EFFECTIVE LOCAL LANGLANDS CORRESPONDENCE

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This is an overview of an extended programme of joint work with Guy Henniart, aimed at rendering the local Langlands correspondence effective as a tool for local investigations. It is intended to serve as an introduction to recent results in [13].

Throughout, F is a non-Archimedean local field, $\mathfrak{o}_F$ is the discrete valuation ring in F, $\mathfrak{p}_F$ is the maximal ideal of $\mathfrak{o}_F$ and $\Bbbk_F = \mathfrak{o}_F/\mathfrak{p}_F$ is the residue field. The characteristic of $\Bbbk_F$ is denoted by p, but *we make no assumptions concerning the characteristic of* F. We fix a separable algebraic closure $\bar F$ of F, and let $\mathcal{W}_F$ be the Weil group of $\bar F/F$.

If $n \geqslant 1$ is an integer, then $\mathcal{G}_n(F)$ will denote the set of equivalence classes of irreducible, smooth representations of $\mathcal{W}_F$ of dimension n. (Here, and throughout, we consider only *complex* representations.) On the other side, $\mathcal{A}_n(F)$ is the set of equivalence classes of irreducible *cuspidal* representations of the locally profinite group $\mathrm{GL}_n(F)$. The Langlands correspondence [18], [21], [27] thus gives a canonical bijection $\mathcal{G}_n(F) \to \mathcal{A}_n(F)$, which we denote $\sigma \mapsto {}^L\sigma$.

The theory of *simple characters,* as developed in [15], provides a complete and explicit classification of the elements of $\mathcal{A}_n(F)$. It is therefore natural to ask how the features of this structure theory are reflected by the representations of $\mathcal{W}_F$. The work summarized here reveals a strong and transparent parallelism.

Let $\mathcal{P}_F$ be the wild inertia, or first ramification, subgroup of $\mathcal{W}_F$ and denote by $\widehat{\mathcal{P}}_F$ the set of equivalence classes of irreducible smooth representations of $\mathcal{P}_F$. The group $\mathcal{W}_F$ acts on $\widehat{\mathcal{P}}_F$ by conjugation. The $\mathcal{W}_F$-isotropy group of $\alpha \in \widehat{\mathcal{P}}_F$ is of the form $\mathcal{W}_E$, where $E = Z_F(\alpha)/F$ is a finite, tamely ramified extension.

Let $\widehat{\mathcal{W}}_F = \bigcup_{n\geqslant 1} \mathcal{G}_n(F)$ be the set of equivalence classes of irreducible smooth representations of $\mathcal{W}_F$. Fixing a representation $\alpha \in \widehat{\mathcal{P}}_F$ and an integer $m \geqslant 1$, let $\mathcal{G}_m(F;\alpha)$ be the set of elements of $\widehat{\mathcal{W}}_F$ which contain α with multiplicity m. Each element σ of $\mathcal{G}_m(F;\alpha)$ then satisfies

$$\dim\sigma = m\,[E{:}F]\,\dim\alpha, \quad E = Z_F(\alpha).$$

Classical Clifford theory yields an explicit description of the elements of $\mathcal{G}_m(F;\alpha)$ in terms of certain (regular) elements of $\mathcal{G}_1(E_m;\alpha)$, where E_m/E is unramified of degree m: see 8.2 below.

On the other hand, the Ramification Theorem of [5] attaches to the pair (α, m) a distinguished conjugacy class $\Theta = \Phi_F^m(\alpha)$ of simple characters in the group $\mathrm{GL}_n(F)$, where $n = m[E{:}F]\dim\alpha$, as above. If $\mathcal{A}_m(F;\Theta)$ denotes the set of $\pi \in \mathcal{A}_n(F)$ which contain Θ, the Langlands correspondence induces a bijection $\mathcal{G}_m(F;\alpha) \to \mathcal{A}_m(F;\Theta)$. The aim is to describe this map. To do this, we use the classification theory from [15] and the theory of tame lifting from [2] and [5]: these give an explicit description of the elements of $\mathcal{A}_m(F;\Theta)$ in terms of (regular) elements of $\mathcal{A}_1(E_m;\Theta_{E_m})$, where $\Theta_{E_m} = \Phi^1_{E_m}(\alpha)$. Combining these two descriptions with the Langlands correspondence $\mathcal{G}_1(E_m;\alpha) \to \mathcal{A}_1(E_m;\Theta_{E_m})$, we get an explicit bijection $\mathcal{G}_m(F;\alpha) \to \mathcal{A}_m(F;\Theta)$. The main result here shows that this bijection differs from the Langlands correspondence by twisting the elements of $\mathcal{G}_1(E_m;\alpha)$ with a fixed, tamely ramified character of $E^\times$. This twisting character is completely computable in many cases; at worst, the Langlands correspondence is determined up to twisting by an unramified character of order dividing the dimension. Such a discrepancy is susceptible to analysis in terms of a finite number of local constant calculations: we summarize that method at the end of §9.

These notes are primarily aimed at giving an overview of recent work, and so are short on detail. The material is based on the theory of simple characters, as laid down in [15] and further developed in [2], [5]. That occupies many pages, so we are constrained to abbreviate it drastically. In the main development in Chapter I, we have omitted many of the definitions, and concentrated on formal or structural properties illuminated by the simplest useful examples. This serves the main purpose of exhibiting the underlying simplicity of the Langlands correspondence (Main Theorem, 9.4) but obscures the essentially explicit nature of simple characters and the simple strata used to describe them. As a partial remedy, we have added an appendix (§6) to Chapter I, giving a skeletal account of the central definitions and constructions. This may be omitted completely, but may equally be useful as a road-map for the reader wishing to pursue the topic further.

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I. Cuspidal representations of GL_n

We review the classification of the irreducible cuspidal representations of $\mathrm{GL}_n(F)$. The account is based firmly on [15] and [2], but we have incorporated some more recent insights, mainly from [13].

1. Intertwining and induction

1.1. Intertwining. For a moment, let G be a group, let K_i be a subgroup of G and ρ_i a representation of K_i, for $i = 1, 2$. An element $g \in G$ *intertwines* ρ_1 *with* ρ_2 if

$$\mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2) \neq 0.$$

Here, K_1^g means $g^{-1}K_1 g$ and ρ_1^g is the representation $x \mapsto \rho_1(gxg^{-1})$, $x \in K_1^g$. Surely this property depends only on the double coset $K_1 g K_2$. In the same vein, we write $I_G(\rho_1)$ for the set of $g \in G$ which intertwine ρ_1 with itself. In particular, $I_G(\rho_1)$ invariably contains K_1.

Suppose next that G is *locally profinite,* and that K is an open subgroup of G. Let ρ be a smooth representation of K on a complex vector space W. The space of functions $f : G \to W$, which satisfy

$$f(kg) = \rho(k)\, f(g), \quad k \in K,\ g \in G,$$

and which are compactly supported modulo K, then carries a natural action of G by right translation. The representation of G so obtained is smooth. It is said to be *compactly induced* from ρ, and is denoted $c\text{-}\mathrm{Ind}_K^G\, \rho$.

We specialize to the case where G is the group of F-points of some connected, reductive algebraic group defined over F: we say that G is a connected reductive F-group. Such a group G carries a locally profinite topology, inherited from F.

Proposition. *Let G be a connected reductive F-group, let K be an open subgroup of G, which is compact modulo the centre of G. Let ρ be an irreducible smooth representation of K. If $I_G(\rho) = K$, then the induced representation $c\text{-}\mathrm{Ind}_K^G\,\rho$ is irreducible and cuspidal.*

The proof in [9], while overtly for $G = \mathrm{GL}_2(F)$, remains valid in the stated degree of generality. It applies, in particular, to the case $G = \mathrm{GL}_n(F)$.

The proposition provides the most effective way we have of exhibiting irreducible cuspidal representations of reductive groups. For a given group G, the aim is always to produce a list $\mathcal{D}$ of inducing data (K,ρ) accounting exactly for the irreducible cuspidal representations of G:

(1) *if $(K,\rho) \in \mathcal{D}$, then $\pi_\rho := c\text{-}\mathrm{Ind}_K^G\,\rho$ is irreducible and cuspidal;*
(2) *if π is an irreducible cuspidal representation of G, there exists $(K,\rho) \in \mathcal{D}$ such that $\pi \cong \pi_\rho$;*
(3) *if $(K_i,\rho_i) \in \mathcal{D}$, $i = 1,2$, and $\pi_{\rho_1} \cong \pi_{\rho_2}$, then (K_1,ρ_1) is G-conjugate to (K_2,ρ_2).*

If $G = \mathrm{GL}_n(F)$, such a list has been obtained [15]. More generally, let D be a central F-division algebra of dimension d^2, $d \geqslant 1$. If $n = md$, for an integer $m \geqslant 1$, the group $G' = \mathrm{GL}_m(D)$ is an *inner form* of G. A similar list for G' is given in [31], implying a classification scheme which is uniform across all of inner forms of G. This uniformity is compatible with the Jacquet-Langlands correspondence, to the extent known [6], [11], [12], [33].

More widely, the inductive approach gives all the cuspidal representations of a classical group G, provided $p \neq 2$ [34], although the structure is more complicated and the classification property (3) is not yet known. J.-K. Yu has used the same sort of method to produce classes of irreducible cuspidal representations of a general connected, reductive F-group G [36]. Yu's method make no pretence at completeness: for example, if $G = \mathrm{GL}_n(F)$, then it yields all irreducible cuspidal representations of G if and only if p does not divide n. In the general case, J.-L. Kim [25] has shown that Yu's construction yields all desired representations provided p is sufficiently large, in a sense depending on G.

1.2. Example. We give the first standard example of the idea of 1.1. It recurs frequently in later pages, despite being rather atypical.

Let V be an F-vector space of dimension n, $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$. Let L be an $\mathfrak{o}_F$-lattice in V and set

$$\mathfrak{m} = \mathfrak{m}_F(L) = \mathrm{End}_{\mathfrak{o}_F}(L) \cong \mathrm{M}_n(\mathfrak{o}_F).$$

Thus $\mathfrak{m}$ is a *maximal* $\mathfrak{o}_F$*-order* in A. The ideal $\mathfrak{p}_\mathfrak{m} = \mathfrak{p}_F\mathfrak{m}$ is the Jacobson radical of $\mathfrak{m}$. Set

$$K = F^\times U_\mathfrak{m}, \quad U^1_\mathfrak{m} = 1+\mathfrak{p}_\mathfrak{m}.$$

In particular, K is an open subgroup of G, compact modulo the centre $F^\times$ of G, and $U^1_\mathfrak{m}$ is an open normal subgroup of K.

Let λ be an irreducible representation of K such that $\lambda|_{U^1_\mathfrak{m}}$ is trivial. The irreducible representation $\lambda|_{U_\mathfrak{m}}$ is therefore inflated from an irreducible representation $\tilde\lambda$ of $U_\mathfrak{m}/U^1_\mathfrak{m} \cong \mathrm{GL}_n(\Bbbk_F)$. We have

$$\boxed{I_G(\lambda) = K \quad\iff\quad \tilde\lambda \text{ is cuspidal.}}$$

So, if $\tilde\lambda$ is cuspidal, then c-$\mathrm{Ind}_K^G\,\lambda$ is irreducible and cuspidal. The boxed statement is a pleasant exercise, *cf.* [9] 14.3. The following assertion, however, lies rather deeper. A proof may be extracted from [15] (6.2 and 8.3.3).

Proposition. *Let $\tilde\lambda$ be an irreducible representation of $U_\mathfrak{m}$, trivial on $U^1_\mathfrak{m}$. The following are equivalent:*

(1) *$\tilde\lambda$ is not cuspidal;*
(2) *the representation c-$\mathrm{Ind}_K^G\,\lambda$ has no irreducible cuspidal sub-quotient;*
(3) *the representation λ occurs in no cuspidal representation of G.*

Elaborating the first exercise, one may further deduce:

Corollary. *Let (π, V) be an irreducible cuspidal representation of G, having a non-zero fixed point for the group $U^1_\mathfrak{m}$. There is a unique irreducible representation λ of K such that $\lambda|_{U_\mathfrak{m}}$ is inflated from an irreducible cuspidal representation of $U_\mathfrak{m}/U^1_\mathfrak{m}$ and $\pi \cong c$-$\mathrm{Ind}_K^G\,\lambda$.*

2. Simple characters

As before, let V be an F-vector space of finite dimension n, and set $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$. Fundamental to the classification theory is the concept of "simple character in G". Simple characters are complex and subtle objects defined explicitly but indirectly. The basic theory is rather technical. It occupies the first three chapters of [15], and is further developed in [2]. We want to concentrate on the implications, for the Langlands correspondence, of certain structural features. We have therefore given the briefest possible account of the background material, appending an overview in §6.

2.1. Hereditary orders. We make much use of a family of special sub-rings of A and a system of subgroups of G derived from them.

An F-lattice chain in V is a non-empty set $\mathfrak{L}$ of $\mathfrak{o}_F$-lattices in V which is both linearly ordered under inclusion and stable under scalar multiplication: if $x \in F^\times$ and $L_1, L_2 \in \mathfrak{L}$, then $xL_i \in \mathfrak{L}$ and either $L_1 \subset L_2$ or $L_2 \subset L_1$.

If $\mathfrak{L}$ is an F-lattice chain in V, the orbit space $F^\times \backslash \mathfrak{L}$ is finite with at most n elements. We set

$$e = e_F(\mathfrak{L}) = \left|F^\times \backslash \mathfrak{L}\right|.$$

This integer $e_F(\mathfrak{L})$ is called the *F-period* of $\mathfrak{L}$.

Let $\mathfrak{L}$ be an $\mathfrak{o}_F$-lattice chain in V. We set

$$\begin{aligned} \mathfrak{a} = \mathfrak{a}_F(\mathfrak{L}) &= \bigcap_{L \in \mathfrak{L}} \mathfrak{m}_F(L), \quad \text{where} \\ \mathfrak{m}_F(L) &= \{x \in A : xL \subset L\} = \mathrm{End}_{\mathfrak{o}_F}(L). \end{aligned}$$

The intersection here is finite, with $e = e_F(\mathfrak{L})$ distinct factors. The set $\mathfrak{a}$ is an $\mathfrak{o}_F$-order in A. An $\mathfrak{o}_F$-order obtained this way is called *hereditary.* (For a full discussion of hereditary orders, see [30] or the early pages of [15].) Observe that the maximal order $\mathfrak{m}_F(L)$ is the hereditary order defined by the lattice chain $\{xL : x \in F^\times\}$.

For $L \in \mathfrak{L}$, let L' be the largest element of $\mathfrak{L}$ such that $L' \subset L$ and $L' \neq L$. The set

$$\mathfrak{p}_{\mathfrak{a}} = \bigcap_{L \in \mathfrak{L}} \mathrm{Hom}_{\mathfrak{o}_F}(L, L')$$

is a two-sided ideal of $\mathfrak{a}$. It is the *Jacobson radical* of $\mathfrak{a}$, $\mathfrak{p}_{\mathfrak{a}} = \mathrm{rad}\,\mathfrak{a}$. It is, moreover, an *invertible* two-sided ideal of $\mathfrak{a}$, its inverse being

$$\mathfrak{p}_{\mathfrak{a}}^{-1} = \bigcap_{L \in \mathfrak{L}} \mathrm{Hom}_{\mathfrak{o}_F}(L', L).$$

One can recover the lattice chain $\mathfrak{L}$ from the hereditary order $\mathfrak{a} = \mathfrak{a}_F(\mathfrak{L})$, because $\mathfrak{L}$ is exactly the set of all $\mathfrak{a}$-lattices in V. When using this viewpoint, the period $e = e(\mathfrak{a}) = e_F(\mathfrak{L})$ appears as a sort of ramification index,

$$\mathfrak{p}_F\mathfrak{a} = \mathfrak{p}_{\mathfrak{a}}^e.$$

Observe that $\mathfrak{a}$ is a maximal order if and only if $e(\mathfrak{a}) = 1$.

Attached to a hereditary $\mathfrak{o}_F$-order $\mathfrak{a}$ in A, we have the unit group $U_{\mathfrak{a}} = \mathfrak{a}^\times$. This is a compact open subgroup of G. It is a maximal compact subgroup if and only if $\mathfrak{a}$ is a maximal order. Inside $U_{\mathfrak{a}}$, we have the "standard filtration subgroups"

$$U_{\mathfrak{a}}^k = 1 + \mathfrak{p}_{\mathfrak{a}}^k, \quad k \geqslant 1.$$

These are again open in G and normal in $U_{\mathfrak{a}}$.

Remark. The groups $U_{\mathfrak{a}}$, attached to the hereditary $\mathfrak{o}_F$-order $\mathfrak{a}$, appear in the theory of the *affine building* of G: see, for example, the exposition in [35]. In this context, the groups $U_{\mathfrak{a}}$ are the *parahoric subgroups* of G. From this point of view, a group $U_{\mathfrak{a}}$ also carries many canonical, non-standard filtrations of interest in representation theory [29], [34].

A further concept is useful. Let $\mathfrak{a} = \mathfrak{a}_F(\mathfrak{L})$ be a hereditary $\mathfrak{o}_F$-order in A and E a subfield of A containing F. One says that $\mathfrak{a}$ is *E-pure* if $x^{-1}\mathfrak{a}x = \mathfrak{a}$, for all $x \in E^\times$. More expansively, this means that V is an E-vector space and $\mathfrak{L}$ is an $\mathfrak{o}_E$-lattice chain in V. If we let $B = \mathrm{End}_E(V)$ be the centralizer of E in A, then $\mathfrak{b} = \mathfrak{a} \cap B$ is the hereditary $\mathfrak{o}_E$-order in B defined by $\mathfrak{L}$: in our earlier notation,

$$\mathfrak{b} = \mathfrak{a} \cap B = \mathfrak{a}_E(\mathfrak{L}).$$

Observe also that $\mathfrak{p}_{\mathfrak{a}} \cap B = \mathrm{rad}\, \mathfrak{b}$.

2.2. Simple strata. A *simple stratum in A* is a pair $[\mathfrak{a}, \beta]$ consisting of a hereditary $\mathfrak{o}_F$-order $\mathfrak{a}$ in A and an element β of G satisfying the following conditions:

(1) the algebra $E = F[\beta]$ is a field and $v_E(\beta) < 0$;
(2) $\mathfrak{a}$ is E-pure;
(3) β is *simple over F.*

The effect of (3) is the following. If we have another pair $[\mathfrak{a},\beta']$ satisfying (1) and (2), such that $\beta'-\beta\in\mathfrak{a}$, then

$$[F[\beta']:F]\geqslant[F[\beta]:F].$$

The formal definition, which is equivalent to (3), is recalled in 6.3 below. The point needed here is that a simple stratum $[\mathfrak{a},\beta]$ gives rise, in a canonical and explicit manner, to an open, therefore compact, subgroup $H^1(\beta,\mathfrak{a})$ of $U^1_{\mathfrak{a}}$. At this stage, we note only that the group $H^1(\beta,\mathfrak{a})$ depends on the *equivalence class* of $[\mathfrak{a},\beta]$: if we have another simple stratum $[\mathfrak{a},\beta']$ such that $\beta'\equiv\beta \pmod{\mathfrak{a}}$, then $H^1(\beta',\mathfrak{a})=H^1(\beta,\mathfrak{a})$.

Remark. The property of "simplicity over F" is necessarily expressed via hereditary orders, but it really depends on β alone. Taking $E=F[\beta]$ as in part (1), let V_i be a finite-dimensional E-vector space and $\mathfrak{a}_i$ an E-pure hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(V_i)$, $i=1,2$. The pair $[\mathfrak{a}_1,\beta]$ is then a simple stratum if and only if $[\mathfrak{a}_2,\beta]$ is a simple stratum.

2.3. Simple characters. To proceed further, we need to choose a smooth character ψ of F *of level one.* That is, ψ is trivial on $\mathfrak{p}_F$ but not on $\mathfrak{o}_F$. If $[\mathfrak{a},\beta]$ is a simple stratum in A, the choice of ψ gives rise to a finite set $\mathcal{C}(\mathfrak{a},\beta,\psi)$ of very particular characters of the group $H^1(\beta,\mathfrak{a})$. These are the *simple characters* attached to $[\mathfrak{a},\beta]$. Again, the set $\mathcal{C}(\mathfrak{a},\beta,\psi)$ depends only on the equivalence class of β.

The choice of ψ does not affect the definition of simple characters: changing ψ only affects the way simple characters are labelled by simple strata. Explicitly, if ψ' is some other character of F of level one, there is a unit $u\in U_F$ such that $\psi'(x)=\psi(ux)$, $x\in F$. We then have $\mathcal{C}(\mathfrak{a},\beta,\psi')=\mathcal{C}(\mathfrak{a},u\beta,\psi)$. For this reason, when treating the relation between simple strata and simple characters, we tend to regard ψ as fixed and use the simpler notation $\mathcal{C}(\mathfrak{a},\beta)=\mathcal{C}(\mathfrak{a},\beta,\psi)$.

3. An example

To illuminate the outline of 2.2, 2.3, we give the simplest useful example.

3.1. Minimal elements. Let E/F be a finite field extension, let $\alpha\in E^\times$ and suppose that $E=F[\alpha]$.

Definition. *The element* α *is* minimal over F *if*

(1) *the integer* $\upsilon = \upsilon_E(\alpha)$ *is relatively prime to* $e = e(E|F)$*;*
(2) *if* ϖ *is a prime element of* F*, the coset* $\alpha^e\varpi^{-\upsilon}+\mathfrak{p}_E \subset U_E$ *generates the residue class field extension* $\Bbbk_E/\Bbbk_F$*.*

Condition (2) is, of course, independent of the choice of ϖ. For us, the key property is:

Proposition. *If* α *is minimal over* F *and* $\upsilon_{F[\alpha]}(\alpha) < 0$*, then* α *is simple over* F*.*

The proof is to be found in [15] 1.4.15.

3.2. Simple characters for minimal elements. Let $[\mathfrak{a}, \alpha]$ be a simple stratum in which α is minimal over F. It is a consequence of parts (1) and (2) in the definition (2.2) that $\alpha^{-1}\mathfrak{a} = \mathfrak{p}_{\mathfrak{a}}^l$, for an integer $l > 0$. Let $E = F[\alpha]$, let B be the A-centralizer of E and set $\mathfrak{b} = \mathfrak{a} \cap B$. In this situation,

$$H^1(\alpha, \mathfrak{a}) = U_{\mathfrak{b}}^1\, U_{\mathfrak{a}}^{[l/2]+1},$$

where $[x]$ denotes the integer part of a real number x.

A character θ of $H^1(\alpha, \mathfrak{a})$ lies in $\mathcal{C}(\mathfrak{a}, \alpha, \psi)$ if and only if $\theta|_{U_{\mathfrak{b}}^1}$ factors through the determinant map $\det_B : B^\times \to E^\times$ and

$$\theta(1+x) = \psi \circ \mathrm{tr}_A(\alpha x), \quad x \in \mathfrak{p}_{\mathfrak{a}}^{[l/2]+1},$$

where $\mathrm{tr}_A : A \to F$ denotes the matrix trace.

3.3. The general case. In general, an element β of G, which is simple over F, is constructed from a finite sequence of pairs (E_i, α_i). Here, the E_i/F are subfields of A of strictly increasing degree, and α_i is minimal over E_i. The definition of $H^1(\beta, \mathfrak{a})$ then follows this sequence step by step, as does the definition of $\mathcal{C}(\mathfrak{a}, \beta)$: see §6 for an overview of the construction.

4. Classification of cuspidal representations

We review the central classification results from [15].

4.1. Intertwining. Let $[\mathfrak{a},\beta]$ be a simple stratum in $A = \mathrm{End}_F(V)$. Let $E = F[\beta]$, let B be the A-centralizer of E and set $\mathfrak{b} = \mathfrak{a}\cap B$. Let $\theta \in \mathcal{C}(\mathfrak{a},\beta)$. We attach to θ the following groups:

$$\begin{aligned} \boldsymbol{J}_\theta &= \text{the } G\text{-normalizer of } \theta, \\ J_\theta^0 &= \boldsymbol{J}_\theta \cap U_\mathfrak{a}, \\ J_\theta^1 &= \boldsymbol{J}_\theta \cap U_\mathfrak{a}^1. \end{aligned}$$

Lemma. *Let $\mathcal{K}_\mathfrak{b}$ denote the group of $y \in B^\times$ such that $y^{-1}\mathfrak{b}y = \mathfrak{b}$.*

(1) *The group $\boldsymbol{J}_\theta$ is open and compact modulo centre in G.*

(2) *The groups $\boldsymbol{J}_\theta$, J_θ^0, J_θ^1 depend only on the equivalence class of $[\mathfrak{a},\beta]$, and satisfy the following relations,*

$$\begin{aligned} \boldsymbol{J}_\theta &= \mathcal{K}_\mathfrak{b}\, J_\theta^1, \\ J_\theta^0 &= U_\mathfrak{b} J_\theta^1, \\ U_\mathfrak{b}^1 &= U_\mathfrak{b} \cap J_\theta^1. \end{aligned}$$

(3) *The set $I_G(\theta)$, of elements of G which intertwine θ, is given by*

$$I_G(\theta) = J_\theta^1\, B^\times\, J_\theta^1.$$

Remarks. We will prefer to label groups by θ rather than the attached simple stratum since, as we shall see in 5.1, the stratum is not a reliable invariant of the situation. So, from now on, we usually write H_θ^1 rather than $H^1(\beta,\mathfrak{a})$, for $\theta \in \mathcal{C}(\mathfrak{a},\beta)$. We observe that J_θ^0 is the unique maximal compact subgroup of $\boldsymbol{J}_\theta$ and J_θ^1 is the pro-p radical of J_θ^0.

Example. If α is *minimal over F* (as in 3.2), the group J_θ^1 is given by

$$J_\theta^1 = J^1(\alpha,\mathfrak{a}) = U_\mathfrak{b}^1\, U_\mathfrak{a}^{[(l+1)/2]}.$$

4.2. Level zero. To get clean statements, we need a variant of the notion of simple character. A *trivial simple character* in G is the trivial character of $U_\mathfrak{a}^1$, for a hereditary $\mathfrak{o}_F$-order $\mathfrak{a}$ in A. We use the notation $1_\mathfrak{a}^1$ for such a character. If $\theta = 1_\mathfrak{a}^1$, the G-normalizer $\boldsymbol{J}_\theta$ of θ is $\mathcal{K}_\mathfrak{a}$ (notation as in 4.1 Lemma), while $J_\theta^0 = U_\mathfrak{a}$ and $H_\theta^1 = J_\theta^1 = U_\mathfrak{a}^1$. The set $I_G(\theta)$ is G itself. All the assertions of 4.1 Lemma thus remain valid in this case, provided we set $E = F$.

4.3. Extended maximal simple types. Let θ be a simple character in G.

Definition. *An* extended maximal simple type over θ *is an irreducible representation Λ of $\boldsymbol{J}_\theta$ such that $\Lambda|_{H^1_\theta}$ contains θ and $I_G(\Lambda) = \boldsymbol{J}_\theta$.*

Let $\mathcal{T}(\theta)$ be the set of equivalence classes of extended maximal simple types over θ.

This concept is only useful for a particular kind of simple character θ. Suppose first that θ is non-trivial, say $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$, for a simple stratum $[\mathfrak{a}, \beta]$ in A. As usual, let B be the A-centralizer of $E = F[\beta]$ and set $\mathfrak{b} = \mathfrak{a} \cap B$. We say that θ is *m-simple* if the hereditary $\mathfrak{o}_E$-order $\mathfrak{b}$ is *maximal,* or, equivalently, if $e(\mathfrak{a}) = e(E|F)$. If, on the other hand, θ is trivial, $\theta = 1^1_{\mathfrak{a}}$ say, then θ is called m-simple if $\mathfrak{a}$ is maximal. The reason for introducing this concept is:

Lemma. *Let θ be a simple character in G. The set $\mathcal{T}(\theta)$ is non-empty if and only if θ is m-simple.*

We have already remarked this property for trivial simple characters, in 1.2 Corollary. In general, one may equally describe the elements of $\mathcal{T}(\theta)$ explicitly: we shall do this in the more suggestive context of 9.3 below.

We may now summarize the main results of [15] concerning the structure of cuspidal representations.

Classification Theorem. *Let π be an irreducible cuspidal representation of G on a complex vector space $\mathcal{V}$.*

(1) *The representation π contains a simple character θ. Any such character is m-simple, and any two are G-conjugate.*
(2) *The natural representation Λ of $\boldsymbol{J}_\theta$ on $\mathcal{V}^\theta$ is irreducible, lies in $\mathcal{T}(\theta)$ and $\pi \cong c\text{-}\mathrm{Ind}_{\boldsymbol{J}_\theta}^G \Lambda$.*
(3) *If θ is an m-simple character in G, the map*
$$\Lambda \longmapsto c\text{-}\mathrm{Ind}_{\boldsymbol{J}_\theta}^G \Lambda$$
is a bijection between $\mathcal{T}(\theta)$ and the set of equivalence classes of irreducible cuspidal representations of G containing θ.

This theorem yields the desired explicit description of the irreducible cuspidal representations of G.

5. Endo-equivalence classes and lifting

We need some further properties of simple characters, in order to state a fundamental result concerning change of base field. As in 2.3, we work relative to a fixed choice of a smooth character ψ of F, of level one.

5.1. Intertwining and conjugacy. We consider the way in which simple characters $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ depend on the two associated parameters $\mathfrak{a}$ and β, within appropriate constraints. We start with a rather weak uniqueness property [15] 3.5.1.

Proposition 1. *Let $[\mathfrak{a}, \beta]$, $[\mathfrak{a}', \beta']$ be simple strata in A, and suppose that*

$$\mathcal{C}(\mathfrak{a}, \beta) \cap \mathcal{C}(\mathfrak{a}', \beta') \neq \emptyset$$

(whence, in particular, $H^1(\beta, \mathfrak{a}) = H^1(\beta', \mathfrak{a}')$). We then have:

1. $\mathfrak{a} = \mathfrak{a}'$,
2. $\mathcal{C}(\mathfrak{a}, \beta) = \mathcal{C}(\mathfrak{a}, \beta')$,
3. $e(F[\beta]|F) = e(F[\beta']|F)$ *and* $[F[\beta] : F] = [F[\beta'] : F]$.

The hypotheses of Proposition 1 imply no further relation between the fields $F[\beta]$, $F[\beta']$. Indeed, it is easy to find examples where any two fields of given degree and ramification index can give rise to the same sets of simple characters.

The second result [15] 3.5.11 of the sequence deals with intertwining of simple characters attached to the same hereditary order.

Proposition 2. *For $i = 1, 2$, let $[\mathfrak{a}, \beta_i]$ be a simple stratum in $A = \mathrm{End}_F(V)$, let $\theta_i \in \mathcal{C}(\mathfrak{a}, \beta_i)$. Suppose that θ_1 intertwines with θ_2 in $G = \mathrm{Aut}_F(V)$. There exists $x \in U_{\mathfrak{a}}$ such that $\theta_2 = \theta_1^x$. Indeed, $\theta \mapsto \theta^x$ is a bijection $\mathcal{C}(\mathfrak{a}, \beta_1) \to \mathcal{C}(\mathfrak{a}, \beta_2)$.*

5.2. Transfer. In another direction, we may fix the element β and vary the order $\mathfrak{a}$. We start from a finite field extension $E = F[\beta]/F$, generated by an element β, of negative valuation and simple over F, as in 2.2.

We suppose given two finite-dimensional E-vector spaces V_1, V_2 and set $A_i = \mathrm{End}_F(V_i)$. Let $\mathfrak{a}_i$ be an E-pure hereditary order in A_i. Thus $[\mathfrak{a}_i, \beta]$ is a simple stratum in A_i. In these circumstances, there is a canonical bijection

$$\tau^{\beta}_{\mathfrak{a}_1, \mathfrak{a}_2} : \mathcal{C}(\mathfrak{a}_1, \beta) \xrightarrow{\approx} \mathcal{C}(\mathfrak{a}_2, \beta).$$

We refer to $\tau^{\beta}_{\mathfrak{a}_1,\mathfrak{a}_2}$ as the *β-transfer* from $\mathfrak{a}_1$ to $\mathfrak{a}_2$. Transfer is natural relative to the orders $\mathfrak{a}_i$: in the obvious notation,

$$\tau^{\beta}_{\mathfrak{a}_1,\mathfrak{a}_3} = \tau^{\beta}_{\mathfrak{a}_2,\mathfrak{a}_3} \circ \tau^{\beta}_{\mathfrak{a}_1,\mathfrak{a}_2}.$$

It may, however, depend on the choice of β, *cf.* 5.1 Proposition 1.

Example. To indicate how $\tau^{\beta}_{\mathfrak{a}_1,\mathfrak{a}_2}$ is constructed, we return to the example of §3, in which the element β is *minimal over F.* Let $\nu = -\upsilon_E(\beta)$, and let $\mathfrak{a}$ be an E-pure hereditary order in some $A = \mathrm{End}_F(V)$. Let B be the A-centralizer of E and $\mathfrak{b} = \mathfrak{a} \cap B$. We have $\beta^{-1}\mathfrak{a} = \mathfrak{p}^l_{\mathfrak{a}}$, where l is the integer $\nu e_F(\mathfrak{a})/e(E|F) = \nu e_E(\mathfrak{b})$. Given $\theta \in \mathcal{C}(\mathfrak{a},\beta)$, there is a unique character χ_θ of U^1_E such that

$$\theta|_{U^1_{\mathfrak{b}}} = \chi_\theta \circ \det_B.$$

The character χ_θ determines θ uniquely. Given simple strata $[\mathfrak{a}_i,\beta]$ as above, the map $\tau = \tau^{\beta}_{\mathfrak{a}_1,\mathfrak{a}_2}$ is defined by the relation

$$\chi_{\tau\theta} = \chi_\theta, \quad \theta \in \mathcal{C}(\mathfrak{a}_1,\beta).$$

5.3. Endo-equivalence. We start with a pair of finite-dimensional F-vector spaces V_1, V_2. We are given a simple stratum $[\mathfrak{a}_i,\beta_i]$ in $A_i = \mathrm{End}_F(V_i)$, $i = 1, 2$. A *common realization* of $[\mathfrak{a}_1,\beta_1]$, $[\mathfrak{a}_2,\beta_2]$ consists of a finite-dimensional F-vector space V, a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $A = \mathrm{End}_F(V)$ and a pair of F-embeddings $f_i : F[\beta_i] \to A$ such that $\mathfrak{A}$ is $f_i(F[\beta_i])$-pure, $i = 1, 2$. Thus each $[\mathfrak{A}, f_i(\beta_i)]$ is a simple stratum in A.

We remark that, for fixed i, any two such embeddings f_i are $U_{\mathfrak{A}}$-conjugate, so the choice of f_i is irrelevant. We therefore speak of the pair $(V,\mathfrak{A})$ as a common realization of the $[\mathfrak{a}_i,\beta_i]$.

Lemma. *Let $[\mathfrak{a}_i,\beta_i]$ be a simple stratum in $A_i = \mathrm{End}_F(V_i)$, and let $\theta_i \in \mathcal{C}(\mathfrak{a}_i,\beta_i)$, $i = 1, 2$. The following are equivalent.*

1. *There exists a common realization $(V,\mathfrak{A})$ of the strata $[\mathfrak{a}_i,\beta_i]$ such that the simple characters $\tau^{\beta_i}_{\mathfrak{a}_i,\mathfrak{A}}\theta_i$ intertwine in $\mathrm{Aut}_F(V)$.*
2. *For any common realization $(V,\mathfrak{A})$ of the strata $[\mathfrak{a}_i,\beta_i]$, the simple characters $\tau^{\beta_i}_{\mathfrak{a}_i,\mathfrak{A}}\theta_i$ intertwine in $\mathrm{Aut}_F(V)$.*

As in 5.1 Proposition 2, the characters $\tau^{\beta_i}_{\mathfrak{a}_i,\mathfrak{A}}\theta_i$ intertwine in $\mathrm{Aut}_F(V)$ if and only if they are $U_{\mathfrak{A}}$-conjugate. For a proof of the lemma, see [2] 8.7.

Continuing in the context of the lemma, we say that θ_1 is *endo-equivalent to* θ_2 if the pair (θ_1, θ_2) satisfies the equivalent conditions (1) and (2). This relation of endo-equivalence is an equivalence relation on the class of all non-trivial simple characters in all groups $\mathrm{Aut}_F(V)$, as V ranges over the class of finite-dimensional F-vector spaces.

A trivial simple character never intertwines with a non-trivial one, so we may extend the notion by deeming that all trivial simple characters belong to one endo-equivalence class. Let $\mathcal{E}(F)$ denote the set of endo-equivalence classes of simple characters over F. We denote by $\mathbf{0}_F \in \mathcal{E}(F)$ the class of trivial ones.

We exhibit some useful consequences of these results.

Proposition.

(1) *A simple character is endo-equivalent to any of its transfers.*
(2) *Two simple characters over F, attached to the same hereditary $\mathfrak{o}_F$-order $\mathfrak{a}$, are endo-equivalent if and only if they are $U_{\mathfrak{a}}$-conjugate.*
(3) *Let $\theta_i \in \mathcal{C}(\mathfrak{a}_i, \beta_i)$, $i = 1, 2$. If θ_1 is endo-equivalent to θ_2, then*

$$[F[\beta_1] : F] = [F[\beta_2] : F].$$

Consequently, if $\Theta \in \mathcal{E}(F)$ is the endo-equivalence class of $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$, the integer

$$\deg \Theta = [F[\beta] : F]$$

depends only on Θ. Conventionally, $\deg \mathbf{0}_F = 1$.

Remark. In the context of part (3) of the proposition, the ramification indices $e(F[\beta_i]|F)$ are equal, as are the inertial degrees $f(F[\beta_i]|F)$. The extensions $F[\beta_i]/F$ need not be isomorphic. However, if T_i/F is the maximal tamely ramified sub-extension of $F[\beta_i]/F$, the fields T_i *are* F-isomorphic [13], 2.4. Indeed, there exists $j \in J^1_\theta$ such that $T_2 = T_1^j$. Any two choices of j induce the same isomorphism $x \mapsto j^{-1}xj$ from T_1 to T_2. Thus θ determines the maximal tamely ramified sub-extension uniquely, *up to distinguished isomorphism.*

5.4. Tame lifting. It is apparent from the definitions that a field isomorphism $F \to F'$ induces a bijection $\mathcal{E}(F) \to \mathcal{E}(F')$. In particular, the group $\operatorname{Aut} F$ acts on $\mathcal{E}(F)$. The operation $F \mapsto \mathcal{E}(F)$ has a more interesting property: if K/F is a finite, *tamely ramified* field extension, there is a canonical map [2]

$$\mathfrak{i}_{K/F} : \mathcal{E}(K) \longrightarrow \mathcal{E}(F).$$

The definition is outlined in §6. Here, we list only the main properties.

Proposition.

(1) *The map $\mathfrak{i}_{K/F}$ is surjective. If L/K is finite and tamely ramified, then*

$$\mathfrak{i}_{L/F} = \mathfrak{i}_{K/F} \circ \mathfrak{i}_{L/K}.$$

(2) *The map $\mathfrak{i}_{K/F}$ has finite fibres. If $\Theta \in \mathcal{E}(F)$, then*

$$\deg \Theta = \sum_{\Phi} \deg \Phi,$$

where Φ ranges over the elements of $\mathcal{E}(K)$ for which $\mathfrak{i}_{K/F}\Phi = \Theta$. Moreover, $\mathfrak{i}_{K/F}\Phi = \mathbf{0}_F$ if and only if $\Phi = \mathbf{0}_K$.

(3) *If K/F is Galois and $\Phi \in \mathcal{E}(K)$, then*

$$\mathfrak{i}_{K/F}^{-1}\big(\mathfrak{i}_{K/F}\Phi\big) = \{\Phi^\gamma : \gamma \in \operatorname{Gal}(K/F)\}.$$

If $\Theta \in \mathcal{E}(F)$, the *$K/F$-lifts of Θ* are the elements of the fibre $\mathfrak{i}_{K/F}^{-1}\Theta$. If Θ is the endo-equivalence class of $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$, there is a canonical bijection between the set of K/F-lifts of Θ and the simple components of the semisimple K-algebra $K \otimes_F F[\beta]$.

5.5. Relation with automorphic induction. We move briefly to a different situation. Let K/F be a finite, *cyclic* extension of degree d. Let ρ be an irreducible cuspidal representation of $\mathrm{GL}_m(K)$. The operation of *automorphic induction* attaches to ρ an irreducible smooth representation $\pi = \mathrm{A}_{K/F}\,\rho$ of the group $\mathrm{GL}_{md}(F)$. This is defined in [22] when F has characteristic zero, and in [23] otherwise. For us, the point is that automorphic induction corresponds, via the Langlands correspondence, to the operation of induction to $\mathcal{W}_F$ of smooth representations of $\mathcal{W}_K$.

For the same ρ, the representation $\pi = \mathrm{A}_{K/F}\,\rho$ is parabolically induced from an irreducible cuspidal representation π_L of the Levi factor L of some (not necessarily proper) parabolic subgroup of $\mathrm{GL}_{md}(F)$. The group L is of the form $G_1 \times G_2 \times \cdots \times G_r$, for a divisor r of d and where $G_i \cong \mathrm{GL}_{md/r}(F)$. Thus $\pi_L \cong \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_r$, where π_j is an irreducible cuspidal representation of G_j. We use the notation

$$\pi = \pi_1 \boxplus \pi_2 \boxplus \cdots \boxplus \pi_r.$$

By the Classification Theorem 4.3, the representation ρ contains a unique conjugacy class of m-simple characters in $\mathrm{GL}_m(K)$, the endo-class of which we denote $\vartheta(\rho)$. We similarly define $\vartheta(\pi_j) \in \mathcal{E}(F)$, $1 \leqslant j \leqslant r$.

Automorphic Induction Theorem. *Let K/F be cyclic and tamely ramified of degree d. Let ρ be an irreducible cuspidal representation of $\mathrm{GL}_m(K)$ and write*

$$\pi = \mathrm{A}_{K/F}\,\rho = \pi_1 \boxplus \pi_2 \boxplus \cdots \boxplus \pi_r,$$

where $r \geqslant 1$ and π_j is an irreducible cuspidal representation of $\mathrm{GL}_{md/r}(F)$, $1 \leqslant j \leqslant r$. We have

$$\vartheta(\pi_j) = \mathrm{i}_{K/F}\,\vartheta(\rho), \quad 1 \leqslant j \leqslant r.$$

The proof of this theorem is given in [5] but relies heavily on some special cases in [2]. In both of those papers, we assumed that F had characteristic zero since, at the time they were written, automorphic induction was known only in that case. The existence, and relevant properties, of automorphic induction in positive characteristic are established in [23]. Once that theory became available, so did the positive characteristic case of the theorem: the proof requires no modification. We remark also that there is a related result connecting tame lifting with *base change,* in the sense of [1] and [23], but we will not use that here.

6. Appendix: a skeleton of definitions

We extract from [15] and [2] the basic definitions pertaining to simple characters and strata, and state the structure theorems giving them their explicit form. The section may be omitted at first reading: we shall only refer to it once in the pages to follow. However, we are guided by the desire to use the Langlands correspondence as a computational tool, and the material here is essential to any such project. This skeleton should prove adequate for most purposes, and may also serve as a short introduction for a reader unfamiliar with these matters.

Let V be a finite-dimensional F-vector space, and set $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$.

6.1. Adjoint and co-restriction. This preliminary material is to be found in [15] 1.3, 1.4.

Let E/F be a subfield of A. Thus V is an E-vector space and $B = \mathrm{End}_E(V)$ is the centralizer of E in A. Let $\mathrm{tr}_A : A \to F$ be the reduced trace. Thus $(x, y) \mapsto \mathrm{tr}_A(xy)$ provides a nondegenerate, symmetric bilinear form $A \times A \to F$. Let C be the orthogonal complement of B with respect to this pairing. In particular, C is a (B, B)-bimodule.

Note. If E/F is separable, then A is the orthogonal sum of B and C. Otherwise, $B \subset C$.

Suppose $E = F[\beta]$, for some $\beta \in E^\times$. For $x \in A$, we define $a_\beta(x) = \beta x - x\beta$. Thus a_β is a (B, B)-homomorphism $A \to C$ with kernel B. It follows that $a_\beta(A) = C$.

In the other direction, *a tame co-restriction on A, relative to E/F,* is a (B, B)-homomorphism $s : A \to B$ with the following property. If $\mathfrak{a}$ is an E-pure, hereditary $\mathfrak{o}_F$-order in A, then $s(\mathfrak{a}) = \mathfrak{a} \cap B$. Such a map s exists, and is unique up to multiplication by a unit of E. In particular, we have an infinite exact sequence

$$\dots \to A \xrightarrow{a_\beta} A \xrightarrow{s} A \xrightarrow{a_\beta} A \xrightarrow{s} \dots$$

Now write $\mathfrak{b} = \mathfrak{a} \cap B$, $\mathfrak{p} = \mathrm{rad}\,\mathfrak{a}$ and $\mathfrak{q} = \mathrm{rad}\,\mathfrak{b}$. A tame co-restriction then has the further property

$$s(\mathfrak{p}^k) = \mathfrak{q}^k, \quad k \in \mathbb{Z}.$$

6.2. Relation with duality. The tame co-restriction appears naturally in the context of duality. Let ψ be a smooth character of F, $\psi \neq 1$, and let ψ_A

denote the smooth character $x \mapsto \psi(\mathrm{tr}_A\, x)$ of A. For $a \in A$, let $a\psi_A$ denote the character $x \mapsto \psi_A(xa)$. The map $a \mapsto a\psi_A$ then gives a topological isomorphism of A with its group $\widehat{A}$ of smooth characters.

If E/F is a subfield of A with centralizer B, and if $\xi \neq 1$ is a smooth character of E, we may similarly define a character $\xi_B \in \widehat{B}$. This yields an isomorphism $B \to \widehat{B}$, $b \mapsto b\xi_B$. The obvious restriction map $\widehat{A} \to \widehat{B}$ is surjective. So, if we identify A with $\widehat{A}$ and B with $\widehat{B}$ via choices of characters $\psi \in \widehat{F}$, $\xi \in \widehat{E}$, this restriction corresponds to a surjective map $s_{\psi,\xi} : A \to B$. If we take both ψ and ξ to be of level one, then $s_{\psi,\xi}$ is a tame co-restriction on A, relative to E/F.

6.3. Strata. We need a looser definition of stratum, as in Chapter 1 of [15]. We recall (2.1) that the Jacobson radical of a hereditary order $\mathfrak{a}$ is *invertible,* as two-sided ideal of $\mathfrak{a}$.

A *stratum in A* is a quadruple $[\mathfrak{a}, l, m, b]$ as follows. First, $\mathfrak{a}$ is a hereditary $\mathfrak{o}_F$-order in A; we set $\mathfrak{p} = \mathrm{rad}\,\mathfrak{a}$. The parameters l, m are integers such that $l > m$. Finally, $b \in \mathfrak{p}^{-l}$. Strata $[\mathfrak{a}, l, m, b_i]$, $i = 1, 2$, are deemed *equivalent* if $b_1 \equiv b_2 \pmod{\mathfrak{p}^{-m}}$. We use the notation

$$[\mathfrak{a}, l, m, b_1] \sim [\mathfrak{a}, l, m, b_2].$$

A stratum $[\mathfrak{a}, l, m, \beta]$ is called *pure* if $F[\beta]$ is a field, $\mathfrak{a}$ is $F[\beta]$-pure, and $\beta\mathfrak{a} = \mathfrak{p}^{-l}$.

Let $[\mathfrak{a}, l, m, \beta]$ be a pure stratum in A, and write $E = F[\beta]$. Let B be the A-centralizer of E, and take $\mathfrak{b}$, $\mathfrak{p}$, $\mathfrak{q}$ as in 6.1. Let k be an integer and define

$$\mathfrak{N}_k = \{x \in \mathfrak{a} : a_\beta(x) \in \mathfrak{p}^k\}.$$

For k sufficiently large, we have $\mathfrak{N}_k \subset \mathfrak{b}{+}\mathfrak{p}$. Assuming $E \neq F$, we define

$$k_0(\beta, \mathfrak{a}) = \max\{k \in \mathbb{Z} : \mathfrak{N}_k \not\subset \mathfrak{b}{+}\mathfrak{p}\}.$$

In the case $E = F$, it is convenient to set $k_0(\beta, \mathfrak{a}) = -\infty$. Otherwise, we have $k_0(\beta, \mathfrak{a}) \geqslant -l$.

A *simple stratum in A* is a pure stratum $[\mathfrak{a}, l, m, \beta]$ such that

$$m < -k_0(\beta, \mathfrak{a}).$$

To describe the dependence of $k_0(\beta, \mathfrak{a})$ on $\mathfrak{a}$, we note that the matrix algebra $\mathrm{End}_F(E)$ contains a unique E-pure hereditary $\mathfrak{o}_F$-order $\mathfrak{a}(E)$: this is defined by the lattice chain $\{\mathfrak{p}_E^j : j \in \mathbb{Z}\}$ in E. Let $\mathfrak{p}(E) = \operatorname{rad} \mathfrak{a}(E)$, so that $\beta\mathfrak{a}(E) = \mathfrak{p}(E)^\upsilon$, where $\upsilon = \upsilon_E(\beta)$. We set

$$k_F(\beta) = k_0(\beta, \mathfrak{a}(E)).$$

We expand a comment made in 2.2.

Proposition. *Let $[\mathfrak{a}, l, m, \beta]$ be a pure stratum in A, with $\beta \notin F$.*

(1) *The quantity $k_0(\beta, \mathfrak{a})$ is given by*

$$k_0(\beta, \mathfrak{a}) = k_F(\beta)\, e_F(\mathfrak{a})/e(F[\beta]|F).$$

(2) *The element β is minimal over F if and only if $k_0(\beta, \mathfrak{a}) = -l$, that is, if and only if $k_F(\beta) = \upsilon_{F[\beta]}(\beta)$. In particular, a pure stratum $[\mathfrak{a}, l, l{-}1, \beta]$ is simple if and only if β is minimal over F.*

These assertions are proved in [15] 1.4.13, 1.4.15 respectively.

6.4. Synthesis of simple strata. All simple strata are built from minimal elements in a systematic manner.

Theorem 1. *Let $[\mathfrak{a}, l, m, \gamma]$ be a simple stratum in A. Let B be the A-centralizer of γ, let $\mathfrak{b} = \mathfrak{a} \cap B$, and let $s_\gamma : A \to B$ be a tame co-restriction on A relative to $F[\gamma]/F$. Let $[\mathfrak{b}, m, m{-}1, \alpha]$ be a simple stratum in B.*

(1) *There is a simple stratum $[\mathfrak{a}, l, m{-}1, \beta]$ in A such that*

$$[\mathfrak{a}, l, m, \beta] \sim [\mathfrak{a}, l, m, \gamma] \quad \text{and}$$
$$[\mathfrak{b}, m, m{-}1, s_\gamma(\beta{-}\gamma)] \sim [\mathfrak{b}, m, m{-}1, \alpha].$$

(2) *For any such β, we have*

$$e(F[\beta]\big|F) = e(F[\gamma]\big|F)\, e(F[\gamma, \alpha]\big|F[\gamma]),$$
$$f(F[\beta]\big|F) = f(F[\gamma]\big|F)\, f(F[\gamma, \alpha]\big|F[\gamma]).$$

(3) *Moreover,*

$$k_0(\beta, \mathfrak{a}) = \begin{cases} -m & \text{if } \alpha \notin F[\gamma], \\ k_0(\gamma, \mathfrak{a}) & \text{otherwise.} \end{cases}$$

Remark. In the context of Theorem 1, the field $F[\gamma]$ need not be F-isomorphic to a subfield of $F[\beta]$.

All simple strata arise from the construction in Theorem 1. Indeed, let $[\mathfrak{a}, l, m, \beta]$ be a simple stratum in A and set $r = -k_0(\beta, \mathfrak{a})$. We assume $\beta \notin F$, so r is an integer satisfying $m < r \leqslant l$. There is nothing to do if $r = l$, since β is then minimal over F. We therefore assume the contrary.

Theorem 2. *There exists a simple stratum $[\mathfrak{a}, l, r, \gamma]$ in A such that*

$$[\mathfrak{a}, l, r, \gamma] \sim [\mathfrak{a}, l, r, \beta].$$

Moreover, if B is the A-centralizer of γ, if $\mathfrak{b} = \mathfrak{a} \cap B$, and if $s_\gamma : A \to B$ is a tame co-restriction on A relative to $F[\gamma]/F$, then $[\mathfrak{b}, r, r-1, s_\gamma(\beta-\gamma)]$ is equivalent to a simple stratum in B.

This sort of technique also allows one to compare simple strata, step by step.

Proposition. *Let $[\mathfrak{a}, l, m, \beta]$ be a simple stratum in A, let B be the A-centralizer of β and let $\mathfrak{b} = \mathfrak{a} \cap b$. Let $[\mathfrak{a}, l, m, \beta']$ be a simple stratum in A, equivalent to $[\mathfrak{a}, l, m, \beta]$. We then have*

(1) $k_0(\beta', \mathfrak{a}) = k_0(\beta, \mathfrak{a})$;
(2) *if s_β is a tame co-restriction on A relative to $F[\beta]/F$, the stratum $[\mathfrak{b}, m, m-1, s_\beta(\beta'-\beta)]$ is equivalent to either $[\mathfrak{b}, m, m-1, 0]$ or a simple stratum $[\mathfrak{b}, m, m-1, \alpha]$, where $\alpha \in F[\beta]^\times$;*
(3) *the first alternative in (2) holds if and only if $[\mathfrak{a}, l, m-1, \beta']$ is equivalent to a G-conjugate of $[\mathfrak{a}, l, m-1, \beta]$.*

6.5. Groups and characters. We start with a simple stratum $[\mathfrak{a}, l, 0, \beta]$ in A, and attach to it a pair $H^1(\beta, \mathfrak{a}) \subset J^1(\beta, \mathfrak{a})$ of open subgroups of $U^1_\mathfrak{a}$. Set $r = -k_0(\beta, \mathfrak{a})$. Thus r is an integer such that $0 < r \leqslant l$, or else $r = \infty$ (corresponding to the case $\beta \in F^\times$). In the case $r \geqslant l$ (so that β is minimal over F), we use the definition from §3:

$$H^1(\beta, \mathfrak{a}) = U^1_\mathfrak{b}\, U^{[l/2]+1}_\mathfrak{a}, \quad J^1(\beta, \mathfrak{a}) = U^1_\mathfrak{b}\, U^{[(l+1)/2]}_\mathfrak{a},$$

where $\mathfrak{b}$ is the $\mathfrak{a}$-centralizer of β.

We therefore assume $0 < r < l$. We choose a simple stratum $[\mathfrak{a}, l, r, \gamma]$ equivalent to $[\mathfrak{a}, l, r, \beta]$. Let B denote the A-centralizer of β and set $\mathfrak{b} = \mathfrak{a} \cap B$. Inductively, the group $H^1(\gamma, \mathfrak{a})$ has been defined. We put $H^k(\beta, \mathfrak{a}) = H^1(\beta, \mathfrak{a}) \cap U_{\mathfrak{a}}^k$, $k \geqslant 1$, and similarly for J^k. We set

$$H^1(\beta, \mathfrak{a}) = U_{\mathfrak{b}}^1 \, H^{[r/2]+1}(\gamma, \mathfrak{a}), \quad J^1(\beta, \mathfrak{a}) = U_{\mathfrak{b}}^1 \, J^{[(r+1)/2]}(\gamma, \mathfrak{a}).$$

These groups depend only on the equivalence class of the stratum $[\mathfrak{a}, l, 0, \beta]$.

Next, we choose a smooth character ψ of F of level one. For $a \in A$, we denote by ψ_a the function $x \mapsto \psi(\mathrm{tr}_A(a(x-1)))$ on A. We define a set $\mathcal{C}(\mathfrak{a}, \beta, \psi)$ of characters of $H^1(\beta, \mathfrak{a})$, following the preceding construction. Suppose first that β is minimal over F. As in §3, a character θ lies in $\mathcal{C}(\mathfrak{a}, \beta, \psi)$ if and only if $\theta|_{U_{\mathfrak{b}}^1}$ factors through $\det_B$ and

$$\theta(y) = \psi_\beta(y), \quad y \in U_{\mathfrak{a}}^{[l/2]+1}.$$

Otherwise, we take r and γ as before. A character θ of $H^1(\beta, \mathfrak{a})$ lies in $\mathcal{C}(\mathfrak{a}, \beta, \psi)$ if and only if $\theta|_{U_{\mathfrak{b}}^1}$ factors through $\det_B$ and there exists $\phi \in \mathcal{C}(\mathfrak{a}, \gamma, \psi)$ such that

$$\theta(y) = \phi(y) \, \psi_{\beta-\gamma}(y), \quad y \in H^{[r/2]+1}(\gamma, \mathfrak{a}).$$

6.6. Tame lifting. We outline a construction from [2]. For this, we need a simple stratum $[\mathfrak{a}, l, 0, \beta]$ in A, and a subfield K/F of A, commuting with β and such that the algebra $K[\beta]$ is a field.

Proposition 1. *Let C denote the A-centralizer of K and $\mathfrak{c} = \mathfrak{a} \cap C$.*

(1) *The quadruple $[\mathfrak{c}, l, 0, \beta]$ is a simple stratum in C.*
(2) *The group $H^1(\beta, \mathfrak{a}) \cap C$ is equal to $H^1(\beta, \mathfrak{c})$.*
(3) *Let $\psi_K = \psi \circ \mathrm{Tr}_{K/F}$. If $\theta \in \mathcal{C}(\mathfrak{a}, \beta, \psi)$, then the restriction*

$$\theta_K := \theta|_{H^1(\beta, \mathfrak{c})}$$

lies in $\mathcal{C}(\mathfrak{c}, \beta, \psi_K)$.

In the situation of the proposition, the field $K[\beta]$ is K-isomorphic to exactly one simple component of the semisimple K-algebra $K \otimes_F F[\beta]$.

Proposition 2. *Let K/F be a finite, tamely ramified field extension. Let V be a finite-dimensional K-vector space and set $C = \mathrm{End}_K(V)$, $A = \mathrm{End}_F(V)$. Let $[\mathfrak{c}, l, 0, \beta]$ be a simple stratum in C and let $\mathfrak{a}$ be the unique K-pure hereditary $\mathfrak{o}_F$-order in A such that $\mathfrak{a} \cap C = \mathfrak{c}$. Let $\psi_K = \psi \circ \mathrm{Tr}_{K/F}$.*

There exists a simple stratum $[\mathfrak{c}, l, 0, \beta']$ in C such that

1. *$\mathcal{C}(\mathfrak{c}, \beta', \psi_K) = \mathcal{C}(\mathfrak{c}, \beta, \psi_K)$, and*
2. *the quadruple $[\mathfrak{a}, l, 0, \beta']$ is a simple stratum in A.*

Let $\phi \in \mathcal{C}(\mathfrak{c}, \beta, \psi_K)$. For any such β', there exists a unique $\theta \in \mathcal{C}(\mathfrak{a}, \beta', \psi)$ such that $\theta_K = \phi$.

In the context of Proposition 2, the endo-equivalence class $\varTheta \in \mathcal{E}(F)$ of θ depends only on the endo-equivalence class $\varPhi \in \mathcal{E}(K)$ of ϕ. The process $\varPhi \mapsto \varTheta$ gives a well-defined map $\mathcal{E}(K) \to \mathcal{E}(F)$ which is independent of the initial choice of ψ. This map is the one denoted $\mathfrak{i}_{K/F}$ in 5.4.

II. Representations of the Weil group

Let $\bar{F}/F$ be a separable algebraic closure of F, and let $\mathcal{W}_F$ be the Weil group of $\bar{F}/F$. If E/F is a finite separable field extension with $E \subset \bar{F}$, we identify the Weil group $\mathcal{W}_E$ of $\bar{F}/E$ with the subgroup of $\mathcal{W}_F$ which fixes E under the natural action of $\mathcal{W}_F$ on $\bar{F}$.

Let $\mathcal{P}_F$ denote the wild inertia, or first ramification, subgroup of $\mathcal{W}_F$. Thus $\mathcal{P}_F$ is a closed, normal subgroup of $\mathcal{W}_F$. It is a pro-p group, and may be identified with the Galois group of $\bar{F}/F^{\mathrm{tr}}$, where F^{tr}/F is the maximal tamely ramified extension of F inside $\bar{F}$. In particular, if $K \subset \bar{F}$ and K/F is finite and tamely ramified, then $\mathcal{P}_K = \mathcal{P}_F$.

7. Application of Clifford theory

7.1. Representations. Let $\mathcal{G}_n(F)$ be the set of equivalence classes of irreducible, smooth representations of $\mathcal{W}_F$ of dimension n and set

$$\widehat{\mathcal{W}}_F = \bigcup_{n \geqslant 1} \mathcal{G}_n(F).$$

Analogously, let $\widehat{\mathcal{P}}_F$ be the set of equivalence classes of irreducible smooth representations of $\mathcal{P}_F$. We use elementary Clifford theory to describe $\widehat{\mathcal{W}}_F$ in terms of $\widehat{\mathcal{P}}_F$.

If $\sigma \in \widehat{\mathcal{W}}_F$, the restriction $\sigma|_{\mathcal{P}_F}$, of σ to $\mathcal{P}_F$, is semisimple. It is a direct sum of various $\alpha \in \widehat{\mathcal{P}}_F$, any two of which are $\mathcal{W}_F$-conjugate and occur with the same multiplicity. We enshrine this in the canonical map

$$r_F^1 : \widehat{\mathcal{W}}_F \longrightarrow \mathcal{W}_F \backslash \widehat{\mathcal{P}}_F$$

which sends $\sigma \in \widehat{\mathcal{W}}_F$ to the $\mathcal{W}_F$-orbit of an irreducible component of $\sigma|_{\mathcal{P}_F}$. For an integer $s \geqslant 1$ and $\alpha \in \widehat{\mathcal{P}}_F$, we accordingly define

$$\mathcal{G}_s(F;\alpha) = \{\sigma \in \widehat{\mathcal{W}}_F : \dim \mathrm{Hom}_{\mathcal{P}_F}(\alpha,\sigma) = s\}.$$

For example, let $\mathbf{1}_F$ be the trivial character of $\mathcal{P}_F$. The elements of $\mathcal{G}_s(F;\mathbf{1}_F)$ are then the irreducible, s-dimensional, *tamely ramified* smooth representations of $\mathcal{W}_F$.

Proposition. *Let $\alpha \in \widehat{\mathcal{P}}_F$.*

(1) *The $\mathcal{W}_F$-isotropy group of α is of the form $\mathcal{W}_E$, where $E = Z_F(\alpha)/F$ is finite and tamely ramified.*
(2) *There exists $\rho \in \widehat{\mathcal{W}}_E$ such that $\rho|_{\mathcal{P}_F} \cong \alpha$. If ρ' is any other such representation, there is a unique tamely ramified character ψ of $\mathcal{W}_E$ such that $\rho' \cong \rho \otimes \psi$.*
(3) *Taking ρ as in (2), let $\tau \in \mathcal{G}_s(E;\mathbf{1}_F)$. The representation*

$$\Sigma_\rho(\tau) = \mathrm{Ind}_{E/F}\, \rho \otimes \tau$$

is irreducible, and lies in $\mathcal{G}_s(F;\alpha)$. The map

$$\Sigma_\rho : \mathcal{G}_s(E;\mathbf{1}_F) \longrightarrow \mathcal{G}_s(F;\alpha)$$

is a bijection.

All assertions here are straightforward, but a complete proof may be found in §1 of [13].

III. Connections

For an integer $n \geqslant 1$, let $\mathcal{A}_n(F)$ denote the set of equivalence classes of irreducible, smooth, *cuspidal* representations of $\mathrm{GL}_n(F)$. It will also be convenient to have the notation

$$\widehat{\mathrm{GL}}_F = \bigcup_{n\geqslant 1} \mathcal{A}_n(F).$$

Thus the Langlands correspondence $\sigma \mapsto {}^L\sigma$ is a bijection $\widehat{\mathcal{W}}_F \to \widehat{\mathrm{GL}}_F$.

If $\pi \in \mathcal{A}_n(F)$, then π contains a unique G-conjugacy class of simple characters in $G = \mathrm{GL}_n(F)$ (4.3). These simple characters all lie in the same endo-equivalence class, which we have denoted $\vartheta(\pi)$. Thus we have a canonical surjective map

$$\vartheta : \widehat{\mathrm{GL}}_F \longrightarrow \mathcal{E}(F).$$

8. Some basic relations

8.1. Ramification theorem. If K/F is a finite, tamely ramified extension, then $\mathcal{P}_K = \mathcal{P}_F$, and there is a canonical surjection $\mathcal{W}_K\backslash\widehat{\mathcal{P}}_F \to \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$. The first step in our description of the Langlands correspondence is:

Ramification Theorem.

(1) *There is a unique map* $\Phi_F : \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F \to \mathcal{E}(F)$ *such that*

$$\begin{array}{ccc} \widehat{\mathcal{W}}_F & \xrightarrow{\ L\ } & \widehat{\mathrm{GL}}_F \\ {\scriptstyle r_F^1}\big\downarrow & & \big\downarrow{\scriptstyle \vartheta} \\ \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F & \xrightarrow[\Phi_F]{} & \mathcal{E}(F) \end{array}$$

commutes. The map Φ_F *is bijective.*

(2) *If* K/F *is a finite, tamely ramified field extension, then*

$$\begin{array}{ccc} \mathcal{W}_K\backslash\widehat{\mathcal{P}}_F & \xrightarrow{\ \Phi_K\ } & \mathcal{E}(K) \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{i}_{K/F}} \\ \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F & \xrightarrow[\Phi_F]{} & \mathcal{E}(F) \end{array}$$

commutes.

Part (1) is proved in [5] §8, under the restriction that F has characteristic zero. As remarked in 5.5, it holds equally, with the same proof, in positive characteristic. Part (2) is 6.2 of [13] (and follows easily from the Automorphic Induction Theorem of 5.5).

Let $\Theta \in \mathcal{E}(F)$ and let $s \geqslant 1$ be a positive integer. We define $\mathcal{A}_s(F;\Theta)$ to be the set of $\pi \in \mathcal{A}_{s\deg\Theta}(F)$ such that $\vartheta(\pi) = \Theta$. As in [13], we have the following corollary.

Tame Parameter Theorem. *Let $\alpha \in \widehat{\mathcal{P}}_F$ and set $E = Z_F(\alpha)$, $\Theta = \Phi_F(\alpha)$.*

(1) *We have*

$$\deg\Theta = [E{:}F]\dim\alpha.$$

(2) *If Θ is the endo-equivalence class of $\theta \in \mathcal{C}(\mathfrak{a},\beta)$, for a simple stratum $[\mathfrak{a},\beta]$ in some matrix algebra, then E is F-isomorphic to the maximal tamely ramified sub-extension T/F of $F[\beta]/F$.*

(3) *The Langlands correspondence induces a bijection*

$$\mathcal{G}_s(F;\alpha) \longrightarrow \mathcal{A}_s(F;\Theta),$$

for all $s \geqslant 1$.

We emphasize that, in part (2), there is no distinguished F-isomorphism of E with T.

Remarks. In the case $\dim\alpha = 1$, part (1) of the Ramification Theorem follows directly from local class field theory: if $\boldsymbol{a}_F : \mathcal{W}_F \to F^\times$ is the Artin Reciprocity map, then $\boldsymbol{a}_F(\mathcal{P}_F) = U_F^1$. In the case $\dim\alpha = p$, one may deduce something of the nature of $\Phi_F(\alpha)$ from Mœglin's treatment [28] of the Langlands correspondence in dimension p, $p \geqslant 5$. For detailed treatment of the case $p = 2$, see [26] or [9], for $p = 3$ see [19]. Otherwise, we have virtually no systematic information concerning the map Φ_F.

8.2. Tamely ramified representations. Let $\mathbf{1}_F$ be the trivial character of $\mathcal{P}_F$, let $n \geqslant 1$, and consider the set $\mathcal{G}_n(F;\mathbf{1}_F)$ of classes of irreducible *tamely ramified* representations of $\mathcal{W}_F$, of dimension n.

Let F_n/F be unramified of degree n, $\Delta = \mathrm{Gal}(F_n/F)$, $X_1(F_n) =$ the group of tamely ramified characters of $F_n^\times$. The group Δ acts on $X_1(F_n)$. We say that

$\chi \in X_1(F_n)$ is *$\mathit{\Delta}$-regular* if the characters χ^δ, $\delta \in \mathit{\Delta}$, are distinct. We denote by $X_1(F_n)^{\mathit{\Delta}\text{-reg}}$ the set of $\mathit{\Delta}$-regular elements of $X_1(F_n)$. The map

$$\begin{aligned} \mathit{\Delta}\backslash X_1(F_n)^{\mathit{\Delta}\text{-reg}} &\longrightarrow \mathcal{G}_n(F;\mathbf{1}_F), \\ \chi &\longmapsto \sigma_\chi = \operatorname{Ind}_{F_n/F}\chi, \end{aligned}$$

is then a canonical bijection.

Recall that $\mathbf{0}_F \in \mathcal{E}(F)$ is the endo-equivalence class of trivial simple characters over F. As an instance of the Remarks in 8.1, we have $\mathbf{0}_F = \Phi_F(\mathbf{1}_F)$. Let $\mathfrak{m} = \mathrm{M}_n(\mathfrak{o}_F)$. We describe canonical bijections

$$\begin{gathered} \mathit{\Delta}\backslash X_1(F_n)^{\mathit{\Delta}\text{-reg}} \longrightarrow \mathcal{T}(1^1_{\mathfrak{m}}) \longrightarrow \mathcal{A}_n(F;\mathbf{0}_F), \\ \chi \longmapsto \mathit{\Lambda}_\chi \longmapsto \pi_\chi. \end{gathered}$$

For the first, let $\boldsymbol{\mu}(F_n)$ denote the group of roots of unity in F_n, of order relatively prime to p. The Galois group $\mathit{\Delta}$ acts on $\boldsymbol{\mu}(F_n)$; an element ζ of $\boldsymbol{\mu}(F_n)$ is called *$\mathit{\Delta}$-regular* if the conjugates ζ^δ, $\delta \in \mathit{\Delta}$, are distinct.

We embed F_n in $\mathrm{M}_n(F)$ so that $\mathfrak{m}$ becomes F_n-pure. This embedding identifies $\boldsymbol{\mu}(F_n)$ with a subgroup of $U_{\mathfrak{m}}$. Reduction modulo $\mathfrak{p}_{\mathfrak{m}}$ then identifies $\boldsymbol{\mu}(F_n)$ with a subgroup of $\mathcal{G} = \mathrm{GL}_n(\Bbbk_F)$, the $\mathit{\Delta}$-regular elements of $\boldsymbol{\mu}(F_n)$ becoming elliptic regular in $\mathcal{G}$. Let $\chi \in X_1(F_n)^{\mathit{\Delta}\text{-reg}}$. As in [17] (*cf.* §2 of [10]), there is a unique irreducible cuspidal representation $\tilde\lambda_\chi$ of $\mathcal{G}$ such that

$$\operatorname{tr}\tilde\lambda_\chi(\zeta) = (-1)^{n-1}\sum_{\delta\in\mathit{\Delta}}\chi^\delta(\zeta),$$

for every $\mathit{\Delta}$-regular element ζ of $\boldsymbol{\mu}(F_n)$. We define an irreducible representation $\mathit{\Lambda}_\chi$ of the group $\boldsymbol{J} = F^\times U_{\mathfrak{m}}$ by deeming that $\mathit{\Lambda}_\chi|_{U_{\mathfrak{m}}}$ be the inflation of $\tilde\lambda_\chi$ and that $\mathit{\Lambda}_\chi|_{F^\times}$ be a multiple of $\chi|_{F^\times}$. The map $\chi \mapsto \mathit{\Lambda}_\chi$ is then the desired bijection $\mathit{\Delta}\backslash X_1(F_n)^{\mathit{\Delta}\text{-reg}} \to \mathcal{T}(1^1_{\mathfrak{m}})$. The second bijection above is then $\mathit{\Lambda}_\chi \mapsto c\text{-}\mathrm{Ind}_{\boldsymbol{J}}^G\,\mathit{\Lambda}_\chi$, $G = \mathrm{GL}_n(F)$, as in 1.2 Corollary.

The representation ${}^L\sigma_\chi$, attached to σ_χ by the Langlands correspondence, is *not* π_χ. It is rather

$${}^L\sigma_\chi = \pi_{\chi'},$$

where $\chi' = \omega^{n-1}\chi$ and ω is the unramified character of $F_n^\times$ of order 2.

8.3. A wild lift. We continue in the situation of 8.2. Let Q/F be a finite, totally wildly ramified, field extension. Thus QF_n/Q is unramified of degree n, and we may identify $\mathrm{Gal}(QF_n/Q)$ with $\varDelta$. Composition with the field norm N_{QF_n/F_n} gives a $\varDelta$-isomorphism $X_1(F_n) \to X_1(QF_n)$, and so leads to a canonical bijection

$$\mathrm{b}_{Q/F} : \mathcal{A}_n(F; \mathbf{0}_F) \xrightarrow{\approx} \mathcal{A}_n(Q; \mathbf{0}_Q).$$

This map is readily described in terms of types. Let $\pi \in \mathcal{A}_n(F; \mathbf{0}_F)$ be given by a $\varDelta$-regular character $\chi \in X_1(F_n)$. Thus π is induced by a representation $\varLambda_\chi \in \mathcal{T}(1^1_{\mathfrak{m}})$ as in 8.2. Likewise, $\mathrm{b}_{Q/F}\,\pi$ is induced by a representation $\varLambda_{\chi_Q}$, where $\chi_Q = \chi \circ \mathrm{N}_{Q_n/F_n}$. By definition, the representation $\varLambda_{\chi_Q}$ is determined by its restriction to $Q^\times$ (which is a multiple of $\chi_Q|_{Q^\times}$) and the restriction of its character to the set of $\varDelta$-regular elements ζ of $\boldsymbol{\mu}(Q_n) = \boldsymbol{\mu}(F_n)$. For such an element ζ,

$$\mathrm{tr}\,\varLambda_{\chi_Q}(\zeta) = \mathrm{tr}\,\varLambda_\chi(\zeta^{[Q:F]}).$$

Note here that the field degree $[Q{:}F]$ is a power of p.

We remark that, in the case where Q/F is also cyclic, this map $\mathrm{b}_{Q/F}$ is *base change,* in the sense of [1], [23].

9. Main Theorem

We return to the context of 4.3, to describe more fully the class of extended maximal types attached to an m-simple character in a group $\mathrm{GL}_n(F)$.

9.1. Notation. We establish notation for the rest of the section. Let θ be a non-trivial m-simple character in $G = \mathrm{GL}_n(F)$. In particular, $\theta \in \mathcal{C}(\mathfrak{a}, \beta, \psi)$, for some simple stratum $[\mathfrak{a}, \beta]$ in $A = \mathrm{M}_n(F)$ and a smooth character ψ of F of level one. We now set $P = F[\beta]$, we let B be the A-centralizer of P and put $\mathfrak{b} = \mathfrak{a} \cap B$.

Attached to θ are the groups $\boldsymbol{J}_\theta$, J^0_θ and J^1_θ of 4.1. Since θ is m-simple, we have $\boldsymbol{J}_\theta = P^\times J^0_\theta = P^\times U_{\mathfrak{b}} J^1_\theta$ and $J^1_\theta \cap P^\times U_{\mathfrak{b}} = U^1_{\mathfrak{b}}$, so the inclusion of $U_{\mathfrak{b}}$ in J^0_θ induces an isomorphism $U_{\mathfrak{b}}/U^1_{\mathfrak{b}} \cong J^0_\theta/J^1_\theta$. Since $\mathfrak{b}$ is a maximal $\mathfrak{o}_P$-order, we have an isomorphism

$$U_{\mathfrak{b}}/U^1_{\mathfrak{b}} \cong \mathrm{GL}_s(\Bbbk_P),$$

where $s = n/[P{:}F]$, uniquely determined up to conjugation by an element of $U_{\mathfrak{b}}$. Altogether, we have isomorphisms

$$(*) \qquad J^0_\theta/J^1_\theta \cong U_{\mathfrak{b}}/U^1_{\mathfrak{b}} \cong \mathrm{GL}_s(\Bbbk_P).$$

Let ξ be an irreducible representation of $P^\times U_{\mathfrak{b}}$, trivial on $U^1_{\mathfrak{b}}$. There is then, by $(*)$, a unique irreducible representation ξ_θ of $\boldsymbol{J}_\theta$ such that $\xi_\theta|_{J^1_\theta}$ is trivial and $\xi_\theta|_{P^\times U_{\mathfrak{b}}} \cong \xi$. The equivalence class of ξ_θ then depends on that of ξ, and not on the choice of isomorphism $(*)$.

9.2. Heisenberg representations. We recall a general result from [15] 5.1.

Lemma. *Let ϕ be a simple character in a group $G' = \mathrm{GL}_r(F)$. There exists a unique irreducible representation $\eta(\phi)$ of J^1_ϕ such that $\eta(\phi)|_{H^1_\phi}$ contains ϕ.*

Since J^1_ϕ normalizes ϕ, the restriction of $\eta(\phi)$ to H^1_ϕ is a multiple of ϕ. One may also show that $I_{G'}(\eta(\phi)) = I_{G'}(\phi)$.

We now revert to the notation of 9.1. The following lies rather deeper, and is proved in [13] 3.2, [15] 5.2.

Proposition 1. *Let θ be an m-simple character in $G = \mathrm{GL}_n(F)$. There exists a representation κ of $\boldsymbol{J}_\theta$ such that $\kappa|_{J^1_\theta} \cong \eta(\theta)$ and $I_G(\kappa) = I_G(\theta)$.*

We denote by $\mathcal{H}(\theta)$ the set of equivalence classes of representations κ of $\boldsymbol{J}_\theta$ satisfying the conditions of the proposition.

We elucidate the structure of the space $\mathcal{H}(\theta)$. Let $X_1(\theta)$ be the group of characters ξ of $\boldsymbol{J}_\theta$ with the following properties:

(1) *ξ is trivial on J^1_θ, and*
(2) *ξ is intertwined by every element of $I_G(\theta)$.*

If $\xi \in X_1(\theta)$ and $\kappa \in \mathcal{H}(\theta)$, then surely $\xi \otimes \kappa \in \mathcal{H}(\theta)$. In this manner *loc. cit.*

Proposition 2. *The set $\mathcal{H}(\theta)$ is a principal homogeneous space over the abelian group $X_1(\theta)$.*

The group $X_1(\theta)$ is easy to describe. Let $X_1(P)$ denote the group of tamely ramified characters of $P^\times$ and $X_0(P)_s$ the subgroup of unramified characters ν such that $\nu^s = 1$. Let $\chi \in X_1(P)$ and let $\det_P : B^\times \to P^\times$ be the determinant map. Thus $\chi_B = \chi\circ\det_P|_{P^\times U_\mathfrak{b}}$ provides a character $P^\times U_\mathfrak{b}$, from which we may form the character $(\chi_B)_\theta$ of $\boldsymbol{J}_\theta$. It is easy to see that $(\chi_B)_\theta$ lies in $X_1(\theta)$ and that the map $\chi\mapsto(\chi_B)_\theta$ gives an isomorphism

$$X_1(P)/X_0(P)_s \xrightarrow{\approx} X_1(\theta).$$

It is sometimes better to view this slightly differently. Let T/F be the maximal tamely ramified sub-extension of P/F. Composition with the field norm $\mathrm{N}_{P/T}$ induces an isomorphism $X_1(T)/X_0(T)_s \to X_1(P)/X_0(P)_s$ and so:

Corollary. *The space $\mathcal{H}(\theta)$ is a principal homogeneous space over the group $X_1(T)/X_0(T)_s$.*

We recall that any two choices of the field T are canonically F-isomorphic, indeed J^1_θ-conjugate. The actions of the groups $X_1(T)$ on $\mathcal{H}(\theta)$ are then related by this conjugation.

9.3. The tensor decomposition. We continue in the same situation. The set $\mathcal{T}(1^1_\mathfrak{b})$ consists of classes of irreducible representations λ of $P^\times U_\mathfrak{b}$ such that $\lambda|_{U_\mathfrak{b}}$ is the inflation of an irreducible cuspidal representation of $U_\mathfrak{b}/U^1_\mathfrak{b}\cong \mathrm{GL}_s(\Bbbk_P)$.

Proposition. *Let Θ be the endo-equivalence class of θ. If $\kappa\in\mathcal{H}(\theta)$ and $\lambda\in\mathcal{T}(1^1_\mathfrak{b})$, then $\kappa\otimes\lambda_\theta\in\mathcal{T}(\theta)$. For any $\kappa\in\mathcal{H}(\theta)$, the map*

$$\begin{aligned}\mathcal{T}(1^1_\mathfrak{b}) &\longrightarrow \mathcal{T}(\theta),\\ \lambda &\longmapsto \kappa\otimes\lambda_\theta,\end{aligned}$$

is a bijection. It induces a bijection

$$\Pi^P_\kappa : \mathcal{A}_s(P;\mathbf{0}_P) \xrightarrow{\approx} \mathcal{A}_s(F;\Theta).$$

The first two assertions come from [13] 3.6 and the final one follows from the Classification Theorem of 4.3.

Let T/F be the maximal tamely ramified sub-extension of P/F. In particular, the extension P/T is totally wildly ramified. Taking account of 8.3, we have a bijection

$$\Pi_\kappa : \mathcal{A}_s(T;\mathbf{0}_T) \xrightarrow{\mathrm{b}_{P/T}} \mathcal{A}_s(P;\mathbf{0}_P) \xrightarrow{\Pi^P_\kappa} \mathcal{A}_s(F;\Theta).$$

As the notation indicates, the map $\mathit{\Pi}_\kappa$ does not depend on the choice of parameter field P/T: this follows easily from the remark following the definition of 8.3. Also, the underlying simple character θ determines the tame parameter field T/F uniquely up to a distinguished isomorphism so $\mathit{\Pi}_\kappa$ is essentially independent of the choice of T.

9.4 Main theorem. In the notation of Chapter II, let $\alpha \in \widehat{\mathcal{P}}_F$ and let $s \geqslant 1$ be an integer. Let $\mathit{\Theta} = \Phi_F(\alpha)$. We describe the Langlands correspondence

$$\mathcal{G}_s(F;\alpha) \xrightarrow{\;\approx\;} \mathcal{A}_s(F;\mathit{\Theta}).$$

Let $E = Z_F(\alpha)$, so that $\deg \mathit{\Theta} = [E{:}F]\dim\alpha$, by the Tame Parameter Theorem. Set $n = s\deg\mathit{\Theta}$, and let θ be an m-simple character in $G = \mathrm{GL}_n(F)$ of endo-equivalence class $\mathit{\Theta}$: this determines θ uniquely, up to G-conjugation. We choose a simple stratum $[\mathfrak{a},\beta]$ in $\mathrm{M}_n(F)$ such that $\theta \in \mathcal{C}(\mathfrak{a},\beta)$, and use the notation set up in 9.1. By the Tame Parameter Theorem again, the field E is F-isomorphic to the maximal tamely ramified sub-extension T/F of P/F.

One needs to specify an F-isomorphism here. The simple character θ gives rise to a simple character θ_T over T, as in 6.6. Let $\mathit{\Theta}_T \in \mathcal{E}(T)$ be the endo-equivalence class of θ_T. We choose the isomorphism $E \to T$ to carry $\Phi_E(\alpha)$ to $\mathit{\Theta}_T$. This determines it uniquely. We henceforward use this isomorphism to identify E with T.

Let $\rho \in \mathcal{G}_1(E;\alpha)$: thus $\rho \in \widehat{\mathcal{W}}_E$ and $\rho|_{\mathcal{P}_F} \cong \alpha$. Using the proposition of 7.1, any $\sigma \in \mathcal{G}_s(F;\alpha)$ is of the form $\mathit{\Sigma}_\rho(\tau)$, for a uniquely determined representation $\tau \in \mathcal{G}_s(E;\mathbf{1}_E)$. In particular, ${}^L\tau \in \mathcal{A}_s(E;\mathbf{0}_E)$.

Main Theorem. *Let $\rho \in \mathcal{G}_1(E;\alpha)$. There exists a unique $\kappa = \kappa_\rho \in \mathcal{H}(\theta)$ such that*

$${}^L\mathit{\Sigma}_\rho(\tau) = \mathit{\Pi}_\kappa({}^L\tau), \quad \tau \in \mathcal{G}_s(E;\mathbf{1}_E).$$

The map

$$\begin{aligned} \mathcal{G}_1(E;\alpha) &\longrightarrow \mathcal{H}(\theta), \\ \rho &\longmapsto \kappa_\rho \end{aligned}$$

is an isomorphism of $X_1(E)$-spaces.

This summarizes the main results of [13], especially 7.3 and 7.6.

9.5. Comments.

9.5.1. In the special case $E = F$, $s = 1$, the theorem says essentially nothing. Each of the sets $\mathcal{G}_1(F;\alpha)$, $\mathcal{A}_1(F;\varTheta)$ is a principal homogeneous space over the abelian group $X_1(F)$ of tamely ramified characters of $F^\times$. The Langlands correspondence provides an $X_1(F)$-bijection $\mathcal{G}_1(F;\alpha) \to \mathcal{A}_1(F;\varTheta)$. Any $X_1(F)$-map $\mathcal{G}_1(F;\alpha) \to \mathcal{A}_1(F;\varTheta)$ is therefore bijective, and differs from the Langlands correspondence by a constant $X_1(F)$-translation.

9.5.2. We return to the general case. Let E_s/E be unramified of degree s and set $\varDelta = \mathrm{Gal}(E_s/E)$. Write $\rho_s = \rho|_{\mathcal{W}_{E_s}}$ and $\pi_s = {}^L\rho_s$. In particular, ρ_s is a $\varDelta$-fixed point of $\mathcal{G}_1(E_s;\alpha)$ and likewise $\pi_s \in \mathcal{A}_1(E_s;\varTheta_s)^\varDelta$, where $\varTheta_s = \Phi_{E_s}(\alpha)$. The representation π_s contains an m-simple character θ_s of endo-equivalence class $\varTheta_s$, lifting an m-simple character θ in $\mathrm{GL}_n(F)$ of endo-equivalence class $\varTheta$. The representation π_s contains an extended maximal simple type $\kappa(\rho_s) \in \mathcal{T}(\theta_s) = \mathcal{H}(\theta_s)$ which is fixed by $\varDelta$.

The first step of the proof uses an explicit construction, based on the Glauberman correspondence [16] from the representation theory of finite groups, to produce a canonical map $\mathfrak{i}_{E_s/F} : \mathcal{H}(\theta_s)^\varDelta \to \mathcal{H}(\theta)$. The representation $\kappa(\rho) = \mathfrak{i}_{E_s/F}\kappa(\rho_s)$ is not the representation κ_ρ required by the theorem. However, for a simple reason as in 9.5.1, there exists $\mu_\rho \in X_1(E_s)^\varDelta$ such that

$$\kappa_\rho = \kappa(\mu_\rho \otimes \rho),$$

for all $\rho \in \mathcal{G}_1(E;\alpha)$. The main labour of the proof is in showing that μ_ρ is *independent of* ρ, and so depends only on s, α and the base field F. We therefore denote it $\mu = \mu^F_{s,\alpha}$.

9.5.3. In the *essentially tame* case, where $\dim\alpha = 1$, the character $\mu^F_{s,\alpha}$ is worked out fully in [7], [8] and [10]. In the general case $\dim\alpha \geqslant 1$, it is constructed as a product following a certain structure tower for the field extension E_s/F:

$$E_s \supset K_0 \supset K_1 \supset \cdots \supset K_r \supset F.$$

Here, K_r/F is unramified and E_s/K_r is totally tamely ramified. Each K_i/K_{i+1}, $0 \leqslant i \leqslant r-1$, is cyclic of prime degree, while E_s has trivial K_0-automorphism group. This yields a decomposition

$$\mu^F_{s,\alpha} = \mu^{E_s/K_0}_\alpha \cdot \mu^{K_0/K_1}_\alpha \cdot \ \ldots \ \cdot \mu^{K_{r-1}/K_r}_\alpha \cdot \mu^{K_r/F}_\alpha.$$

The unramified contribution $\mu_\alpha^{K_r/F}$ can be worked out completely, in terms of simple combinatorial invariants. It has order $\leqslant 2$ but may be ramified [13] 10.7. At the other end, $\mu_\alpha^{E_s/K_0}$ is somewhat mysterious: it is unramified of order dividing $2[E_s{:}K_0]\dim\alpha$. The remaining factors are given by various explicit formulæ involving transfer factors and certain constants derived from automorphic induction.

9.5.4. The proof of the Main Theorem exposes a structure of some independent interest. If $\alpha \in \widehat{\mathcal{P}}_F$ and $E = Z_F(\alpha)$, the set $\mathcal{G}_s(F;\alpha)$ carries a natural action of the group $X_1(E)$. Let $\sigma \in \mathcal{G}_s(F;\alpha)$, and write $\sigma = \Sigma_\rho(\tau)$, for some $\rho \in \mathcal{G}_1(E;\alpha)$, $\tau \in \mathcal{G}_s(E;\mathbf{1}_F)$. We set

$$\chi \odot_\alpha \sigma = \Sigma_\rho(\chi\otimes\tau), \quad \chi \in X_1(E).$$

On the other side, let θ be an m-simple character in $G = \mathrm{GL}_n(F)$, say $\theta \in \mathcal{C}(\mathfrak{a},\beta)$. Set $P = F[\beta]$, $s = n/[P{:}F]$ and let Θ be the endo-equivalence class of θ. Let T/F be the maximal tamely ramified sub-extension of P/F. Let $\pi \in \mathcal{A}_s(F;\Theta)$. Thus $\pi = \Pi_\kappa(\xi)$, for $\kappa \in \mathcal{H}(\theta)$ and $\xi \in \mathcal{A}_s(T;\mathbf{0}_T)$. For $\chi \in X_1(T)$, we set

$$\chi \odot_T \pi = \Pi_\kappa(\chi\xi).$$

In the case $\Theta = \Phi_F(\alpha)$, the F-isomorphism $E \cong T$ chosen in 9.4 yields

$${}^L(\chi \odot_\alpha \sigma) = \chi \odot_E {}^L\sigma, \quad \chi \in X_1(E),\ \sigma \in \mathcal{G}_s(F;\alpha).$$

This $\odot_T$-action may be defined more transparently via extended maximal simple types, in the manner of 9.2.

9.5.5. The version of the Langlands correspondence given by the Main Theorem is well-adapted to describing congruence behaviour, modulo a prime number $l \neq p$. See [14] for a simple treatment of this topic.

9.6. Local constant comparisons. We recall briefly a different method with some claim to effectiveness. It works more generally, but we shall consider only the most interesting case of totally ramified representations.

Let ψ be a non-trivial smooth character of F and s a complex variable. For $\pi_1,\pi_2 \in \widehat{\mathrm{GL}}_F$, let $\varepsilon(\pi_1\times\pi_2,s,\psi)$ be the local constant of [24], [32]. Likewise, for $\sigma \in \widehat{\mathcal{W}}_F$, let $\varepsilon(\sigma,s,\psi)$ be the Langlands-Deligne local constant.

Let $n > 1$. Under the standard characterization [20], if $\sigma \in \mathcal{G}_n(F)$ and $\pi \in \mathcal{A}_n(F)$, then $\pi = {}^L\sigma$ if and only if

$$\varepsilon(\tau \otimes \sigma, s, \psi) = \varepsilon({}^L\tau \times \pi, s, \psi),$$

for all $\tau \in \widehat{\mathcal{W}}_F$ such that $\dim \tau < n$. We use the Tame Parameter Theorem to refine this criterion.

We rely on a result from [3], as follows. Let $c(\psi)$ be the greatest integer k such that $\mathfrak{p}_F^{-k} \subset \operatorname{Ker} \psi$. For $\pi_i \in \mathcal{A}_{n_i}(F)$, $i = 1, 2$, the local constant takes the form

$$\varepsilon(\pi_1 \times \pi_2, s, \psi) = q^{-s(a(\pi_1 \times \pi_2) + n_1 n_2 c(\psi))}\, \varepsilon(\pi_1 \times \pi_2, 0, \psi),$$

where $q = |\Bbbk_F|$ and $a(\pi_1 \times \pi_2)$ is an integer independent of ψ. In particular, if χ is an unramified character of $F^\times$, then

$$\begin{aligned} \varepsilon(\chi\pi_1 \times \pi_2, s, \psi) &= \varepsilon(\pi_1 \times \chi\pi_2, s, \psi) \\ &= \chi(\varpi)^{a(\pi_1 \times \pi_2) + n_1 n_2 c(\psi)}\, \varepsilon(\pi_1 \times \pi_2, s, \psi), \end{aligned}$$

where ϖ is a prime element of F.

For $\pi \in \widehat{\mathrm{GL}}_F$, let $d(\pi)$ be the number of unramified characters χ of $F^\times$ for which $\chi\pi \cong \pi$. We say that π is *totally ramified* if $d(\pi) = 1$. Similarly for representations $\sigma \in \widehat{\mathcal{W}}_F$.

From [3], we obtain:

Lemma. *Let $\pi \in \mathcal{A}_n(F)$ be totally ramified, and let l be a prime divisor of n. There exists a positive divisor n_l of n/l and a totally ramified representation $\pi_l \in \mathcal{A}_{n_l}(F)$ such that $a(\pi_l \times \pi)$ is not divisible by l.*

The defining property of π_l depends only on the endo-equivalence class $\vartheta(\pi_l)$. One can construct an endo-equivalence class, with the desired properties, directly from $\vartheta(\pi)$.

We return to our usual situation with $\alpha \in \widehat{\mathcal{P}}_F$, $E = Z_F(\alpha)$, but we now assume E/F is totally ramified. Set $\Theta = \Phi_F(\alpha)$. If $\pi \in \mathcal{A}_1(F; \Theta)$, then π is totally ramified. We now obtain:

Theorem. *Let S be the set of prime divisors of $[E{:}F]\dim\alpha$. There is a subset $\{\sigma_l : l \in S\}$ of $\widehat{\mathcal{W}}_F$ with the following properties.*

(1) *σ_l is totally ramified and $\dim\sigma_l$ divides n/l.*

(2) *Let $\sigma \in \mathcal{G}_1(F;\alpha)$, $\pi \in \mathcal{A}_1(F;\Theta)$ and suppose that $\det\sigma = \omega_\pi$, the central character of π. The following are equivalent:*

(a) $\pi = {}^L\sigma$;

(b) $\varepsilon(\sigma_l \otimes \sigma, s, \psi) = \varepsilon({}^L\sigma_l \times \pi, s, \psi)$, *for all* $l \in S$.

The hypothesis in (2) implies that ${}^L\sigma = \chi\pi$, where χ is unramified of order dividing n. The theorem follows on taking π_l as in the lemma and defining σ_l by ${}^L\sigma_l = \pi_l$.

For an application of this result, see [4].

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