Geometry of the fundamental lemma

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Contents

1 Some words about the fundamental lemma.

1.1. Origin. — The global Langlands correspondence predicts deep relations between automorphic forms on a reductive group over a number field and representations of the absolute Galois group of this number field. It implies the Langlands functoriality, a family of transfers between automorphic representations on different groups. A general strategy to attack Langlands functoriality is still missing. However, in some specific but basic cases, the *endoscopic* ones, Langlands suggested to use the Arthur-Selberg trace formula. Roughly speaking, the trace formula relates the trace of a test function on the automorphic spectrum to more geometric distributions, namely the (global) orbital integrals attached to rational conjugacy class. The point is that there exists a explicit transfer of (regular semisimple) conjugacy classes from an endoscopic group to the group itself and this transfer is expected to be dual to the transfer of automorphic forms. In particular, there should be deep relations between orbital integrals on a group and on its endoscopic groups. Conversely, if one knows these relations, one should get, by the trace formula, character identities between automorphic forms which should characterize the automorphic transfer. The global orbital integrals are products of local ones and we expect also relations between local orbital integrals. The simplest relation and the most important one is the fundamental lemma, a combinatorial identity between orbital integrals for the units of Hecke algebras. It appears in the works [18] of Labesse-Langlands and [19] of Langlands, and it is stated in general in the work [20] of Langlands-Shelstad.

1.2. Geometry and cohomology. — Ngô proved the fundamental lemma in [24]. More precisely, Ngô proved a variant of this statement for Lie algebra over local fields of positive characteristic. It is known by the work of Waldspurger that it suffices to prove this variant (cf. [26],[27], and cf. [10] for different methods). The advantage of the latter situation is that orbital integrals (both local and global) then have a geometric meaning : they count the number of rational points of some varieties over finite fields. By Grothendieck-Lefschetz trace formula, the fundamental lemma admits a cohomological interpretation. Ngô indeed proves the fundamental lemma through a deep cohomological study of the elliptic part of the Hitchin fibration.

1.3. The fundamental lemma for $GL(n)$. — In the rest of the paper, we will focus only on the group $GL(n)$. In this case, the fundamental lemma is a tautological statement. Nonetheless, Ngô's main cohomological theorem is still very deep in this situation. Moreover, his geometric and cohomological arguments become easier to understand. We will also give some words about the extension of Ngô's work by Laumon and myself (cf. [7] and [8]) which is the key of our proof of the *weighted* fundamental lemma (a generalized form of the fundamental lemma stated by Arthur which is also needed in the endoscopic program).

Let's describe quickly how the paper is organized. We begin in section 2 by the geometric interpretation of local orbital integrals through affine Springer fibers. Then we introduce Hitchin's fibration in section 3. We give a geometric description of Hitchin fibers in section 4 using the Hitchin-Beauville-Narasimhan-Ramanan correspondence. Then in section 5 we study an example of a non-separated Hitchin fiber. In section 6, we explain the truncation of the Hitchin fibration we used in [7]. In the more technical section 7, we state and explain Ngô's main cohomological theorem and its extension. We discuss at length some good open subset of the base of the Hitchin fibration. We hope that this makes the constructions of section 9 of [8] more accessible.

1.4. Acknowledgement. — This expository article is largely based on the one hand on the papers [23] and [24] of Ngô Bao Châu and on the other hand on my work [7] and [8] with Gérard Laumon on the weighted fundamental lemma. This text takes also great profit of many talks Laumon or me gave on this subject. I thank Gérard Laumon for having shared with me some of his notes and for his help for the figures. Finally I would like to thank the referee for carefully reading the article.

2 Local orbital integrals and their geometric interpretation.

2.1. Notations. — Let $\mathbb{F}_q$ be a finite field with q elements. Let n be an integer. We assume that the characteristic of $\mathbb{F}_q$ satisfies the following inequality

$$\mathrm{char}(\mathbb{F}_q) > n.$$

Let

$$G = GL(n)$$

over $\mathbb{F}_q$ and let $\mathfrak{g} = \mathfrak{gl}(n)$ be its Lie algebra. The adjoint action of G on $\mathfrak{g}$ is simply the action by conjugation of $GL(n)$ on $\mathfrak{gl}(n)$.

Let

$$\mathcal{O} = \mathbb{F}_q[[\varepsilon]]$$

be the ring of power series with coefficients in $\mathbb{F}_q$ and let F be its fraction field. So

$$F = \mathbb{F}_q((\varepsilon))$$

is the field of Laurent power series with coefficients in $\mathbb{F}_q$.

We will frequently use left quotients denoted by $\backslash$.

2.2. Orbital integrals. — Let $\gamma \in \mathfrak{g}(F)$. We assume that γ is *regular semisimple* which means that its characteristic polynomial has n distinct roots in a suitable extension of F. The centralizer of γ in G (by abuse of notations, we will not distinguish G and $G \times_{\mathbb{F}_q} F$) is then a F-maximal subtorus of G. It is denoted by T_γ or simply T.

Let

$$\mathbf{1}_{\mathfrak{g}(\mathcal{O})}$$

be the characteristic function of $\mathfrak{g}(\mathcal{O})$. We can attach to the element γ and the function $\mathbf{1}_{\mathfrak{g}(\mathcal{O})}$ the following *orbital integral*

$$\mathcal{O}_\gamma = \int_{T(F)\backslash G(F)} \mathbf{1}_{\mathfrak{g}(\mathcal{O})}(g^{-1}\gamma g)\, dg$$

where the measure dg is the quotient of Haar measures on $G(F)$ and $T(F)$ respectively normalized by

$$\mathrm{vol}(G(\mathcal{O})) = 1$$

and

$$\mathrm{vol}(X_*(T)\backslash T(F)) = 1$$

where $X_*(T)$ is the group of F-rational cocharacters of T. Here the choice of the uniformizer ε gives an injective morphism

$$\begin{array}{lcl} X_*(T) & \to & T(F) \\ \lambda & \mapsto & \lambda(\varepsilon) \end{array}$$

so that we can view the group $X_*(T)$ as a subgroup of $T(F)$.

Remark 2.2.1. — The orbital integral is finite : since the function $\mathbf{1}_{\mathfrak{g}(\mathcal{O})}$ is compactly supported and the orbit of γ is closed, the integral can be taken over a compact subset of $T(F)\backslash G(F)$.

The integral $\mathcal{O}_\gamma$ is the Lie algebra analog of an orbital integral attached to the unit function of the spherical algebra of $G(F)$.

2.3. Orbital integral as a counting function. — A subset $\mathcal{L} \subset F^n$ is called a lattice if it is a sub-$\mathcal{O}$-module free of rank n. For example, $\mathcal{L}_0 = \mathcal{O}^n$ is a such a lattice and it will called "standard" in the sequel. Let us denote by $\mathfrak{X}$ the set of lattices.

The group $G(F)$ acts on F^n and thus acts on the set $\mathfrak{X}$ of lattices. It is easy to see that the action is transitive. Moreover the stabilizer of the standard lattice is $G(\mathcal{O})$.

For any endomorphism $\gamma \in \mathfrak{g}(F)$ of F^n, we say that a lattice is γ-stable if $\gamma(\mathcal{L}) \subset \mathcal{L}$. Let

$$\mathfrak{X}_\gamma := \{\mathcal{L} \in \mathfrak{X} \mid \gamma(\mathcal{L}) \subset \mathcal{L}\}$$

be the set of γ-stable lattices. For example, the standard lattice $\mathcal{L}_0$ is γ-stable if and only if $\gamma \in \mathfrak{g}(\mathcal{O})$. If $g \in G(F)$ and $\mathcal{L} = g \cdot \mathcal{L}_0$ then $\mathcal{L}$ is γ-stable if and only if $\mathcal{L}_0$ is $g^{-1}\gamma g$-stable that is if and only if $g^{-1}\gamma g \in \mathfrak{g}(\mathcal{O})$.

Proposition 2.3.1. — *The map $g \mapsto g\mathcal{O}^n$ gives bijections*
- $G(F)/G(\mathcal{O}) \simeq \mathfrak{X}$
- $\{g \in G(F)/G(\mathcal{O}) \mid g^{-1}\gamma g \in \mathfrak{g}(\mathcal{O})\} \simeq \mathfrak{X}_\gamma$.

From now on, let us assume that $\gamma \in \mathfrak{g}(F)$ is *regular semisimple*. Remember we have defined the orbital integral $\mathcal{O}_\gamma$.

Proposition 2.3.2. — *The group $X_*(T)$ acts on $\mathfrak{X}_\gamma$ without fixed points and*

$$\mathcal{O}_\gamma = |X_*(T)\backslash\mathfrak{X}_\gamma|$$

Proof. — The subgroup $T(F)$ of $G(F)$ acts also on $\mathfrak{X}$. Since any element of $T(F)$ commutes with γ, the action preserves $\mathfrak{X}_\gamma$. By restriction, we get an action of the subgroup $X_*(T)$. The stabilizer of a lattice in $X_*(T)$ is a compact discrete subgroup thus a finite group : it must be trivial since the group $X_*(T)$ is a free $\mathbb{Z}$-module. Hence the action of $X_*(T)$ does not have fixed points. For the remaining equality, using our normalization of measures, we can write

$$\begin{aligned} \mathcal{O}_\gamma &= \int_{X_*(T)\backslash G(F)} \mathbf{1}_{\mathfrak{g}(\mathcal{O})}(g^{-1}\gamma g)\, dg \\ &= \sum_{g \in X_*(T)\backslash G(F)/G(\mathcal{O})} \int_{H_g\backslash G(\mathcal{O})} \mathbf{1}_{\mathfrak{g}(\mathcal{O})}((gk)^{-1}\gamma gk)\, dk \end{aligned}$$

where $H_g = G(\mathcal{O}) \cap g^{-1}X_*(T)g$ is a torsion-free, discrete and compact group and hence must be trivial. Moreover, using $\mathrm{vol}(G(\mathcal{O})) = 1$ and the fact that the map $g \mapsto \mathbf{1}_{\mathfrak{g}(\mathcal{O})}(g^{-1}\gamma g)$ is clearly invariant on the left by $G(\mathcal{O})$, we get

$$\begin{aligned}\mathcal{O}_\gamma &= \sum_{g\in X_*(T)\backslash G(F)/G(\mathcal{O})} \int_{G(\mathcal{O})} \mathbf{1}_{\mathfrak{g}(\mathcal{O})}(((gk)^{-1}\gamma gk) \\ &= \sum_{g\in X_*(T)\backslash G(F)/G(\mathcal{O})} \mathbf{1}_{\mathfrak{g}(\mathcal{O})}(g^{-1}\gamma g) \\ &= |X_*(T)\backslash \mathfrak{X}_\gamma|.\end{aligned}$$

□

2.4. Affine Grassmannian. — We would like to view the set $\mathfrak{X}$ of lattices as the set of $\mathbb{F}_q$-points of an algebraic variety. This is possible for each set $\mathfrak{X}^{i,j}$ for integers $i \geqslant 0$ and $j \in \mathbb{Z}$ where

$$\mathfrak{X}^{i,j} = \{\mathcal{L} \in \mathfrak{X} \mid \varepsilon^i\mathcal{O}^n \subset \mathcal{L} \subset \varepsilon^{-i}\mathcal{O}^n \text{ and } \wedge^n \mathcal{L} = \varepsilon^j\mathcal{O}\}.$$

Here $\wedge^n\mathcal{L}$ is the maximal exterior power of $\mathcal{L}$: this is a fractional ideal of $\mathcal{O}$. Let V_i be the left quotient $\varepsilon^i\mathcal{O}^n\backslash\varepsilon^{-i}\mathcal{O}^n$: it is a $\mathbb{F}_q$-vector space. Note that the uniformizer induces on V_i a nilpotent endomorphism (still denoted by ε) which satisfies $\varepsilon^{2i} = 0$. The map

$$\mathcal{L} \to V_{\mathcal{L}} = \varepsilon^i\mathcal{O}^n\backslash\mathcal{L}$$

induces a bijection of $\mathfrak{X}^{i,j}$ onto the set

$$\{W \subset V_i \mid \dim(W) = ni - j \text{ and } \varepsilon(W) \subset W\}.$$

of subspaces of V_i of dimension $ni - j$ which are stable under the nilpotent endomorphism ε. This latter set is the set of $\mathbb{F}_q$-points of a projective variety. Indeed, we can consider the Grassmanniann of linear subspaces W of dimension $ni - j$ in V (this is a projective variety) and inside the Grassmannian, the condition $\varepsilon(W) \subset W$ defines a closed subvariety which is classically known as a Springer fiber. But we have more : the inclusion $\mathfrak{X}^{i,j} \subset \mathfrak{X}^{i+1,j}$ gives a closed immersion of the corresponding Springer fibers. Thus the set

$$\mathfrak{X}^j = \{\mathcal{L} \in \mathfrak{X} \mid \wedge^n\mathcal{L} = \varepsilon^j\mathcal{O}\}$$

is the set of $\mathbb{F}_q$-points of an ind-variety. Finally, $\mathfrak{X}$ is the set of $\mathbb{F}_q$-points of the disjoint union (over $j \in \mathbb{Z}$) of the ind-varieties corresponding to $\mathfrak{X}^j$. This ind-variety is called the affine Grassmannian.

In the sequel, we change slightly our notations : $\mathfrak{X}$ denotes the affine Grassmannian. The set of $\mathbb{F}_q$-points is denoted by $\mathfrak{X}(\mathbb{F}_q)$.

2.5. Affine Springer fiber. — Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element. The condition of being γ-stable defines a closed ind-subvariety still denoted $\mathfrak{X}_\gamma \subset \mathfrak{X}$. The variety $\mathfrak{X}_\gamma$ was introduced by Kazhdan and Lusztig in [17] and it is by now called an affine Springer fiber.

Theorem 2.5.1. —*(Kazhdan-Lusztig, cf. [17])*
- *The reduced ind-scheme of $\mathfrak{X}_\gamma$ is represented by a variety, locally of finite type and of finite dimension.*
- *The quotient $X_*(T)\backslash\mathfrak{X}_\gamma$ is a projective variety (where T is the centralizer of γ in G)*

Remark 2.5.2. — A formula for the dimension of $\mathfrak{X}_\gamma$ was stated by Kazhdan-Lusztig and proved by Bezrukavnikov in [5].

2.6. An example of affine Springer fiber. — Take $G = GL(2)$ and $\gamma = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}$. The set of connected components of the affine Springer fiber $\mathfrak{X}_\gamma$ is indexed by $\mathbb{Z}$. For $j \in \mathbb{Z}$, the connected component $\mathfrak{X}^j_\gamma$ is such that $\mathfrak{X}^j_\gamma(\mathbb{F}_q)$ is the set of lattices $\mathcal{L}$ in F^2 which are γ-stable and of index $\wedge^2\mathcal{L} = \varepsilon^j\mathcal{O}$. Let $p_1 : F^2 \to F$ (resp. p_2) be the first (resp. second) projection. For any lattice $\mathcal{L} \subset F^2$, let us consider the free $\mathcal{O}$-sub-modules of F defined by $\mathcal{L}_1 = p_1(\mathcal{L})$, $\mathcal{L}_2 = p_2(\mathcal{L})$, $\mathcal{L}^1 = \mathcal{L} \cap \mathrm{Ker}(p_2)$ and $\mathcal{L}^2 = \mathcal{L} \cap \mathrm{Ker}(p_1)$. We have $\wedge^2 \mathcal{L} \simeq \mathcal{L}_1 \otimes_{\mathcal{O}} \mathcal{L}^2 \simeq \mathcal{L}_2 \otimes_{\mathcal{O}} \mathcal{L}^1$ and a commutative diagram

$$\begin{tikzcd}
 & \mathcal{L}^1 \arrow[d, hook] \arrow[dr] & & \\
\mathcal{L}^2 \arrow[r, hook] \arrow[dr] & \mathcal{L} \arrow[r, two heads] \arrow[d, two heads] & \mathcal{L}_1 \arrow[dr] & \\
 & \mathcal{L}_2 \arrow[dr] & & \mathcal{L}^1 \backslash \mathcal{L}_1 \arrow[dl, "\psi"] \\
 & & \mathcal{L}^2 \backslash \mathcal{L}_2 &
\end{tikzcd}$$

where the arrow ψ is an isomorphism of $\mathcal{O}$-module defined by $\psi(x_1) = x_2$ for any $x_1 \in \mathcal{L}_1$ and $x_2 \in \mathcal{L}_2$ such that $x_1 + x_2 \in \mathcal{L}$. If moreover the lattice $\mathcal{L}$ is γ-stable, the morphism ψ is also γ-equivariant for the actions induced by γ, namely multiplication by ε on $\mathcal{L}^1 \backslash \mathcal{L}_1$, resp. $-\varepsilon$ on $\mathcal{L}^2 \backslash \mathcal{L}_2$. It follows that $\mathcal{L}^1 \backslash \mathcal{L}_1$ is killed by ε. Let $i \in \mathbb{Z}$ be such that $\mathcal{L}^1 = \varepsilon^i \mathcal{O}$. Since we have

$$\mathcal{L}^1 \oplus \mathcal{L}^2 \subset \mathcal{L} \subset \mathcal{L}_1 \oplus \mathcal{L}_2,$$

we also have

$$\varepsilon^i \mathcal{O} \oplus \varepsilon^{j-i+1} \mathcal{O} \subset \mathcal{L} \subset \varepsilon^{i-1} \mathcal{O} \oplus \varepsilon^{j-i} \mathcal{O}. \tag{2.6.1}$$

Let $\mathbb{P}_i$ be the set of lattices $\mathcal{L}$ which satisfies (2.6.1) and $\wedge^2 \mathcal{L} = \varepsilon^j \mathcal{O}$. Clearly, $\mathbb{P}_i$ is the set of $\mathbb{F}_q$-points of a projective line. Any lattice $\mathcal{L} \in \mathbb{P}_i$ is γ-stable. The intersection $\mathbb{P}_i \cap \mathbb{P}_{i'}$ is empty unless $i' \in \{i+1, i, i-1\}$. Moreover $\mathbb{P}_i \cap \mathbb{P}_{i+1}$ is reduced to a single split lattice $\varepsilon^i \mathcal{O} \oplus \varepsilon^{j-i} \mathcal{O}$ (which is the point ∞ in $\mathbb{P}_i$ and the point 0 in $\mathbb{P}_{i+1}$).

In this way, we see that $\mathfrak{X}_\gamma^j$ is an infinite chain of projective lines as in picture 1. The black nodes are the split lattices $\varepsilon^i \mathcal{O} \oplus \varepsilon^{j-i} \mathcal{O}$.

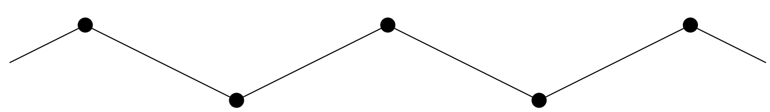

Figure 1
Infinite chain of projective lines

The centralizer of γ in G is a split torus of rank 2. We have $X_*(T) \simeq \mathbb{Z}^2$ and the action of $\mathbb{Z}^2$ on $\mathfrak{X}_\gamma$ permutes the connected components. The stabilizer of a connected component is isomorphic to $\mathbb{Z}$ and acts on it by translation on the chain. The quotient $X_*(T) \backslash \mathfrak{X}_\gamma$ looks like a projective line with a node (cf. figure 2 below).

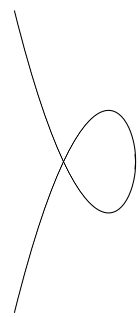

Figure 2
A projective line with a node

2.7. The work of Goresky-Kottwitz-MacPherson. — In some cases, Goresky, Kottwitz and MacPherson were able to compute the cohomology of the affine Springer fiber $\mathfrak{X}_\gamma$. In their work [15], they assume that the regular semisimple element $\gamma \in \mathfrak{g}(F)$ is "equivalued" and unramified. Let us explain their first hypothesis "equivalued". Technically this essentially means the following : over an algebraic closure, the element γ is conjugated to a diagonal matrix $diag(\lambda_1, \ldots, \lambda_n)$ and there exists $r \in \mathbb{Q}$ such that $\mathrm{val}(\lambda_i - \lambda_j) = r$ for any $i \neq j$. The point is that they can prove that the cohomology of affine Springer fibers $\mathfrak{X}_\gamma$ associated to equivalued elements γ is pure (cf. [16]). A first consequence of the purity is that it is possible to deduce the cohomology from the equivariant cohomology for some torus action. Here enters their second hypothesis "unramified". Technically γ is unramified means that γ can be conjugated to a diagonal matrix on the maximal unramified extension of F. In particular, its centralizer is a split torus of rank n over such an extension. From a geometric point of view, this implies that, after extension of scalars to an algebraic closure k of $\mathbb{F}_q$, a torus of rank n over k acts on $\mathfrak{X}_\gamma$. Moreover, there's a combinatorial way to get the equivariant cohomology of $\mathfrak{X}_\gamma$ for this action from the knowledge of orbits of dimension less or equal to one.

We have seen that orbital integrals count the number of rational points of quotients of affine Springer fibers. Thanks to the Grothendieck-Lefschetz fixed point formula, this gives a cohomological interpretation to orbital integrals and also to the fundamental lemma. From their computation of the cohomology of affine Springer fibers of reductive groups (still for "equivalued" and unramified elements), Goresky, Kottwitz and MacPherson proved the fundamental in these cases. In fact, it is conjectured (but until now not known) that the cohomology of affine Springer fibers is always pure (for some recent progress on this purity problem see [9]). So their hypothesis of equivaluation could be removed. Nonetheless their use of equivariant cohomology needs a "big" torus action which appears only in the unramified case.

3 Hitchin fibration.

3.1. Notation. — Let $k = \overline{\mathbb{F}_q}$ be an algebraic closure of the finite field $\mathbb{F}_q$. Let C be a connected, smooth, projective curve with over $k = \overline{\mathbb{F}_q}$ of genus g_C. Let $D = 2D'$ be an even and effective divisor on C. We assume that

$$\deg(D) > 2g_C.$$

Let n be an integer such that $\mathrm{char}(\mathbb{F}_q) > n$.

3.2. Hitchin bundles. — A *Hitchin bundle* is a pair $(\mathcal{E}, \theta)$ where

– $\mathcal{E}$ is a vector bundle on C of rank n and degree 0

– $\theta : \mathcal{E} \to \mathcal{E}(D) = \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{O}_C(D)$ is a homomorphism of $\mathcal{O}_C$-modules.

Remark 3.2.1. — When D is a canonical divisor (which we do not assume), a Hitchin pair is called a Higgs bundle in the literature. The setting we consider appears in Ngô's paper [23].

3.3. Characteristic polynomial. — Let $(\mathcal{E}, \theta)$ be a Hitchin bundle. The trace of θ is the section of $\mathcal{O}(D)$ defined by

$$\mathrm{trace}(\theta) : \mathcal{O}_C \overset{id}{\to} \mathcal{E}nd(\mathcal{E}) \overset{\theta}{\to} \mathcal{O}_C(D) \in H^0(C, \mathcal{O}_C(D)),$$

where the first arrow is the identity section and the second one is θ viewed as an element of $\mathrm{Hom}(\mathcal{E}nd(\mathcal{E}), \mathcal{O}_C(D))$. For any $1 \leqslant i \leqslant n$, we also get sections

$$\mathrm{trace}(\wedge^i \theta) \in H^0(C, \mathcal{O}_C(iD))$$

and the *characteristic polynomial* of θ is the polynomial

$$\chi_\theta(X) = X^n - \mathrm{trace}(\theta) X^{n-1} + \ldots + (-1)^n \,\mathrm{trace}(\wedge^n \theta)$$

3.4. Hitchin fibration. — Let $\mathbb{M}$ be the *algebraic k-stack* which classifies Hitchin bundles $(\mathcal{E}, \theta)$.

Let $\mathbb{A}$ be the *affine space* of characteristic polynomials

$$X^n - a_1 X^{n-1} + \ldots + (-1)^n a_n$$

with $a_i \in H^0(C, \mathcal{O}_C(iD))$. By Riemann-Roch theorem, it is easy to compute its dimension

$$\begin{aligned} \dim_k(\mathbb{A}) &= \sum_{i=1}^n \dim_k(H^0(C, \mathcal{O}_C(iD))) \\ &= \sum_{i=1}^n (1 - g_C + i \deg(D)) \\ &= \frac{n(n+1)}{2} \deg(D) + n(1 - g_C) \end{aligned} \tag{3.4.1}$$

The *Hitchin fibration* is the morphism

$$f : \mathbb{M} \to \mathbb{A}$$

defined by

$$f(\mathcal{E}, \theta) = \chi_\theta$$

3.5. Adelic description of Hitchin fibers. —Let F be the function field of C. Let $|C|$ be the set of closed points of C. For any $c \in |C|$, let $\mathcal{O}_c$ be the completion of the local ring at c and let F_c be the fraction field of $\mathcal{O}_c$. Let

$$\mathbb{A}_F = \varinjlim_{S} \prod_{c \in S} F_c \prod_{c \notin S} \mathcal{O}_c$$

be the ring of adèles of C where the direct limit is taken over finite subsets $S \subset |C|$. The field F is diagonally embedded in $\mathbb{A}_F$. Let

$$\mathcal{O} = \prod_{c \in |C|} \mathcal{O}_c \subset \mathbb{A}_F.$$

To the divisor $D = \sum_{c \in |C|} n_c [c]$, we attach an idèle $\varpi_D = (\varpi_c^{n_c})_c \in \mathbb{A}_F^\times$. We have a degree morphism

$$\deg : \mathbb{A}_F^\times \to \mathbb{Z}$$

which is trivial on $\mathcal{O}^\times$ and $F^\times$. We have

$$\deg(\varpi_D) = \deg(D).$$

Let

$$\chi = X^n - a_1 X^{n-1} + \ldots + (-1)^n a_n$$

be a characteristic polynomial in the Hitchin base $\mathbb{A}(k)$. We can view it as a polynomial with coefficients in F. Let $G = GL(n)$ and $\mathfrak{g} = \mathfrak{gl}(n)$ its Lie algebra. Let $\mathcal{H}_\chi$ be the set of pairs

$$(g, \gamma) \in G(\mathbb{A}_F)/G(\mathcal{O}) \times \mathfrak{g}(F)$$

such that

1. $\deg(\det(g)) = 0$;
2. the characteristic polynomial of γ is χ ;
3. we have the following integral condition

$$g^{-1}\gamma g \in \varpi_D^{-1}\mathfrak{g}(\mathcal{O}).$$

The map

$$\begin{array}{ccc} G(F) \times G(\mathbb{A}_F)/G(\mathcal{O}) \times \mathfrak{g}(F) & \to & G(\mathbb{A}_F)/G(\mathcal{O}) \times \mathfrak{g}(F) \\ (\delta, g, \gamma) & \mapsto & \delta \cdot (g, \gamma) := (\delta g, \delta\gamma\delta^{-1}) \end{array}$$

defines an action on the left of the group $G(F)$ on the set $\mathcal{H}_\chi$. We can form the quotient groupoid $[G(F)\backslash\mathcal{H}_\chi]$. It is a small category where the objects are elements of $\mathcal{H}_\chi$ and for (g, γ) and (g', γ') the set $\mathrm{Hom}((g, \gamma), (g', \gamma'))$ is the set of $\delta \in G(F)$ such that $(g', \gamma') = \delta \cdot (g, \gamma)$. The next proposition gives a description "à la Weil" of a Hitchin fiber.

Proposition 3.5.1. — *The Hitchin fiber $f^{-1}(\chi)(k)$ is equivalent to the quotient groupoid $[G(F)\backslash\mathcal{H}_\chi]$.*

Let us denote

$$\mathbb{A}^{\mathrm{rss}} \tag{3.5.1}$$

the open subset of $\chi \in \mathbb{A}$ which are *square-free* in the ring $F[X]$ of polynomials with coefficients in F. The exponent rss means regular semi-simple since the characteristic polynomial of $\gamma \in \mathfrak{g}(F)$ is square-free if and only γ is regular semi-simple. Let $\chi \in \mathbb{A}^{\mathrm{rss}}$ and $\gamma \in \mathfrak{g}(F)$ with characteristic polynomial $\chi_\gamma = \chi$. Then the centralizer of γ in G is a maximal torus T and the set of $\gamma' \in \mathfrak{g}(F)$ such that $\chi_{\gamma'} = \chi$ is simply the conjugacy class of γ under $G(F)$. That's why the Hitchin fiber $f^{-1}(\chi)(k)$ is also equivalent to the quotient groupoid

$$[T(F)\backslash\mathcal{H}_\gamma]$$

where $\mathcal{H}_\gamma$ is the set of $g \in G(\mathbb{A}_F)/G(\mathcal{O})$ such that $\deg(\det(g)) = 0$ and $g^{-1}\gamma g \in \varpi_D^{-1}\mathfrak{g}(\mathcal{O})$. There is a more suggestive way to write this quotient. Let

$$(\gamma_c)_{c\in|C|} = \varpi_D\gamma \in \mathfrak{g}(\mathbb{A}_F). \tag{3.5.2}$$

The integrality condition 3 for $g = (g_c)_{c\in|C|} \in G(\mathbb{A})/G(\mathcal{O})$ is equivalent to

$$g_c^{-1}\gamma_c g_c \in \mathfrak{g}(\mathcal{O}_c).$$

Thanks to the proposition 2.3.1, such a coset g_c is nothing else but a k-point in the affine Springer fiber $\mathfrak{X}_{\gamma_c}$. The Hitchin fiber $f^{-1}(\chi)(k)$ is thus equivalent to the quotient groupoid

$$[T(F)\backslash \prod_{c\in|C|}{}' \mathfrak{X}_{\gamma_c}(k)]$$

where the restricted product $\prod'$ means that at almost every point c we take the standard lattice. Note that since $T(F)$ centralizes γ it also centralizes γ_c and it does act on $\mathfrak{X}_{\gamma_c}$.

3.6. Counting points of Hitchin fibers. — Suppose in this § that the curve C and the divisor D come from a curve C_0 and a divisor D_0 defined over a finite field $\mathbb{F}_q$. In the same way as in §3.5, we get an adelic description of the category of $\mathbb{F}_q$-points of a Hitchin fiber : one has to replace the objects relative to C, D and k by the analogous objects relative to C_0, D_0 and $\mathbb{F}_q$ which are denoted by a subscript 0.

We can count the number of points : this is precisely the sum over isomorphism classes weighted by the inverse of the order of the group of automorphisms. Like in the proposition 3.6.1, the result is expressed as an orbital integral. Let $\chi \in \mathbb{A}^{\mathrm{rss}}(\mathbb{F}_q)$ and $\gamma \in \mathfrak{g}(F_0)$ such that $\chi_\gamma = \chi$. Let T be the centralizer of γ in G. For any $c \in |C_0|$, the groups $G(F_c)$ and $T(F_c)$ are provided with Haar measures normalized as in §2.2. Let

$$T(\mathbb{A}_0)^0 = \{t \in T(\mathbb{A}_0) \mid \deg(t) = 0\}.$$

The following proposition is the starting observation of Ngô (cf. [23]).

Proposition 3.6.1. — *The number of $\mathbb{F}_q$-points of the Hitchin fiber $f^{-1}(\chi)$ is equal to the following product of orbital integrals*

$$\mathrm{vol}(T(F_0)\backslash T(\mathbb{A}_0)^0) \prod_{c\in|C_0|} \int_{T(F_c)\backslash G(F_c)} \mathbf{1}_{\mathfrak{g}(\mathcal{O}_c)}(g^{-1}\gamma_c g)\, dg_c. \tag{3.6.1}$$

where the element γ_c is defined as in (3.5.2)*, the measure is the quotient of Haar measures on $G(F_c)$ and $T(F_c)$, the function $\mathbf{1}_{\mathfrak{g}(\mathcal{O}_c)}$ is the characteristic function of $\mathfrak{g}(\mathcal{O}_c)$. The expression* (3.6.1) *is finite if and only if the Hitchin fiber $f^{-1}(\chi)$ has finitely many $\mathbb{F}_q$-points.*

Remark 3.6.2. — For almost every $c \in |C_0|$, the element γ_c belongs to $\mathfrak{g}(\mathcal{O}_c)$ and moreover its image in the residue field is still regular semisimple. This implies that the orbital integral

$$\int_{T(F_c)\backslash G(F_c)} \mathbf{1}_{\mathfrak{g}(\mathcal{O}_c)}(g^{-1}\gamma_c g)\, dg_c$$

is 1. At the other places, the orbital integral is finite (since the conjugacy class of a semisimple element is closed, the integral is in fact taken over a compact subset of $T(F_c)\backslash G(F_c)$). Thus the product is finite. This is not always the case for the volume $\mathrm{vol}(T(F_0)\backslash T(\mathbb{A}_0)^0)$. In general, one can write

$$F_0[X]/(\chi) = \prod_{i=1}^r E_i$$

as a product of finite extensions E_i of F_0. We have

$$T(F_0)\backslash T(\mathbb{A}_0) = \prod_{i=1}^r E_i^\times \backslash \mathbb{A}_{E_i}^\times.$$

The degree morphism gives a surjection of the right hand side onto $\mathbb{Z}^r$. It restricts to a surjection of $T(F_0)\backslash T(\mathbb{A}_0)^0$ onto the sublattice of $(n_1, \dots, n_r) \in \mathbb{Z}^r$ such that $n_1 + \ldots + n_r = 0$. Thus $T(F_0)\backslash T(\mathbb{A}_0)^0$ cannot be of finite volume unless $r = 1$. In this case it is compact. The condition $r = 1$ means that the polynomial χ is irreducible over F_0.

4 Geometric description of Hitchin fibers.

4.1. A slight variant of the Hitchin fibration. — In the following, we will focus on the (open) regular semisimple locus of the Hitchin fibration

$$\mathbb{M}^{\mathrm{rss}} = \mathbb{M} \times_{\mathbb{A}} \mathbb{A}^{\mathrm{rss}}$$

where the open set $\mathbb{A}^{\mathrm{rss}}$ of the base has been defined in (3.5.1). In fact, we are going to introduce some étale open subset of $\mathbb{M}^{\mathrm{rss}}$. For this, we introduce another datum, namely a closed point of C, denoted by ∞. We assume that the point ∞ does not belong to the support of the divisor D. Let

$$\mathbb{A}^{\infty} \subset \mathbb{A}^{\mathrm{rss}}$$

be the open subset of characteristic polynomials $\chi = X^n - a_1 X^{n-1} + \ldots + (-1)^n a_n \in \mathbb{A}$ such that the polynomial

$$\chi_{\infty} = X^n - a_1(\infty) X^{n-1} + \ldots + (-1)^n a_n(\infty) \in k[X]$$

has only simple roots. Let

$$\mathcal{A} \to \mathbb{A}^{\infty}$$

be the étale Galois cover of $\mathbb{A}^{\infty}$ of group the symetric group $\mathfrak{S}_n$ given by

$$\mathcal{A} = \{(\chi, \tau) \in \mathbb{A}^{\infty} \times k^n \mid \chi_{\infty} = \prod_{i=1}^{n} (X - \tau_i)\}.$$

The fiber product $\mathbb{M} \times_{\mathbb{A}} \mathcal{A}$ classifies quadruples $(\mathcal{E}, \theta, \chi, \tau)$ where $(\mathcal{E}, \theta)$ is a Hitchin bundle and $(\chi, \tau) \in \mathcal{A}$ is such that $\chi = f(\mathcal{E}, \theta)$. This implies that the endomorphism θ_{∞} of the k-vector-space $\mathcal{E}_{\infty}$ must has n distinct eigenvalues. In particular, θ_{∞} is regular semisimple and θ is generically regular semisimple.

Let

$$\mathcal{M} \to \mathbb{M} \times_{\mathbb{A}} \mathcal{A}$$

be the $\mathbb{G}_m$-torsor obtained by choosing an eigenvector e_1 in the line $\mathrm{Ker}(\theta_{\infty} - \tau_1 \mathrm{Id}_{\mathcal{E}_{\infty}})$. We get a Hitchin morphism (still denoted by f) by base change

$$f : \mathcal{M} \longrightarrow \mathbb{M} \times_{\mathbb{A}} \mathcal{A} \longrightarrow \mathcal{A}$$

By an argument from deformation theory, we can show the following theorem.

Theorem 4.1.1. —*(Biswas-Ramanan [6], cf. also [24] §4.14). The algebraic stack $\mathcal{M}$ is smooth over k.*

Remark 4.1.2. — The main point in the above constructions is to avoid the singularities of the global nilpotent cone. The additional datum $\tau \in k^n$ will later play more or less the role of a parabolic structure. The datum e_1 is not at all essential but it is convenient here in order to rigidify the situation.

4.2. The spectral curve of Hitchin-Beauville-Narasimhan-Ramanan. — Let

$$\pi_{\Sigma} : \Sigma_D = \mathbf{Spec}\big(\bigoplus_{i=0}^{\infty} \mathcal{O}_C(-iD) X^i\big) \to C.$$

be the total space of the line bundle $\mathcal{O}_C(D)$.

Let $a = (\chi, \tau) \in \mathcal{A}$. We write

$$\chi(X) = X^n - a_1 X^{n-1} + \ldots + (-1)^n a_n$$

with $a_i \in H^0(C, \mathcal{O}_C(iD))$.

The *spectral curve* Y_a (cf. [3]) is the closed curve in Σ_D defined by

$$Y_a = \mathbf{Spec}\big(\bigoplus_{i=0}^{\infty} \mathcal{O}_C(-iD) X^i\big) / \mathcal{I}_a)$$

where the sheaf of ideals $\mathcal{I}_a$ of $\mathcal{O}_{\Sigma}$ is generated by $\mathcal{O}_C(-nD)\chi$. The canonical projection

$$\pi_a : Y_a \to C$$

is a finite cover of degree n, which is *étale over* the point ∞. But we have the datum $\tau = (\tau_1, \ldots, \tau_n)$ which is an ordering on the (simple) roots of χ_∞. Thus we have also an ordering on the fiber of π_a over ∞ :

$$\pi_a^{-1}(\infty) = \{\infty_1, \ldots, \infty_n\}.$$

Proposition 4.2.1. — *The spectral curve Y_a is*

1. *reduced ;*
2. *connected ;*
3. *not always irreducible : there is a $1-1$ correspondence between the set of irreducible components of Y_a and the set of irreducible factors of the characterictic polynomial $\chi \in F[X]$.*

The assertion 1 is due to the fact that χ belongs to the regular semisimple locus $\mathbb{A}^{\mathrm{rss}}$ by construction of $\mathcal{A}$. Let

$$\mathcal{A}^{\mathrm{ell}}$$

be the open subset of (χ, τ) such that $\chi \in F[X]$ is irreducible. By the assertion 3 above, it is also the open subset of $a \in \mathcal{A}$ such that Y_a is integral.

The arithmetic genus of Y_a is defined by

$$q_{Y_a} = \dim_k(H^1(Y_a, \mathcal{O}_{Y_a})) = \dim_k(H^1(C, \pi_{a,*}\mathcal{O}_{Y_a}))$$

One can compute ,

$$\pi_{a,*}\mathcal{O}_{Y_a} = \mathcal{O}_C \oplus \mathcal{O}_C(-D) \oplus \ldots \oplus \mathcal{O}((-n+1)D)$$

and by the theorem of Riemann-Roch, one gets

Proposition 4.2.2. — *The arithmetic genus of Y_a does not depend on a and it is equal to*

$$q_{Y_a} = \frac{n(n-1)}{2}\deg(D) + n(g_C - 1) + 1.$$

4.3. Hitchin-Beauville-Narasimhan-Ramanan correspondence.

— This the following theorem (cf. [3]).

Theorem 4.3.1. — *Let $a \in \mathcal{A}$. The Hitchin fiber $\mathcal{M}_a = f^{-1}(a)$ is isomorphic to the stack of torsion-free coherent $\mathcal{O}_{Y_a}$-modules $\mathcal{F}$ of degree 0 and rank 1 at generic points of Y_a, equipped with a trivialization of their stalk at ∞_1.*

Let us recall briefly how one can construct a Hitchin bundle from a torsion-free $\mathcal{O}_{Y_a}$-module of rank 1. The multiplication by X on $\mathcal{O}_\Sigma$ gives the universal section

$$\mathcal{O}_\Sigma \to \pi_\Sigma^*\mathcal{O}_C(D)$$

and for any $a \in \mathcal{A}$ a section

$$\mathcal{O}_{Y_a} \to \pi_a^*\mathcal{O}_C(D).$$

By tensoring by a coherent $\mathcal{O}_{Y_a}$-modules $\mathcal{F}$, we get a morphism $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{Y_a}} \pi_a^*\mathcal{O}_C(D)$ and by the projection formula a twisted endomorphism of $\pi_{a,*}\mathcal{F}$

$$\theta : \pi_{a,*}\mathcal{F} \to \pi_{a,*}(\mathcal{F} \otimes_{\mathcal{O}_{Y_a}} \pi_a^*\mathcal{O}_C(D)) = \pi_{a,*}(\mathcal{F})(D).$$

If $\mathcal{F}$ is moreover torsion-free of rank 1, then $\pi_{a,*}\mathcal{F}$ is torsion-free $\mathcal{O}_C$-module of rank n. But since C is smooth, it is a locally free $\mathcal{O}_C$-module of rank n thus a vector bundle of rank n on C. So we have a pair $(\pi_{a,*}\mathcal{F}, \theta)$. We can compute its degree :

$$\begin{aligned}
\deg(\pi_{a,*}\mathcal{F}) &= \chi(C, \pi_{a,*}\mathcal{F}) + n(g_C - 1) \\
&= \chi(Y_a, \mathcal{F}) + n(g_C - 1) \\
&= \deg(\mathcal{F}) + \chi(Y_a, \mathcal{O}_{Y_a}) + n(g_C - 1) \\
&= \deg(\mathcal{F}) + 1 - q_{Y_a} + n(g_C - 1)
\end{aligned}$$

If we assume $\deg(\mathcal{F}) = 0$, using proposition 4.2.2, we get

$$\deg(\pi_{a,*}\mathcal{F}) = -\frac{n(n-1)}{2}\deg(D).$$

But remember that the divisor $D = 2D'$ is even. So the pair $(\pi_{a,*}\mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_C((n-1)D'), \theta)$ is a Hitchin bundle (the underlying vector bundle is now of degree 0). Finally the trivialization at ∞_1 gives our fourth datum e_1.

4.4. First consequences of theorem 4.3.1. — Let $\mathcal{A}^{\mathrm{sm}}$ the open set of a such that Y_a is smooth. In fact, one can show that $\mathcal{A}^{\mathrm{sm}}$ is not empty. When Y_a is smooth, a torsion-free $\mathcal{O}_{Y_a}$-module of rank 1 is a line bundle. Thus we get :

Corollary 4.4.1. — *For $a \in \mathcal{A}^{sm}$, the Hitchin fiber $\mathcal{M}_a$ is the Jacobian of Y_a. In particular, it is an abelian variety (and as such a scheme).*

For any $a \in \mathcal{A}$, we can consider the smooth commutative group scheme

$$\mathrm{Pic}^0(Y_a)$$

of line bundles on Y_a of degree 0, equipped with a trivialization of their stalk at ∞_1.

By tensor product, this groups acts on (and is an open substack of) the stack of torsion-free $\mathcal{O}_{Y_a}$-modules of degree 0 and rank 1. By theorem 4.3.1, we get an action of $\mathrm{Pic}^0(Y_a)$ on the Hitchin fiber $\mathcal{M}_a$.

Let $\mathcal{M}_a^{\mathrm{reg}} \subset \mathcal{M}_a$ be the open sub-stack $(\mathcal{E}, \theta, \tau, e_1) \in \mathcal{M}_a$ such that θ_c is regular for any $c \in C$. Recall that an endomorphism of k^n is regular if its centralizer is of minimal dimension, namely n. In the correspondence of theorem 4.3.1, this open substack corresponds to $\mathrm{Pic}^0(Y_a)$. We can state :

Corollary 4.4.2. — *$\mathcal{M}_a^{\mathrm{reg}}$ is a $\mathrm{Pic}^0(Y_a)$-torsor.*

4.5. Dimension of Hitchin fibers. — As a consequence of the work [1] of Altman-Iarrobino-Kleiman on compactified Jacobian, we have the following theorem

Theorem 4.5.1. — *Let $a \in \mathcal{A}$.*

1. *The open substack $\mathcal{M}_a^{\mathrm{reg}}$ is dense in $\mathcal{M}_a$.*
2. *We have*
$$\dim(\mathcal{M}_a) = \dim(\mathcal{M}_a^{\mathrm{reg}}) = \dim(\mathrm{Pic}^0(Y_a))$$
and this dimension is q_{Y_a}, the arithmetic genus of Y_a. In particular, $\dim(\mathcal{M}_a)$ does not depend on a.
3. *The set of irreducible components of $\mathcal{M}_a$ is a torsor under the abelian group of connected component of $\mathrm{Pic}^0(Y_a)$ and we have*
$$\pi_0(\mathrm{Pic}^0(Y_a)) \simeq \{(n_i) \in \mathbb{Z}^{Irr(Y_a)} \mid \sum_i n_i = 0\}$$
where $Irr(Y_a)$ is the set of irreducible components of Y_a.

Recall that Y_a is irreducible if and only if a belongs to the elliptic set $\mathcal{A}^{\mathrm{ell}}$. From the assertion 3 of theorem 4.5.1, we get the corollary :

Corollary 4.5.2. — *The Hitchin fiber $\mathcal{M}_a$ is irreducible if and only if a belongs to the elliptic set $\mathcal{A}^{\mathrm{ell}}$.*

One can also compute the dimension of the stack $\mathcal{M}$. We have

$$\dim(\mathcal{M}) = \dim(\mathcal{A}) + \dim(f).$$

The dimensions of $\mathcal{A}$ and f are respectively given by (3.4.1) and the combination of proposition 4.2.2 and the assertion 2 of theorem 4.5.1. We get :

Corollary 4.5.3. — *We have*

$$\dim(\mathcal{M}) = n^2 \deg(D) + 1.$$

5 Examples of Hitchin fibers in rank 2

5.1. The situation. — In this section, we take

$$C = \mathbb{P}^1_k$$

We write $\mathbb{P}^1_k = \mathrm{Spec}(k[y]) \cup \{0\}$ and in the affine chart $\mathrm{Spec}(k[y])$, the point ∞ is defined by the equation $y = 0$. We take

$$D = 2[0].$$

5.2. The base $\mathcal{A}$. — The scheme $\mathcal{A}$ classifies pairs

$$(X^2 - a_1(y)X + a_2(y), (\tau_1, \tau_2))$$

where, for $i = 1, 2$

$$a_i(y) \in H^0(\mathbb{P}^1_k, \mathcal{O}(2i[0])) = \{a_i(y) \in k[y] \mid \deg(a_i) \leqslant 2i\}$$

are such that the discriminant $(a_1)^2 - 4a_2$ does not vanish at $y = 0$ and $(\tau_1, \tau_2) \in k^2$ is the ordered pair of distinct root of

$$X^2 - a_1(0)X + a_2(0).$$

In the following, we will restrict ourselves to the case

$$a_1 = 0,$$

that is to the case of Hitchin pairs with traceless endomorphism.

Remember the open subsets

$$\mathcal{A}^{\mathrm{ell}} \subset \mathcal{A}^{\mathrm{sm}} \subset \mathcal{A}$$

Let $a \in \mathcal{A}$ with $a_1 = 0$. Then

1. $a \in \mathcal{A}^{\mathrm{sm}}$ if and only if a_2 has only simple roots.
2. $a \in \mathcal{A}^{\mathrm{ell}}$ if and only if a_2 is not a square in $k[y]$.

5.3. Spectral curves. — In our situation, all spectral curves are of arithmetic genus

$$q_{Y_a} = 1.$$

In figure 3 below, one finds some examples of spectral curve. From the left to the right, one has the following cases

- a_2 has only simple roots; the curve Y_a is a smooth projective curve of genus 1 : it is an elliptic curve.
- a_2 has a double root and two simple roots. Then Y_a is integral and has only one singularity which is a node.
- a_2 has a triple root. Then Y_a is integral and has only one singularity which is a cusp.
- a_2 is the square of a polynomial with two distinct roots. Then Y_a has two irreducible components which intersect transversally.

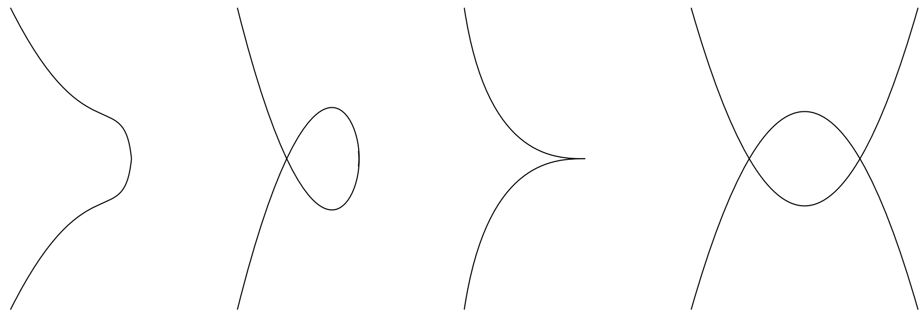

Figure 3
Examples of spectral curve

5.4. Examples of elliptic Hitchin fibers. — When $a \in \mathcal{A}^{\mathrm{ell}}$, the spectral curve is an integral curve of arithmeric genus 1 thus an elliptic curve or a plane cubic with a node or a cusp. The theorem 4.3.1 identifies the Hitchin fiber $\mathcal{M}_a$ with the compactified Jacobian of Y_a (which is simply the Jacobian when Y_a is smooth). But it is known (cf. [2]) that the compactified Jacobian for an curve integral curve of genus 1 is (non-canonically) isomorphic to the curve. So the first three pictures in figure 3 are also typical examples of elliptic Hitchin fibers.

5.5. A non-elliptic Hitchin fiber. — In this paragraph, we take $a_1(y) = 0$ and

$$a_2(y) = (y^2 - 1)^2.$$

In this case, the spectral curve Y_a is the union of two projective lines which intersect transversally. Let

$$U = \mathbb{P}^1_k - \{-1, 1\} = \mathrm{Spec}(k[(\frac{y+1}{y-1})^{\pm 1}]).$$

Let $y_\pm \in Y_a$ be the points over ± 1. To understand the Hitchin fiber $\mathcal{M}_a$, we will use theorem 4.3.1. So let $\mathcal{F}$ be a torsion-free $\mathcal{O}_{Y_a}$-module of generic ranks 1 and degree 0. Outside the points $y_\pm$, the $\mathcal{O}_{Y_a}$-module $\mathcal{F}$ is invertible and may be trivialized. We fix such a trivialization. Then the $\mathcal{O}_{Y_a}$-module $\mathcal{F}$ is determined by its restriction to the formal neighborhoods of $y_\pm$, which is a torsion-free

$$k[[y \pm 1]][X]/(X^2 - a_2(y))$$

-module of generic rank 1. By the obvious local analog of theorem 4.3.1, this is nothing else but a free $k[[y \pm 1]]$-submodule of $k((y \pm 1))$ of rank 2 stable under the endomorphism given by the matrix

$$\begin{pmatrix} y^2 - 1 & 0 \\ 0 & 1 - y^2 \end{pmatrix}.$$

So the choice of the local modules at $y_\pm$ amounts to a choice of points in a product of two affine Springer fibers. These affine Springer fibers are isomorphic to the one we studied in §5. Conversely, a point in this product gives two local modules at $y_\pm$ and thus a torsion-free $\mathcal{O}_{Y_a}$-module $\mathcal{F}$ of rank 1 with a trivialization of $\mathcal{F}$ on $Y_a - \{y_\pm\}$. We have to get rid of this trivialization. The group

$$H^0(U, \mathbb{G}^2_{m,k}) = (k^\times)^2((\frac{y+1}{y-1})^{\mathbb{Z}})^2 \simeq (k^\times)^2 \times (\mathbb{Z}^2),$$

which is the group of automorphism of the trivial line bundle on $Y_a - \{y_\pm\}$, acts on the product of affine Springer fibers. But we also want $\mathcal{F}$ to be of degree 0 and be trivialized at the marked point $\infty_1 \in Y_a$. For the first condition, we can assume that the point belongs to a fixed connected component of the product of affine Springer fibers, namely the product of connected components of "degree 0" of the two affine Springer fibers : this is a product of two infinite chains of projective lines (cf. figure 4 below).

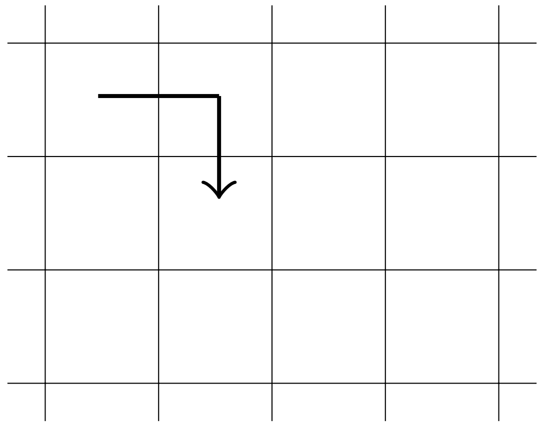

Figure 4
A product of two infinite chains of $\mathbb{P}^1$
In bold the action of $\mathbb{Z}$

The stabilizer in $H^0(U, \mathbb{G}_{m,k}^2)$ of both the point ∞_1 and the fixed connected component is identified with

$$\mathbb{G}_{m,k} \times \mathbb{Z}.$$

The group $\mathbb{Z}$ acts on each chain of projective line by translation (the generator $1 \in \mathbb{Z}$ sends a line to the next one) and the group $\mathbb{Z}$ acts anti-diagonally on the product (cf. the bold arrow in figure 4). The action of $\mathbb{G}_{m,k}$ on a chain of projective lines fixes the split lattices and the connected components of the complement are precisely the orbits of dimension 1. The group $\mathbb{G}_{m,k}$ acts diagonally on the product of the two chains. It preserves each square in figure 4. On the left in figure 5 below, we extracted a square from figure 4 and we drew with a circle the four fixed points for the $\mathbb{G}_{m,k}$-action. Moreover we drew some 1-dimensional orbits for this action. The main point to observe here is that upper orbits tend to the *two* bold orbits in the upper left corner. Similarly, lower orbits tend to the *two* bold orbits in the lower right corner. So when we take the quotient of the square by the action of $\mathbb{G}_{m,k}$, each fixed point gives a $\mathrm{B}\mathbb{G}_{m,k}$ and the quotient of the complement is a projective line with two non-separeted 0 and two non-separeted ∞ (pictured in figure 5 on the right).

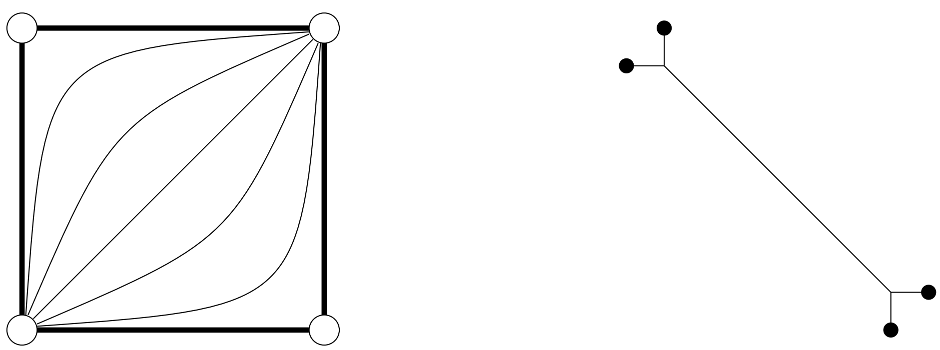

Figure 5
On the left : action of $\mathbb{G}_{m,k}$ on a square. On the right : the quotient (up to some $\mathrm{B}\mathbb{G}_{m,k}$).

Finally, the Hitchin fiber $\mathcal{M}_a$ can be identified with the quotient of the product of the two chains by the action of $\mathbb{G}_{m,k} \times \mathbb{Z}$. So, up to some $\mathrm{B}\mathbb{G}_{m,k}$, the Hitchin fiber $\mathcal{M}_a$ is an infinite chain of non-separeted projective lines (as pictured in figure 6). Note that it is neither of finite type nor separeted.

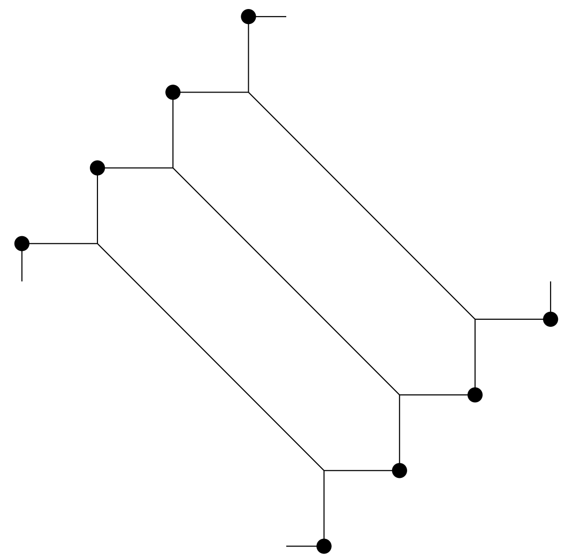

Figure 6
The Hitchin fiber $\mathcal{M}_a$ (up to $\mathrm{B}\mathbb{G}_{m,k}$)
An infinite chain of non-separeted projective lines

6 A truncated Hitchin fibration.

6.1. Properness over the elliptic locus. — Let $a \in \mathcal{A}^{\mathrm{ell}}$. The spectral curve Y_a is then an integral curve. The theorem 4.3.1 identifies the Hitchin fiber $\mathcal{M}_a$ with the compactified Jacobian of Y_a. We can introduce the relative spectral curve Y over the elliptic set $\mathcal{A}^{\mathrm{ell}}$ and the relative compactified Jacobian of Y over $\mathcal{A}^{\mathrm{ell}}$ can be identified with the elliptic Hitchin morphism $\mathcal{M}^{\mathrm{ell}} \to \mathcal{A}^{\mathrm{ell}}$. As a consequence of the work [2] of Altman-Kleiman on the compactified Jacobian, we have the following theorem.

Theorem 6.1.1. — *The elliptic Hitchin morphism*

$$f^{\mathrm{ell}} : \mathcal{M}^{\mathrm{ell}} = \mathcal{M} \times_{\mathcal{A}} \mathcal{A}^{\mathrm{ell}} \to \mathcal{A}^{\mathrm{ell}}$$

is proper and $\mathcal{M}^{\mathrm{ell}}$ is a smooth scheme over k.

Remark 6.1.2. — The properness of f^{ell} enables Ngô to apply the decomposition theorem for the morphism f^{ell} : so it is crucial in Ngô's proof of the fundamental lemma.

But outside the elliptic locus $\mathcal{A}^{\mathrm{ell}}$, the Hitchin fibration is neither of finite type nor separeted (as we saw in the example of §5.5). In our work with Laumon, we wanted to prove an advanced version of the fundamental lemma introduced by Arthur (the so-called weighted fundamental lemma). For this, we need to look outside the elliptic locus. This is why we needed to introduce a truncated version of the Hitchin fibration.

6.2. The notion of ξ-stability. — We first introduce a parameter of stability : let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} \xi_i = 0.$$

Let us introduce the following definition.

Definition 6.2.1. — Let $m = (\mathcal{E}, \theta, \tau, e_1) \in \mathcal{M}$. One says that m is ξ-stable if for any subbundle

$$0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$$

such that $\theta(\mathcal{F}) \subset \mathcal{F}(D)$ one has

$$\deg(\mathcal{F}) + \sum_i \xi_i < 0 \tag{6.2.1}$$

where the sum is over $1 \leqslant i \leqslant n$ such that τ_i is an eigenvalue of $\theta_{|\mathcal{F}_\infty}$.

Remark 6.2.2. — When $\xi = 0$, on gets the usual stability for the underlying Hitchin pairs $(\mathcal{E}, \theta)$. But here for generic ξ, we take advantage of our third datum τ.

The notion of ξ-stability is reminiscent (through theorem 4.3.1) of the work [14] of Estèves on the compactified Jacobian.

Remark 6.2.3. — The subbundles $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$ such that $\theta(\mathcal{F}) \subset \mathcal{F}(D)$ are determined by their generic fiber which must be a θ-stable linear subspace of the generic fiber of $\mathcal{E}$. However, by construction of $\mathcal{M}$, the endomorphism θ is generically regular semi-simple and there is only a finite number of θ-stable linear subspaces. Hence there is only a finite number of subbundles $\mathcal{F}$ satisfying $\theta(\mathcal{F}) \subset \mathcal{F}(D)$. When $m \in \mathcal{M}$ is elliptic, the characteristic polynomial of θ is generically irreducible and there is no such subspace and no such subbundle $\mathcal{F}$. So, any $m \in \mathcal{M}^{\mathrm{ell}}$ is ξ-stable for any ξ.

6.3. Properness of $\mathcal{M}^\xi$. — Let $\mathcal{M}^\xi$ be the ξ-stable sub-stack of $\mathcal{M}$. We say that ξ is generic if

$$\sum_{i \in I} \xi_i \notin \mathbb{Z}$$

for any $\emptyset \neq I \subsetneq \{1, \ldots, n\}$. Concretely, for generic ξ, there is no difference between the notions of ξ-stability and ξ-semistability (defined by large inequality in (6.2.1)).

Through theorem 4.3.1, we can deduce the following theorem from the work [14] of Estèves (the properness can also be proved directly from methods of the paper [21] by Langton).

Theorem 6.3.1. —

1. *The stack $\mathcal{M}^\xi$ is a smooth open sub-stack of $\mathcal{M}$ which contains $\mathcal{M}^{\mathrm{ell}}$. It is even an algebraic space.*
2. *If ξ is generic, the ξ-stable Hitchin fibration*

$$f^\xi : \mathcal{M}^\xi \to \mathcal{A}$$

is proper.

Remark 6.3.2. — As we shall see in the next example, if we take $\xi = 0$, the stack $\mathcal{M}^\xi$ is of finite type but one cannot check the existence part of the valuative criterium of properness for f^ξ. If we use a substack defined by the usual semi-stability (for Hitchin pairs) then we get a stack of finite type which is not separeted.

Remark 6.3.3. — Over finite field we can compute the number of rational points of a truncated Hitchin fiber $\mathcal{M}_a^\xi$. Perhaps a more surprising fact is that, for ξ generic, we get essentially a global weighted orbital integral constructed by Arthur (cf. section 11 of [7]). This generalizes proposition 3.6.1.

6.4. An example of a truncated Hitchin fiber. — Let's go back to the example of §5.5. In the pictures 7,8 and 9 below, we draw on the left the effect of stability, semi-stability and ξ-stability (for a generic ξ). In picture 7, we find that the stability condition isolates in the figure

4 a stair of width one (open) square . The stable Hitchin fiber $\mathcal{M}_a^{\xi=0}$ is a $\mathbb{G}_{m,k}$, thus not proper. In picture 8, the semi-stability condition defines a stair of width three squares ; the bold lines (which are $\mathbb{G}_{m,k}$) and also the vertices common to four bold lines belong to the stair. Even up to a $\mathrm{B}\mathbb{G}_{m,k}$ corresponding to these vertices, the quotient is not separeted. Finally, on picture 9, the effect of ξ-stability (for ξ generic) is to take a stair of width two squares ; the stair contains the bold lines but no vertex. Then the quotient has no more non-separeted points and the truncated Hitchin fiber $\mathcal{M}_a^{\xi}$ is isomorphic to two projective lines intersecting transversally. Not surprisingly, the Hitchin fiber $\mathcal{M}_a^{\xi}$ is thus isomorphic to the spectral curve.

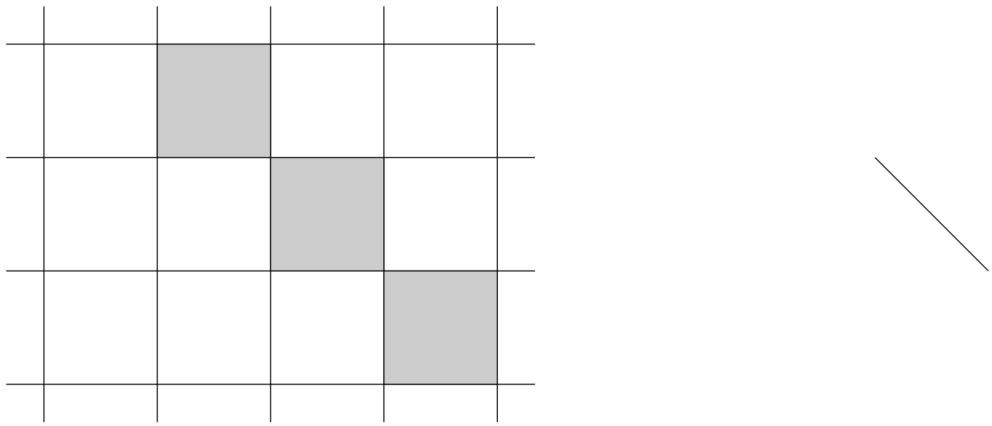

Figure 7
On the left, in grey the stable region and on the right the quotient ($\simeq \mathbb{G}_{m,k}$)

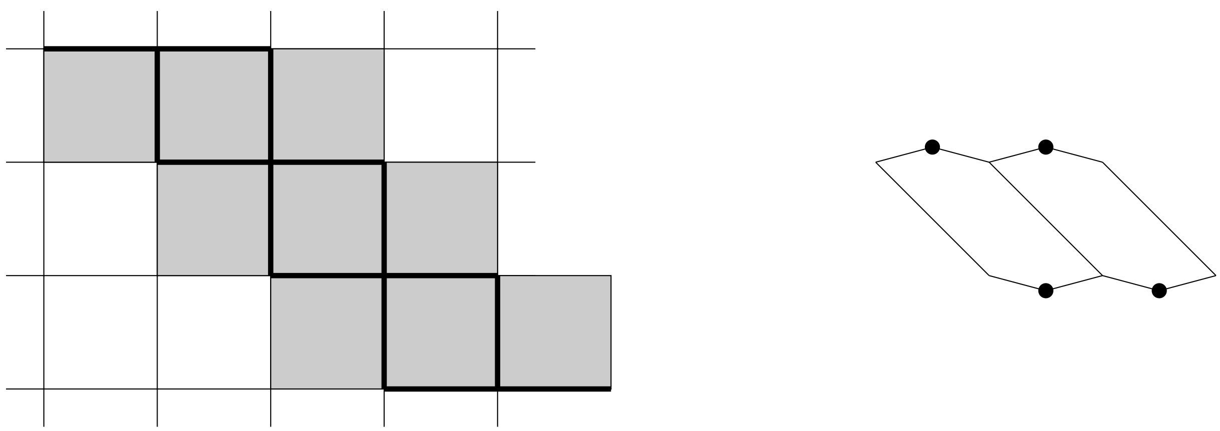

Figure 8
On the left, in grey the stable region and on the right the (non-separeted) quotient
There are two pairs of non-separeted bold points

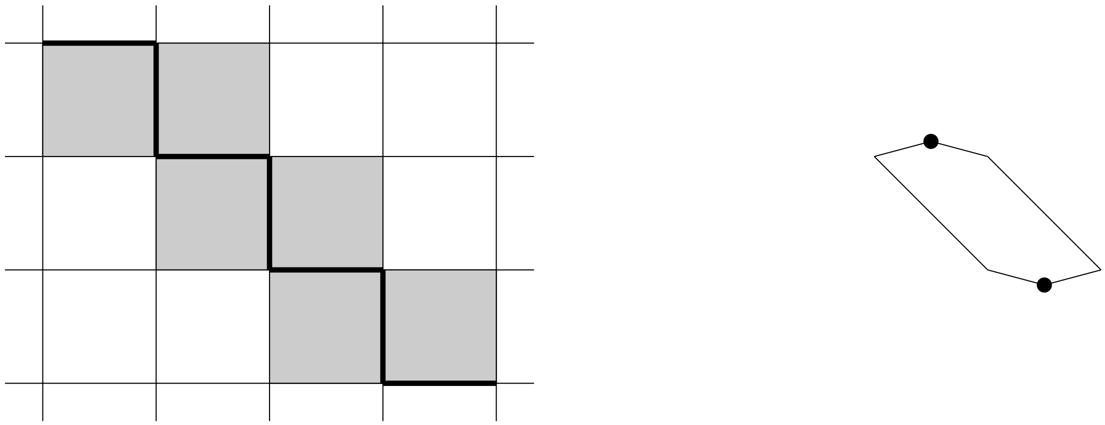

Figure 9
On the left, in grey the ξ-stable region (for generic ξ) and on the right the quotient

7 The main cohomological result

7.1. Invariant δ. — Let $\mathrm{Spf}(A)$ be a formal germ of a reduced curve. Let $\tilde{A}$ be its normalization in the total ring of fractions of A. We can attach to A the following invariants :

1. Serre's invariant
$$\delta_A = \mathrm{length}(\tilde{A}/A) \ ;$$
2. The number r_A of branches (which is also the number of connected components of $\mathrm{Spf}(\tilde{A})$) ;
3. The multiplicity m_A ;
4. The κ-invariant defined by
$$\kappa_A = \mathrm{length}(\Omega^1_{\tilde{A}/A})$$

Note that both δ_A and κ_A vanish for a non-singular germ. Note also that we have $\kappa_A = m_A - r_A$, at least when the characteristic of A is greater than the multiplicity m_A.

Let $a \in \mathcal{A}$ and Y_a be the corresponding spectral curve. We define the global invariants δ_a and κ_a by
$$\delta_a = \sum_{y \in |Y_a|} \delta_{\hat{\mathcal{O}}_{Y_a,y}}$$
and
$$\kappa_a = \sum_{y \in |Y_a|} \kappa_{\hat{\mathcal{O}}_{Y_a,y}}$$
where the sum is over the set of closed points of Y_a and the ring $\hat{\mathcal{O}}_{Y_a,y}$ is the completion of the local ring of Y_a at y. Thus we get two constructible functions $a \mapsto \delta_a$ and $a \mapsto \kappa_a$ on $\mathcal{A}$. The first one is moreover upper semi-continuous. Let $\delta \in \mathbb{N}$ and $\mathcal{A}^\delta$ be the locally closed subset of $a \in \mathcal{A}$ such that $\delta_a = \delta$. For $\delta = 0$, we have $\mathcal{A}^0 = \mathcal{A}^{\mathrm{sm}}$ the open dense subset of $a \in \mathcal{A}$ such that Y_a is smooth.

7.2. The codimension of $\mathcal{A}^\delta$. — For fields k of characteristic 0, in [13] Diaz and Harris have shown the equality for the codimension of $\mathcal{A}^\delta$:

(7.2.1) $$\mathrm{codim}_{\mathcal{A}}(\mathcal{A}^\delta) = \delta,$$

but their argument which uses the tangent cone does not work in positive characteristic (see also [22] sect. 3.3). Nonetheless, to get his main cohomological result, Ngô needs at least the inequality $\geqslant$ in (7.2.1) (which is unknown in positive characteristic). In his paper, Ngô is able to prove the desired inequality when δ is smaller than $\deg(D)$. But, outside the elliptic set, this last condition is never satisfied. In fact, in our work with Laumon, we use an open subset of $\mathcal{A}$ where the codimension of $\mathcal{A}^\delta$ can be computed by deformation theory. Let us sketch briefly our approach. In [8] section 9, we treated the analogous problem for the so-called cameral curve. We hope that the following discussion will make the arguments in [8] more transparent.

It is possible to consider a slight variant $\mathcal{B}$ of the moduli space of pairs

$$(a, \tilde{Y}_a \to_\varphi \Sigma_D)$$

such that $a \in \mathcal{A}$, the curve $\tilde{Y}_a$ is smooth and projective and the morphism φ is the normalization of the spectral curve $Y_a \subset \Sigma_D$. On any connected component of $\mathcal{B}$, the invariant δ_a of Y_a must be constant. Conversely, by the work of Teissier (cf. [12]), $\mathcal{A}^\delta$ is in one-one bijection with some connected components of $\mathcal{B}$. In general, one does not know how to compute the dimension of $\mathcal{B}$. However, when one has

$$\kappa_a < \deg(D) - 2g_C + 2, \tag{7.2.2}$$

the space $\mathcal{B}$ is smooth at $(a, \tilde{Y}_a \to_\varphi \Sigma_D)$. Indeed it suffices to check that

$$\mathrm{Ext}^2(L, \mathcal{O}_{\tilde{Y}_a}) = 0 \tag{7.2.3}$$

where L is the cotangent complex (in degrees -1 and 0)

$$L = [\varphi^*\Omega^1_{\Sigma_D/C} \to \Omega^1_{\tilde{Y}_a/C}].$$

One can write $\mathrm{Ext}^2(L, \mathcal{O}_{\tilde{Y}_a}) = \mathrm{Ext}^1(\mathcal{H}^{-1}(L), \mathcal{O}_{\tilde{Y}_a})$ where $\mathcal{H}^{-1}(L)$ is the sheaf of cohomology of L in degree -1. By Serre duality, it suffices to check that

$$H^0(\tilde{Y}_a, \mathcal{H}^{-1}(L) \otimes \Omega^1_{\tilde{Y}_a/k}) = 0 \tag{7.2.4}$$

Take $\pi = \pi_\Sigma \circ \varphi$. One can compute the degree

$$\begin{aligned}
\deg(\mathcal{H}^{-1}(L) \otimes \Omega^1_{\tilde{Y}_a/k}) &= \deg(\varphi^*\Omega^1_{\Sigma_D/C}) - \deg(\Omega^1_{\tilde{Y}_a/C}) + \deg(\Omega^1_{\tilde{Y}_a/Y_a}) + \deg(\Omega^1_{\tilde{Y}_a/k}).\\
&= \deg(\pi^*\mathcal{O}(-D)) + \kappa_a + \deg(\pi^*\Omega^1_{C/k})\\
&= -n\deg(D) + \kappa_a + n(2g_C - 2)
\end{aligned}$$

which is negative if

$$\kappa_a < n(\deg(D) - 2g_C + 2).$$

This condition is satisfied by (7.2.2) and gives the vanishing assertion (7.2.3) when $\tilde{Y}_a$ is connected. In general, a similar computation for each connected component of $\tilde{Y}_a$ gives the same result under the condition (7.2.2).

Once we know the smoothness at $(a, \tilde{Y}_a \to_\varphi \Sigma_D)$, the dimension is given by

$$\dim(\mathrm{Ext}^1(L, \mathcal{O}_{\tilde{Y}_a})) = \dim(\mathrm{Ext}^0(\mathcal{H}^{-1}(L), \mathcal{O}_{\tilde{Y}_a}) + \dim(\mathrm{Ext}^1(\mathcal{H}^0(L), \mathcal{O}_{\tilde{Y}_a})). \tag{7.2.5}$$

The sheaf $\mathcal{H}^0(L) = \Omega^1_{\tilde{Y}_a/Y_a}$ is a torsion sheaf of length κ_a. Denoting by r_a the number of connected components of $\tilde{Y}_a$, we can compute (7.2.5) by Serre duality, the Riemann-Roch formula, the vanishing property (7.2.4) and the dimension (3.4.1) of $\mathcal{A}$: this gives

$$\begin{aligned}
\dim(\mathrm{Ext}^1(L, \mathcal{O}_{\tilde{Y}_a})) &= n(\deg(D) - 2g_C + 2) - \kappa_a + g_{\tilde{Y}_a} - r_a + \kappa_a\\
&= n(\deg(D) - 2g_C + 2) + g_{\tilde{Y}_a} - r_a\\
&= \dim(\mathcal{A}) - (\frac{n(n-1)}{2}\deg(D) + n(g_C - 1) - g_{\tilde{Y}_a} + r_a)\\
&= \dim(\mathcal{A}) - q_a + 1 + g_{\tilde{Y}_a} - r_a\\
&= \dim(\mathcal{A}) - \delta_a
\end{aligned}$$

The last two equalities come on the one hand from the formula for the arithmetic genus q_a of Y_a (cf. proposition 4.2.2) and on the other hand from the long exact sequence in cohomology associated to

$$0 \to \mathcal{O}_{Y_a} \to \mathcal{O}_{\tilde{Y}_a} \to \mathcal{O}_{\tilde{Y}_a}/\mathcal{O}_{Y_a} \to 0. \tag{7.2.6}$$

This is precisely expected by (7.2.1).

7.3. The open subset $\mathcal{A}^{\mathrm{good}}$. — It is defined as the biggest open subset of $\mathcal{A}$ such that for any closed point $a \in \mathcal{A}^{\mathrm{good}}$ the inequality (7.2.2) is true. It contains for example all a such that the irreducible components of the spectral curve Y_a are smooth.

7.4. The decomposition theorem. — Let ℓ be a prime number *invertible* in k. Let ξ be a generic stability parameter (cf. §§ 6.2 and 6.3). By theorem 6.3.1, the algebraic space $\mathcal{M}^\xi$ is smooth over k and the Hitchin morphism f^ξ is proper. By Deligne theorem (cf. [11]), this implies that the complexe of ℓ-adic sheaves

$$Rf^\xi_*\overline{\mathbb{Q}}_\ell$$

is pure. By the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber [4], the direct sum of perverse cohomology sheaves

$$^p\mathcal{H}^\bullet(Rf^\xi_*\overline{\mathbb{Q}}_\ell) = \bigoplus_i {}^p\mathcal{H}^i(Rf^\xi_*\overline{\mathbb{Q}}_\ell)$$

is *semi-simple*. So we can write

$$^p\mathcal{H}^\bullet(Rf^\xi_*\overline{\mathbb{Q}}_\ell[\dim(\mathcal{M})]) = \bigoplus_{a\in\mathcal{A}} i_{a,*}j_{a,!*}\mathcal{F}^\bullet_a[\dim(a)], \tag{7.4.1}$$

where

- the sum is over Zariski points in $\mathcal{A}$;
- $i_a : \overline{\{a\}} \hookrightarrow \mathcal{A}$ is the canonical inclusion of the closure of a in $\mathcal{A}$;
- $\mathcal{F}^\bullet_a$ is a graded local system on a smooth open subset of $\overline{\{a\}}$ and $j_{a,!*}\mathcal{F}^\bullet_a$ is its middle extension to $\overline{\{a\}}$.

7.5. Properties of the socle. — The socle of ${}^p\mathcal{H}^\bullet(Rf^\xi_*\overline{\mathbb{Q}}_\ell[\dim(\mathcal{M})])$ is the finite set of $a \in \mathcal{A}$ such that $\mathcal{F}^\bullet_a \neq 0$. A fundamental problem is to determine the socle. We shall see in the next paragraph that it is possible to determine it at least if the Hitchin morphism f^ξ is restricted to the open subset $\mathcal{A}^{\mathrm{good}}$. Besides Ngô's article [24], we refer also the reader to [25] another article of Ngô.

Meanwhile, we would like to give some properties of the socle. Let a be an element of the socle. The amplitude of a is defined by

$$\mathrm{Ampl}(a) = m_a - m'_a$$

where m_a, resp. m'_a, is the maximum, resp. the minimum, of integers m such that $\mathcal{F}^m_a \neq 0$. By Poincaré duality, we have

$$\mathcal{F}^{-m}_a = (\mathcal{F}^m_a)^\vee[\dim(a)]$$

so $m'_a = -m_a$ and

$$\mathrm{Ampl}(a) = 2m_a.$$

Moreover since $\mathcal{F}^{m_a}_a$ appears in $R^{\dim(\mathcal{M}^\xi)+m_a-\dim(a)}f^\xi_*\overline{\mathbb{Q}}_\ell$, we must have

$$\dim(\mathcal{M}^\xi) + m_a - \dim(a) \leqslant 2\dim(f^\xi) \tag{7.5.1}$$

and we have equality in (7.5.1) if and only if $\mathcal{F}^{m_a}_a$ appears in $R^{2\dim(f^\xi)}f^\xi_*\overline{\mathbb{Q}}_\ell$. We get

$$m_a \leqslant \dim(f^\xi) - \dim(\mathcal{A}) + \dim(a). \tag{7.5.2}$$

So the amplitude satisfies the first inequality (with the same case of equality as that for (7.5.1))

$$\text{(7.5.3)} \qquad \mathrm{Ampl}(a) \leqslant 2(\dim(f^\xi) - \dim(\mathcal{A}) + \dim(a)).$$

In what follows we will be a little sketchy (for example, we do not specify when we have to take geometric points instead of Zariski ones). Recall (cf. §4.4) that the Picard group scheme $\mathrm{Pic}^0(Y_a)$ acts on the Hitchin fiber $\mathcal{M}_a$. In fact, this action does not preserve the open substack $\mathcal{M}_a^\xi$. But the neutral component of $\mathrm{Pic}^0(Y_a)$, denoted by P_a, does act on $\mathcal{M}_a^\xi$. It is then possible to deduce from it an action of the homology of P_a on the stalk of $\mathcal{F}_a^\bullet$ at a. But we have the usual Chevalley dévissage

$$0 \to P_a^{\mathrm{aff}} \to P_a \to P_a^{\mathrm{ab}} \to 0$$

where P_a^{aff} is affine and the quotient is an abelian scheme. By a weight argument, (cf. [24] §7.4.8), the action of the homology of P_a factors through an action of the homology of P_a^{ab}; moreover the stalk of $\mathcal{F}_a^\bullet$ at a is a free graded module over this homology (proposition 7.4.10 de [24], cf. also [8] proposition 10.3.1 and proof of theorem 10.5.1). So the amplitude satisfies the second inequality

$$\text{(7.5.4)} \qquad \mathrm{Ampl}(a) \geqslant 2\dim(P_a^{\mathrm{ab}}).$$

In fact, the group scheme P_a^{ab} can be identified with the the neutral component of the Picard group scheme of the normalization $\tilde{Y}_a$ of Y_a. We thus have

$$\dim(P_a^{\mathrm{ab}}) = \dim(H^1(\tilde{Y}_a, \mathcal{O}_{\tilde{Y}_a}))$$

and by the long exact sequence in cohomology associated to the short exact sequence (7.2.6) and theorem 4.5.1, we get

$$\text{(7.5.5)} \qquad \begin{aligned} \dim(P_a^{\mathrm{ab}}) &= -1 + r_a - \delta_a + \dim(P_a) \\ &= -1 + r_a - \delta_a + \dim(f^\xi) \end{aligned}$$

where r_a is the number of connected components of $\tilde{Y}_a$. Combining (7.5.4) and (7.5.3), we get

$$\text{(7.5.6)} \qquad \mathrm{codim}_{\mathcal{A}}(a) \leqslant \delta_a - r_a + 1.$$

7.6. The support theorem on $\mathcal{A}^{\mathrm{good}}$. — *From now on, we assume that we are working on the open subset* $\mathcal{A}^{\mathrm{good}}$. So we should introduce new notations $\mathcal{M}^{\xi,\mathrm{good}} = \mathcal{M}^\xi \times_{\mathcal{A}} \mathcal{A}^{\mathrm{good}}$ and so on. By abuse, we will keep the former ones. We can state the main cohomological theorem which is the key of the fundamental lemma.

Theorem 7.6.1. — *The socle of ${}^p\mathcal{H}^\bullet(Rf_*^\xi\overline{\mathbb{Q}}_\ell)$ contains a single element which is the generic point of $\mathcal{A}$.*

In other words, the only support of a simple constituent of ${}^p\mathcal{H}^\bullet(Rf_*^\xi\overline{\mathbb{Q}}_\ell)$ is $\mathcal{A}$ itself.

Remark 7.6.2. — This theorem is due to Ngô on the elliptic set and to Laumon and myself for the extension outside the elliptic set. By the theorem, the perverse cohomology of the Hitchin fiber is determined by its restriction to any open dense subset of $\mathcal{A}$. Thanks to the Grothendieck-Lefschetz trace formula and the countings of proposition 3.6.1 and remark 6.3.3, the stalks of the perverse cohomology are related to global (weighted) orbital integrals. So the theorem gives a technical sense to the vague assertion that global (weighted) orbital integrals are "limits" of the simplest orbital integrals (those associated to smooth spectral curves which can be computed "by hands").

Remark 7.6.3. — For general groups, the support theorem is not true as stated : in general, there are other supports besides $\mathcal{A}$. But all new supports fit perfectly in the theory of endoscopy : they are the bases of Hitchin fibration associated to endoscopic groups. The determination of the supports is the key to the solution of the fundamental lemma. Indeed, Ngô first checks by hand a global variant of the fundamental lemma on a smaller set. Then, by his support theorem, the identity extends to a larger set. Finally he gets from it the local statement by local-global methods.

Let us briefly explain how to get theorem 7.6.1. Let a be an element in the socle. If a does not belong to the elliptic set $\mathcal{A}^{\mathrm{ell}}$, the spectral curve has at least two irreducible components so we have $r_a > 1$. Thus (7.5.6) gives

$$\operatorname{codim}_{\mathcal{A}}(a) < \delta_a$$

in contradiction with (7.2.1) (here we use that $a \in \mathcal{A}^{\mathrm{good}}$). If $a \in \mathcal{A}^{\mathrm{ell}}$ then $r_a = 1$. By (7.2.1), the inequalities (7.5.6) and (7.5.1) must be equalities. Hence $\mathcal{F}_a^{m_a}$ must appear in $R^{2\dim(f^\xi)} f_*^\xi \overline{\mathbb{Q}}_\ell$. But on the elliptic set, the Hitchin fibers are irreducible (cf. theorem 4.5.1) and this sheaf is simply the constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\mathcal{A}^{\mathrm{ell}}$. So a must be the generic point of $\mathcal{A}$.

Remark 7.6.4. — For Hitchin fibrations for general reductive groups, the elliptic fibers are in general not irreducible : this explains in part why there are new supports besides $\mathcal{A}$ in the decomposition theorem.

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