Non-equilateral deformed triangle groups

James M. Thompson

University of Durham

Geometry and Arithmetic of Lattices
London Mathematical Society - EPSRC Durham symposium
2011

(Deformed) Hyperbolic triangle groups

A hyperbolic triangle group is a faithful and discrete representation of a Coxeter group

$$\triangle(p,q,r) = \langle R_1, R_2, R_3 \mid R_i^2, (R_2 R_3)^p, (R_3 R_1)^q, (R_1 R_2)^r \rangle$$

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When we change to the complex hyperbolic plane ($\mathbf{H}^2_{\mathbb{C}}$) with isometry group $\mathbf{PU}(2,1)$, we find there is a one dimensional family of non-isometric hyperbolic triangles with angles π/p, π/q and π/r. These correspond to a one dimensional family of non-conjugate representations:

$$\rho_t : \triangle(p,q,r) \to \mathbf{PU}(2,1)$$

(These representations are not necessarily discrete nor faithful).

A Representation for a deformed triangle group in $\mathbf{PU}(2,1)$

Representation

For ρ, σ, $\tau \in \mathbb{C}$ we define the matrices

$$R_1 = \begin{pmatrix} 1 & \rho & \overline{\tau} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -1 & 0 & 0 \\ \overline{\rho} & 1 & \sigma \\ 0 & 0 & -1 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \tau & \overline{\sigma} & 1 \end{pmatrix}.$$

These preserve the Hermitian form

$$H = \begin{pmatrix} 2 & \rho & \overline{\tau} \\ \overline{\rho} & 2 & \sigma \\ \tau & \overline{\sigma} & 2 \end{pmatrix}$$

The parameter space of deformed triangle groups

Conjecture / 'working hypothesis'

A discrete deformed triangle group is a lattice if and only if $R_1 R_2$, $R_2 R_3$ and $R_3 R_1$ are non-loxodromic and $R_1 R_2 R_3$ is (finite order) regular elliptic.

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We want $\mathrm{ord}(R_2R_3) = p$, $\mathrm{ord}(R_3R_1) = q$ and $\mathrm{ord}(R_1R_2) = r$.
This forces:

$$|\sigma| = 2\cos(\pi/p), \quad |\tau| = 2\cos(\pi/q), \quad |\rho| = 2\cos(\pi/r).$$

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This accounts for the first 3 parameters. The final parameter is $t = \arg(\rho\sigma\tau) \in [0, \pi]$ (or something equivalent).

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The trace of $R_1R_2R_3$ is

$$1 + \rho\sigma\tau - |\rho|^2 - |\sigma|^2 - |\tau|^2.$$

So $R_1R_2R_3$ is finite order, regular elliptic iff ρ, σ, τ satisfy:

$$1 + \rho\sigma\tau - |\rho|^2 - |\sigma|^2 - |\tau|^2 = e^{Ai\pi} + e^{Bi\pi} + e^{Ci\pi}.$$

for some rational A, B, C

Lattice candidates

It is difficult to find triples ρ, σ, τ arithmetically. A brute force search on a computer yields the following triples:

T	ρ	σ	τ	$\Gamma(p, q, r; n)$	$\mathrm{ord}(R_{123})$
$\mathbf{S}_1$	$\zeta_7 + \zeta_7^2 + \zeta_7^4$	1	1	$\Gamma(3, 3, 4; 4)$	7
$\mathbf{S}_2$	$1 + \omega\phi$	1	1	$\Gamma(3, 3, 4; 5)$	5
$\mathbf{E}_1$	$\sqrt{-2}$	1	1	$\Gamma(3, 3, 4; 6)$	8
$\mathbf{E}_2$	$\sqrt{2}$	$-\overline{\omega}$	$\sqrt{2}$	$\Gamma(3, 4, 4; 4)$	6
$\mathbf{H}_1$	$\zeta_7 + \zeta_7^2 + \zeta_7^4$	ζ_7^5	ζ_7^5	$\Gamma(3, 3, 4; 7)$	42
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This search produced two 'new' reflection group lattices: $\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$.

Two new lattices

$\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$ are commensurable with known arithmetic Deligne-Mostow lattices. The groups have the presentations:

$$\Gamma(3,3,4;7) = \left\langle R_1, R_2, R_3 \;\middle|\; \begin{array}{c} R_i^2, (R_2R_3)^3, (R_3R_1)^3, (R_1R_2)^4 \\ (R_1R_3R_2R_3)^7, (R_1R_2R_3)^{42} \end{array} \right\rangle,$$

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The Euler-Poincaré characteristic of the lattices are

$$\chi(\Gamma(3,3,4;7)) = \frac{1}{49} \quad \text{and} \quad \chi(\Gamma(3,3,5;5)) = \frac{1}{100}.$$

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'Conjecture'

$\Gamma(3,3,4;7)$, $\Gamma(3,3,5;5)$, $\Gamma(4,4,4;5)$ and $\Gamma(5,5,5;5)$ are the only deformed triangle group lattices generated by order 2 reflections.

Idea for a fundamental domain for $\Gamma(3,3,5;5)$

Our fundamental domain for $\Gamma(3,3,5;5)$ will consist of two things:

- A finite order regular elliptic isometry P with fixed point o_P,
- A carefully constructed codimension 1 polyhedra D whose orbit under P is homeomorphic to a 3-sphere containing o_p

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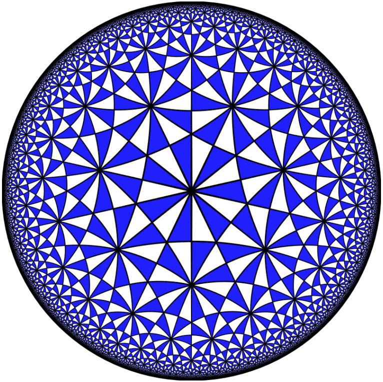

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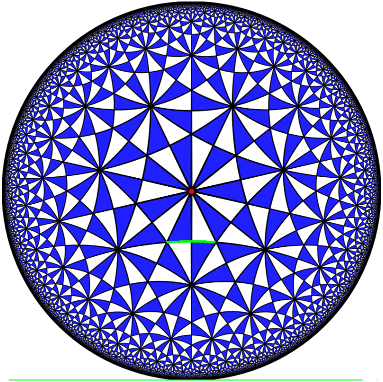

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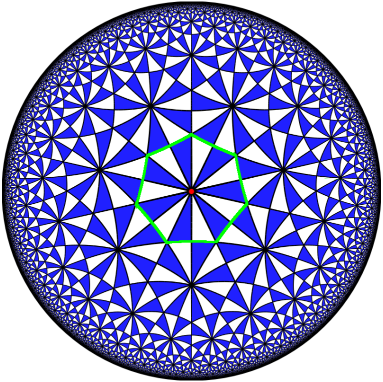

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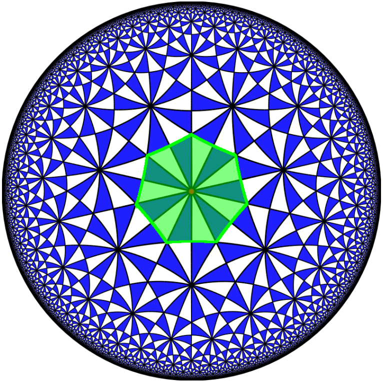

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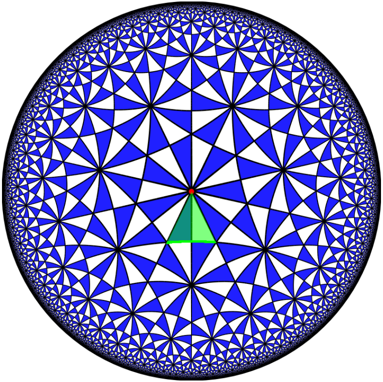

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In our case, $P = R_1 R_2 R_3$ and is D is...

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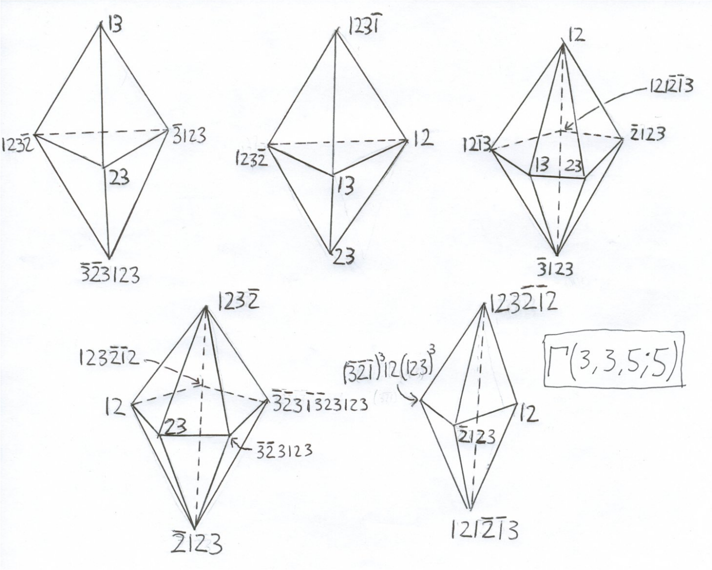

Higher order reflections

A representation for a triangle group generated by higher order reflections is:

Representation

Let ρ, σ, $\tau \in \mathbb{C}$ and $\psi = 2\pi/p$,

$$R_1 = e^{-i\psi/3}\begin{pmatrix} e^{i\psi} & \rho & -\overline{\tau} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = e^{-i\psi/3}\begin{pmatrix} 1 & 0 & 0 \\ -e^{i\psi}\overline{\rho} & e^{i\psi} & \sigma \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_3 = e^{-i\psi/3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{i\psi}\tau & -e^{i\psi}\overline{\sigma} & e^{i\psi} \end{pmatrix}.$$

$$H = \begin{pmatrix} 2-2\mathrm{Re}\left(e^{i\psi}\right) & \rho(e^{-i\psi}-1) & \overline{\tau}(1-e^{-i\psi}) \\ \overline{\rho}(e^{i\psi}-1) & 2-2\mathrm{Re}\left(e^{i\psi}\right) & \sigma(e^{-i\psi}-1) \\ \tau(1-e^{i\psi}) & \overline{\sigma}(e^{i\psi}-1) & 2-2\mathrm{Re}\left(e^{i\psi}\right) \end{pmatrix}$$

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We are still interested in groups where $R_1 R_2 R_3$ is finite order regular elliptic.

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$$R_1 R_2 R_3 = \begin{pmatrix} 1 - |\rho|^2 - |\tau|^2 + \rho\sigma\tau & \rho(1 - |\sigma|^2) + \overline{\sigma\tau} & \sigma\rho - \overline{\tau} \\ \overline{\rho} + \tau & 1 - |\sigma|^2 & \sigma \\ \tau & \overline{\sigma} & 1 \end{pmatrix}$$

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The $e^{i\psi}$s terms cancel out.
Choose a triple ρ, σ, τ. The word $R_1R_2R_3$ is regular elliptic in a higher order deformed triangle group iff it is regular elliptic in the group generated by order 2 reflections. Similarly, R_iR_j will be non-loxodromic in the higher order reflection group iff and only if R_iR_j are non-loxodromic in the order 2 case.

Higher order reflections

Recall, the values of ρ, σ, τ satisfying the conditions are:

T	ρ	σ	τ	$\mathrm{ord}(R_{123})$
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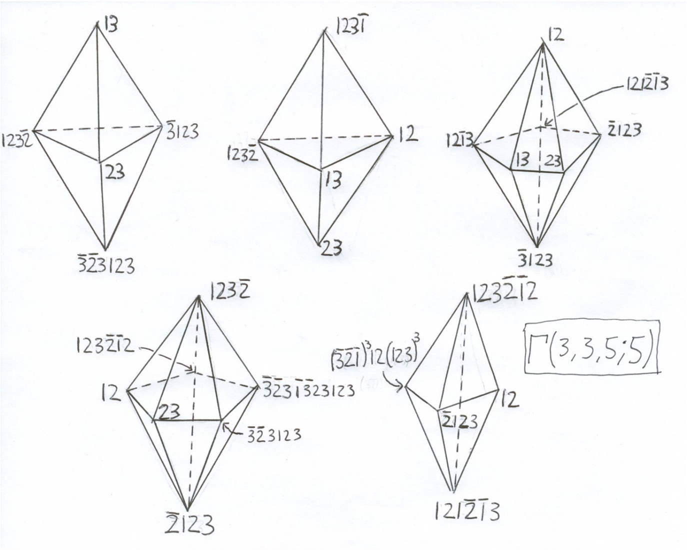
For each of these triples we have a new infinite family of groups, $\Gamma\left(\frac{2\pi}{p}, \mathbf{T}\right)$. However all but finitely many of the groups are non-discrete.

Higher order reflections:New lattices

Using Martin Deraux's computer program the following 14+2 groups appear to be lattices:

p	$\mathbf{T}$	A/NA?	compact?
2	$\mathbf{H}_1$	A	C
2	$\mathbf{H}_2$	A	C
3	$\mathbf{S}_2$	A	C
3	$\mathbf{E}_1$	NA	NC
3	$\mathbf{H}_2$	NA	C
4	$\mathbf{S}_2$	NA	NC
4	$\mathbf{E}_1$	NA	NC
4	$\mathbf{E}_2$	NA	NC
5	$\mathbf{S}_2$	NA	C
5	$\mathbf{H}_2$	NA	C
5	$\overline{\mathbf{H}}_2$	NA	C
6	$\mathbf{E}_1$	NA	C
6	$\mathbf{E}_2$	A	C
7	$\overline{\mathbf{H}}_1$	A	C
10	$\mathbf{H}_2$	A	C
12	$\mathbf{E}_2$	A	C

Fundamental domains for $\Gamma\left(\frac{2\pi}{p}, \mathbf{H}_2\right)$



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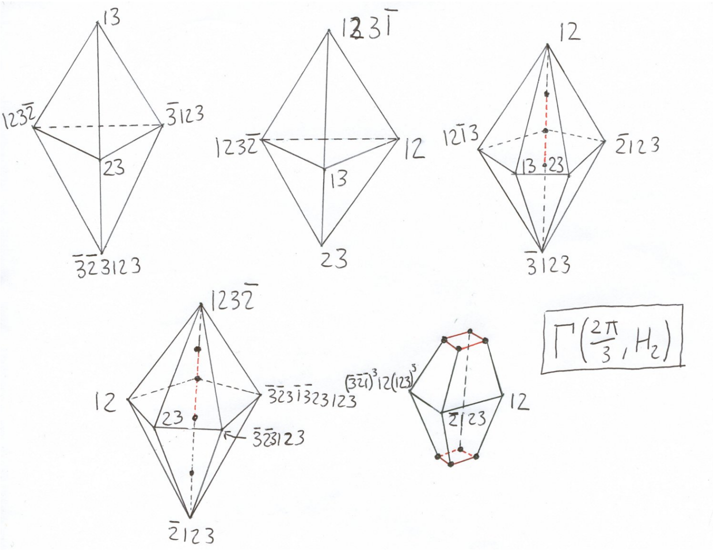

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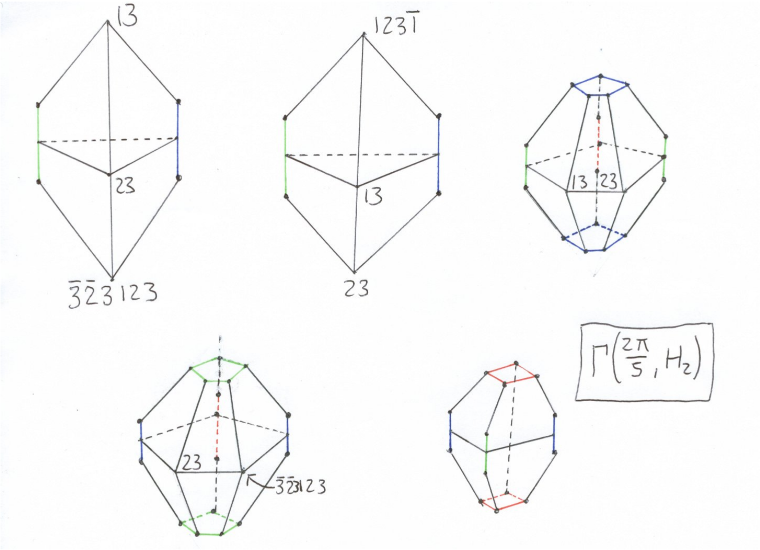

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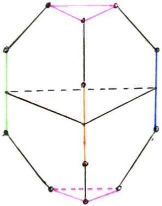

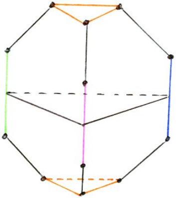

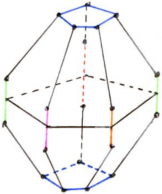

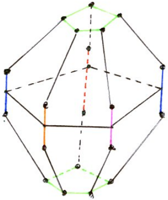

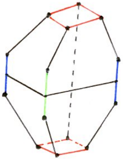

$\Gamma\left(\frac{2\pi}{10}, H_2\right)$

$\Gamma\left(\frac{2\pi}{p}, \mathbf{H}_2\right)$

Presentations

Common relations:

$$1^p, 2^p, 3^p, 131 = 313, 232 = 313, 21212 = 12121,$$
$$(123)^{15}, (1(23\bar{2}))^{5/2} = ((23\bar{2})1)^{5/2}$$

Extra relations:

- $p = 2$: no extra relations.
- $p = 3$: $(\bar{2}12123123)^{15}$.
- $p = 5$: $(12)^{10}$, $(1(23\bar{2}))^{10}$, $(\bar{2}12123123)^{10}$.
- $p = 10$: $(13)^{15}$, $(23)^{15}$, $(12)^5$, $(1(23\bar{2}))^5$, $(\bar{2}12123123)^{10}$.

Euler-Poincaré characteristics

$p = 2$, $\chi = \frac{1}{100}$; $p = 3$, $\chi = \frac{26}{75}$; $p = 5$, $\chi = \frac{73}{100}$; $p = 10$, $\chi = \frac{13}{100}$.