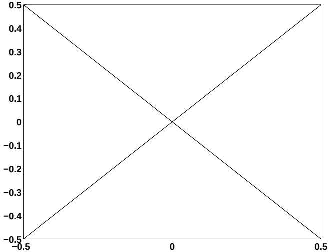
Lemma

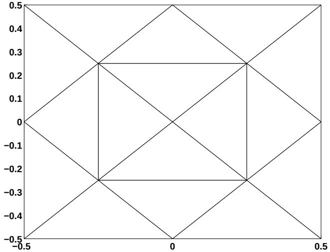
If p is fixed on all elements and h is small enough, the discrete FE-UWVF has a unique solution. Furthermore the local solution computed from $\mathcal{X}_h$ coincides with the solution of the standard Raviart-Thomas method for this problem.

Remark: We have thus accomplished a hybridization of the RT system. This turns out to be exactly equivalent to an HDG method (see our paper).

Meshes

Exact solution: $u = \exp(\mathrm{i}\kappa \boldsymbol{d}\cdot\boldsymbol{x})$ with $\kappa = 10$ and $\boldsymbol{d} = (\cos(1), \sin(1))$.





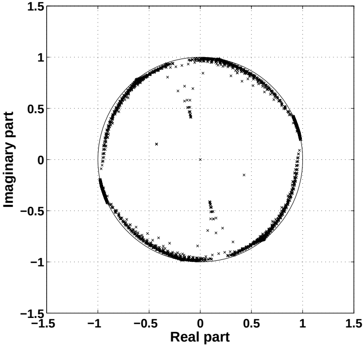
Left: Initial mesh ($h = 1$). Right: one refinement ($h = 1/2$).

Dependence on mesh width h for RT_0

Mesh size h	Relative $L^2(\Omega)$ error (%)	Number of biCG iterations
1	100	5
1/2	100	26
1/4	28	61
1/8	5.8	95
1/16	1.4	188
1/32	0.35	371
1/64	8.7×10^{-2}	742

The error is computed by quadrature at the centroid of each element. Error is $O(h^2)$, number of iterations is $O(1/h)$.

Distribution of eigenvalues



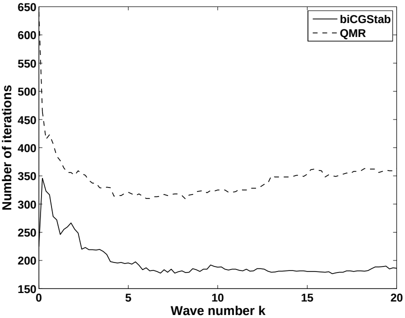
Eigenvalues of $D^{-1}C$ when $h = 1/16$. The eigenvalues are known to lie in the unit disc with $\lambda = 1$ excluded.

Dependence on wave number

Fixed mesh with $h = 1/16$.

Iteration count as a function of k for biCGStab and QMR applied to solve

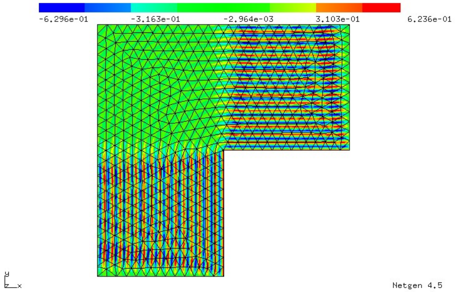
$$\vec{\mathcal{X}} - D_h^{-1} C_h \vec{\mathcal{X}} = D_h^{-1} \vec{G}.$$

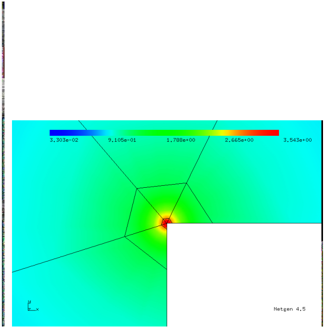


BiCGStab is faster in wall-clock time also.

L-shaped domain - RT_p elements

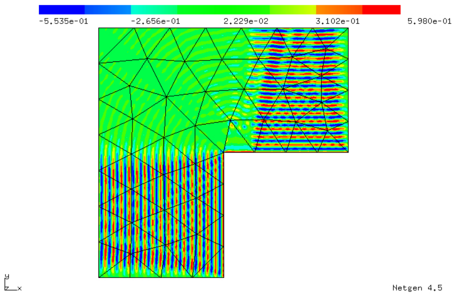
Dirichlet boundary condition at the reentrant corner produces a singularity that requires a refined mesh near that point.

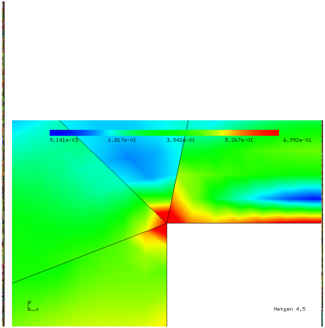




Solved via NETGEN

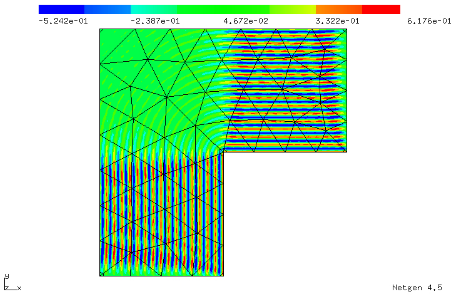
L-shaped domain - PW - UWVF without refinement

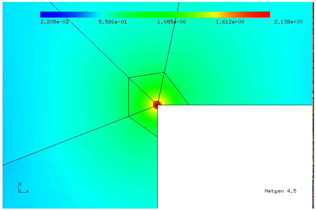




L-shaped domain - combined PW-FE UWVF

Using RT elements near the re-entrant corner and classical PW UWVF further away gives the "best" of both worlds.





Maxwell's equations: Cavity problem

$\boldsymbol{E}, \boldsymbol{H}$:	Unknown electric/magnetic field
k :	Wave-number
ϵ_r:	Relative permittivity (piecewise constant)
μ_r:	Relative permeability (piecewise constant)
Ω:	Bounded domain

$$
\begin{aligned}
-\mathrm{i}k\epsilon_r \boldsymbol{E} - \nabla \times \boldsymbol{H} &= 0 \text{ in } \Omega \\
-\mathrm{i}k\mu_r \boldsymbol{H} + \nabla \times \boldsymbol{E} &= 0 \text{ in } \Omega \\
\boldsymbol{H} \times \boldsymbol{n} - \eta \boldsymbol{E}_T &= Q(\boldsymbol{H} \times \boldsymbol{n} + \eta \boldsymbol{E}_T) - \sqrt{2\eta}\boldsymbol{g} \text{ on } \Gamma = \partial\Omega
\end{aligned}
$$

Variational Problem

Let ξ_k, ψ_k satisfy the *adjoint* Maxwell system on Ω_k

$$-\mathrm{i}k\overline{\epsilon_r}\xi_k - \nabla\times\psi_k = 0,\ -\mathrm{i}k\overline{\mu_r}\psi_k + \nabla\times\xi_k = 0,$$

The unknown traces are

$$\mathcal{X}_k = \boldsymbol{E}_k \times \boldsymbol{n}^{K_k} - \eta(\boldsymbol{H}_k)_T\Big|_{\partial\Omega_k} \text{ and if} \mathcal{Y}_k = \xi_k \times \boldsymbol{n}^{K_k} - \eta(\psi_k)_T\Big|_{\partial\Omega_k}$$

then

$$F_k(\mathcal{Y}_k) = -\xi_k \times \boldsymbol{n}^{K_k} - \eta(\psi_k)_T\Big|_{\partial\Omega_k}$$

then, for a tetrahedron surrounded by four other tetrahedra

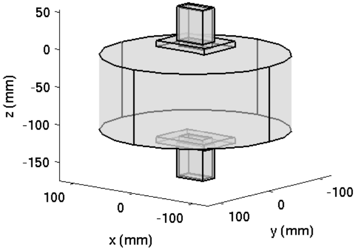
$$\int_{\partial\Omega_k} \frac{1}{\eta}\mathcal{X}_k\overline{\mathcal{Y}_k}\,ds = -\sum_j \int_{\Sigma_{k,j}} \frac{1}{\eta}\mathcal{X}_j\overline{F_k(\mathcal{Y}_k)}\,ds.$$

Vector plane waves are used to discretize on each element.

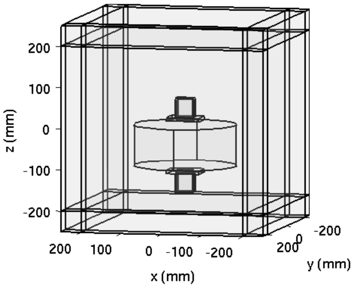
Maxwell's equations: Typical Application [thanks to Tomi Huttunen]

Example: Simulate microwave interaction with wood.

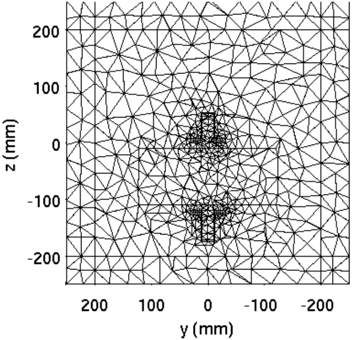
A transmitting and receiving antenna are shown.



Modeling

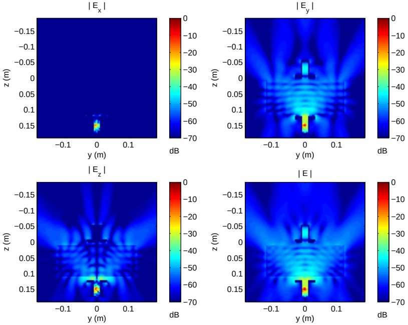


Truncation by a suitable.
layer and boundary condition.

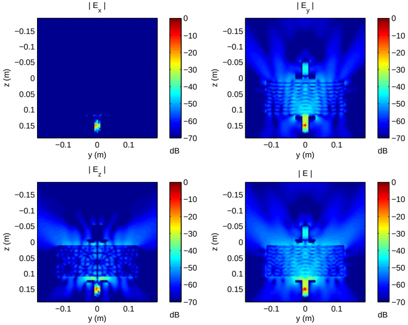


Discretization using
tetrahedra.

Typical Results at 5GHz



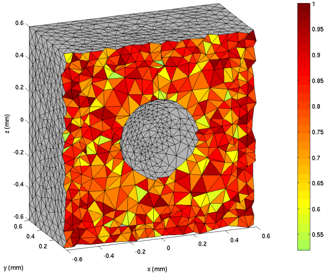
$\epsilon_r = 3$

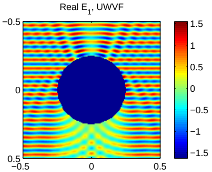


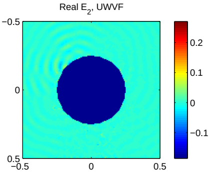
$\epsilon_r = 8$

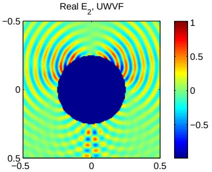
Scattering from a sphere (electromagnetic!)

Sphere of radius 0.25 inside cube $[-.5, .5]^3$, $\kappa = 100$, $\epsilon = \mu = 1$ ($\lambda = 0.06$). PML width 0.1 (uses 3,474,770 degrees of freedom).



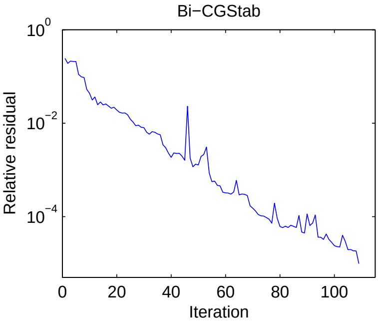






Iterative solution

The UWVF linear system can be solved by simple iterative scheme. We use BiCGStab.



BiCGStab convergence for a problem having 3,474,770 degrees of freedom using a 24 processor cluster (2.8GHz P4, 48Gb memory total, 1000BaseT). Solution time is 451s using 25.3 GB memory (109 iterations).

A model scattering problem

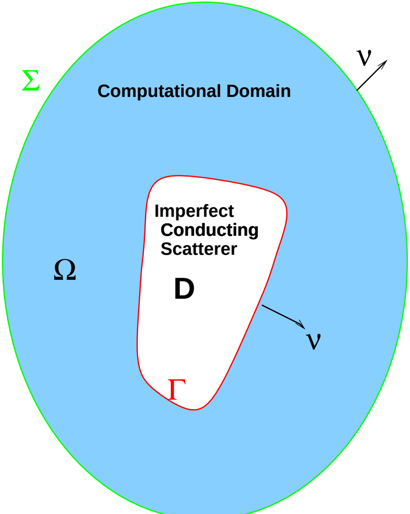
The unknown total field $\boldsymbol{E}$ and scattered field $\boldsymbol{E}^s$ satisfy

$$\begin{aligned}
\nabla \times \left(\mu_r^{-1} \nabla \times \boldsymbol{E}\right) - k^2 \epsilon_r \boldsymbol{E} &= \boldsymbol{F} \text{ in } \mathbb{R}^3 \setminus \overline{D}, \\
\boldsymbol{E} &= \boldsymbol{E}^i + \boldsymbol{E}^s \text{ in } \mathbb{R}^3 \setminus \overline{D}, \\
\boldsymbol{E} \times \boldsymbol{\nu} &= 0 \text{ on } \Gamma, \\
\lim_{\rho \to \infty} \rho \left((\nabla \times \boldsymbol{E}^s) \times \hat{\boldsymbol{x}} - ik\boldsymbol{E}^s \right) &= 0 \text{ as } r \to \infty.
\end{aligned}$$

For ease of exposition $\epsilon_r = \mu_r = 1$. D is bounded, simply connected with simply connected complement.

How to truncate the problem?

Introduce a surface Σ containing the scatterer in it's interior.



Let Ω be the region inside Σ and outside *D*

Need an appropriate boundary condition on the *artificial boundary* Σ.

Hazard and Lenoir overlapping formulation

The integral representation outside Γ is provided by an extension of the *Stratton-Chu* formula. Let

$$\mathbb{G}(\boldsymbol{x},\boldsymbol{y}) = \Phi(\boldsymbol{x},\boldsymbol{y})I + k^{-2}Hess(\Phi)(\boldsymbol{x},\boldsymbol{y}),$$

where I is the identity matrix. For $\boldsymbol{x}$ outside C

$$\begin{aligned}\boldsymbol{E}^s(\boldsymbol{x},\boldsymbol{y}) &= \int_\Gamma (\mathbb{G}(\boldsymbol{x},\boldsymbol{y}))^T \boldsymbol{\nu}_y \times (\nabla \times \boldsymbol{E}^s(\boldsymbol{y})) \\ &\quad + (\nabla \times \mathbb{G}(\boldsymbol{x},\boldsymbol{y}))^T (\boldsymbol{\nu}_y \times \boldsymbol{E}^s(\boldsymbol{y}))\, dA(\boldsymbol{y}) \\ &=: \mathcal{I}(\boldsymbol{E}^s).\end{aligned}$$

Using the fact $\mathcal{I}(\boldsymbol{E}^i) = 0$, $\boldsymbol{E} = \boldsymbol{E}^i + \mathcal{I}(\boldsymbol{E})$ in the neighborhood Σ

An iterative scheme

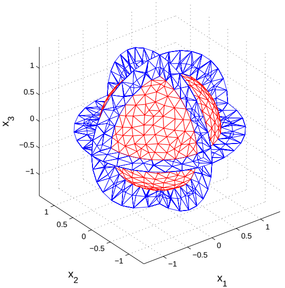
We use the Ultra Weak Variational Formulation in the computational domain. Note:

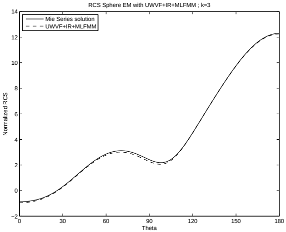
- Both fields are available on Γ.
- Solving the resulting coupled problem by a biConjugate Gradient method (biCGStab) requires to evaluate $\mathcal{I}(\boldsymbol{E})$ and this can be done using the multilevel fast multipole method.

This algorithm is described in a paper with Eric Darrigrand.

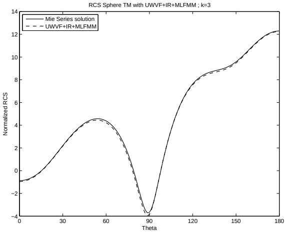
Unit Sphere: $k = 3$, $\lambda = 2.1$

The finite element grid is chosen approximately two elements think, and each element is approximately $\lambda/10$ in diameter. This does not exercise the high order capability of UWVF.





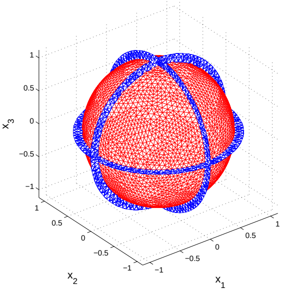
RCS: TE Mode

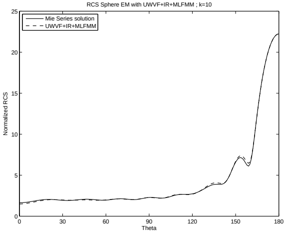


TM Mode

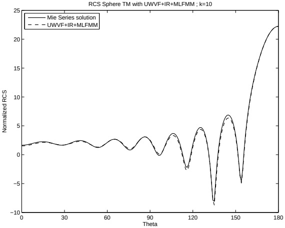
Unit Sphere: $k = 10$, $\lambda = 0.63$

The mesh becomes finer but still only two elements thick.





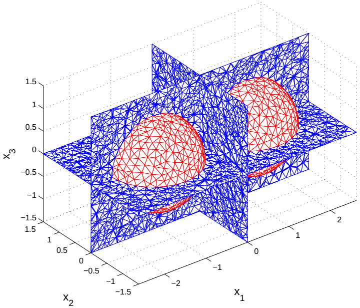
RCS: TE Mode

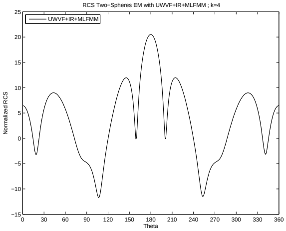


TM Mode

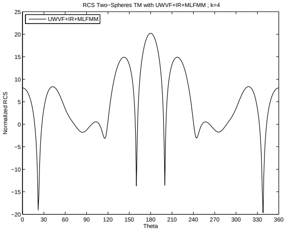
Two Spheres: $k = 4$, $\lambda = 1.6$

When objects are close together (in terms of wavelengths), the space between must be meshed.





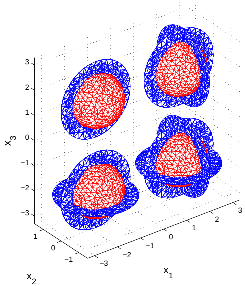
RCS: TE Mode

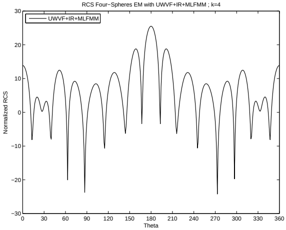


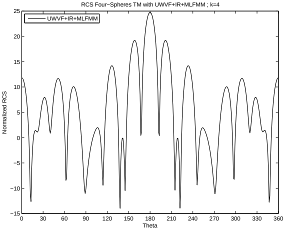
TM Mode

Four Spheres: $k = 4$, $\lambda = 1.6$

When objects are sufficiently far apart (in terms of wavelengths), the meshes can be disjoint.







RCS: TE Mode

TM Mode

Conclusions

- Accuracy can be obtained if the complete family is well matched to the problem
- Robustness is an issue (particularly ill-conditioning)
- These techniques can help with the numerical linear algebra aspects. But a better solver would be useful.
- Need to choose plane wave directions carefully.

A partial bibliography of the UWVF

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